PACIFIC JOURNAL OF MATHEMATICS
Vol. 80, No. 1, 1979

OSCILLATION RESULTS FOR A NONHOMOGENEOUS EQUATION

SAMUEL M. RANKIN, III

The purpose of this note is to investigate oscillatory properties of solutions of the equation

$$(1) \qquad y''+p(t)y=f(t)$$

via the transformation $y(t)=u(t)z(t)$ where $u(t)$ is a solution of the equation

$$(2) \qquad u''+p(t)u=0\ .$$

Equation (2) is assumed to be nonoscillatory throughout the paper. This represents a distinct change from most of the recent work concerning oscillation in equation (1).

The transformation $y(t) = \phi(t)z(t)$ transforms equation (1) into

$$(3) \qquad (\phi^2 z')' + \phi(t)(\phi''(t) + p(t)\phi(t))z = f(t)\phi(t)\ .$$

If $\phi(t)$ is a solution of (2) then (3) becomes

$$(3') \qquad (\phi^2 z')' = f(t)\phi(t)\ .$$

Equation (3′) enables us to characterize the oscillatory behavior of solutions of (1) in terms of the forcing function $f(t)$ and the nonoscillatory solutions of equation (2). The need for "explicit" sign conditions on $p(t)$ is eliminated. However, some implicit sign conditions will be assumed, that is, the solution $\phi(t)$ of equation (2) will be given properties that are implied by specific sign conditions on $p(t)$.

In recent articles Macki [10] and Komkov [7] have pointed out the usefulness of the transformation $u(t) = \phi(t)z(t)$ in studying qualitative properties of the differential equation

$$(r(t)u')' + p(t)u = 0\ .$$

As usual a nontrivial solution $y(t)(u(t))$ of equation (1) [resp. (2)] is oscillatory if on each ray $(a, \infty)(a > 0)$ there exists a $t_0 \in (a, \infty)$ with $y(t_0) = 0$ $(u(t_0) = 0)$. Equation (1) [resp. (2)] is oscillatory if all solutions are oscillatory. A solution $y(t)$ [resp. $u(t)$] of equation (1) [resp. (2)] is nonoscillatory if it is eventually nonzero. It is well known that all solutions of equation (2) are either oscillatory or nonoscillatory. The functions $p(t)$ and $f(t)$ are assumed to be continuous on $[0, \infty)$, so only solutions on the interval $[0, \infty)$ will be considered.

There has been considerable interest in the oscillatory properties of equation (1) and some of its nonlinear analogues, for example, Abramovich [1], Grimmer and Patula [2], Graef and Spikes [3] [4], Hammett [5], Jones and Rankin [6], Lovelady [8] [9], Rankin [11] [12], Singh [13], Skidmore and Bowers [14], Tefteller [15] and Wallgren [16]. In each of these papers, except [2] and [11], a sign condition is imposed on $p(t)$, and in all but [6] and [8] the unforced equation is either implicitly or explicitly assumed oscillatory.

To motivate our first theorem, consider the following examples:

EXAMPLE 1. $u'' + (1/4)t^{-2}u = 0$ $y'' + (1/4)t^{-2}y = t(1/2)\sin t$ and

EXAMPLE 2. $u'' = 0$ $y'' = t\sin t$.

It is seen below that the nonhomogeneous equations in the above examples are oscillatory.

THEOREM 1. *If there exists a positive solution $\phi(t)$ of equation* (2) *such that for each $T > 0$ and for some $M > 0$*

(i) $\underline{\lim}_{t\to\infty}\int_T^t f(s)\phi(s)ds = -\infty$ *and* $\overline{\lim}_{t\to\infty}\int_T^t f(s)\phi(s)ds = \infty$,

(ii) $\left|\int_T^t 1/\phi^2(s)\int_T^s f(r)\phi(r)\,drds\right| \leqq M\int_T^t ds/\phi^2(s)$ *and*

(iii) $\lim_{t\to\infty}\int_T^t ds/\phi^2(s) = \infty$, *then equation* (1) *is oscillatory.*

REMARK. In Theorem 1 and the theorems given below, it is easily seen that if $f(t)$ satisfies our hypothesis, so does $-f(t)$. The transformation $v = -y$ changes (1) into an equation of the same form preserving the assumptions of the theorems. Therefore, when we assume a solution $y(t)$ of equation (1) is nonoscillatory, we will assume $y(t) > 0$ on some ray (a, ∞).

Proof of Theorem 1. Suppose equation (1) is nonoscillatory so that there exists a solution $y(t)$ of equation (1) such that $y(t) > 0$ on (a, ∞) for some $a > 0$. The function $z(t)$, defined by $y(t) = \phi(t)z(t)$, is a nonoscillatory solution of equation (3′). After integrating (3′) and applying (i), we have that $\underline{\lim}_{t\to\infty}\phi^2(t)z'(t) = -\infty$. Now choosing $T_1 > T$ such that $\phi^2(T_1)z'(T_1) < -2M$, we have by integration that

$$z(t) = z(T_1) + \phi^2(T_1)z'(T_1)\int_{T_1}^t ds/\phi^2(s) + \int_{T_1}^t 1/\phi^2(s)\int_{T_1}^s f(r)\phi(r)drds\,.$$

From (ii) we obtain

$$z(t) < z(T_1) - M\int_{T_1}^{t} ds/\phi^2(s)\ ,$$

and by (iii) the solution $z(t)$ is eventually negative. This contradicts $y(t) > 0$ on $[T, \infty)$.

REMARK. In Example (1), choose $\phi(t) = t^{1/2}$ and in Example (2), $\phi(t) = 1$.

THEOREM 2. *If there exists a positive solution* $\phi(t)$ *of equation* (2) *such that for* T *sufficiently large*

(i) $\underline{\lim}_{t\to\infty}\int_{T}^{t} 1/\phi^2(s)\int_{T}^{s} f(r)\phi(r)drds = -\infty$ *and*
$\overline{\lim}_{t\to\infty}\int_{T}^{t} 1/\phi^2(s)\int_{T}^{s} f(r)\phi(r)drds = \infty$ *and*

(ii) $\lim_{t\to\infty}\int_{T}^{t} ds/\phi^2(s) < \infty$ *then equation* (1) *is oscillatory.*

Proof. Suppose there exists a solution $y(t)$ of equation (2) such that $y(t) > 0$ on (a, ∞) for some $a > 0$, then the function $z(t)$, defined by $y(t) = \phi(t)z(t)$, is a positive solution of equation (3′) on $[T, \infty)$ for some $T > a$. Integrating equation (3′) twice we have

$$z(t) = z(T) + \phi^2(T)z'(T)\int_{T}^{t} ds/\phi^2(s) + \int_{T}^{t} 1/\phi^2(s)\int_{T}^{s} f(r)\phi(r)drds\ .$$

By conditions (i) and (ii), $z(t)$ satisfies $z(t_0) < 0$ for some $t_0 > T$, thus contradicting the positivity of $y(t)$ on (a, ∞).

EXAMPLE 3. The equation $y'' - y = e^{3t}\sin t$ illustrates Theorem 2 where $\phi(t) = e^t$. Also for $y'' = t^3\cos t$ choose $\phi(t) = t$.

EXAMPLE 4. For the equation $y'' - y = \sin t$ all of the conditions of Theorems 1 and 2 are not met. This equation has the general solution $y(t) = -1/2\sin t + c_1e^{-t} + c_2e^t$. Notice that all bounded solutions on $[0, \infty)$ can be written in the form $y(t) = -1/2\sin t + c_1e^{-t}$ for some c_1. It is easily seen that these solutions are oscillatory. The following theorem can now be stated.

THEOREM 3. *If there exists a positive bounded solution* $\phi(t)$ *of equation* (2) *and an* $a > 0$ *such that*

(i) $\lim_{t\to\infty}\phi(t)\int_{T}^{t} ds/\phi^2(s) = \lim_{t\to\infty}\int_{T}^{t} ds/\phi^2(s) = \infty$ *for each* $T > a$ *and*

(ii) *there exists a sequence* $\{T_n\}_{n=1}^{\infty}$ *such that* $\lim_{n\to\infty} T_n = \infty$,

$\lim_{t\to\infty}\int_{T_n}^{t} f(s)\phi(s)ds = 0$, $\underline{\lim}_{t\to\infty}\int_{T_n}^{t} 1/\phi^2(s)\int_{T_n}^{s} f(r)\phi(r)drds = -\infty$, $\overline{\lim}_{t\to\infty}\int_{T_n}^{t} 1/\phi^2(s)$ $\int_{T_n}^{s} f(r)\phi(r)drds = \infty$, *and* $\left|\phi(t)\int_{T_n}^{t} 1/\phi^2(s)\int_{T_n}^{s} f(r)\phi(r)drds\right|$ *is bounded, then all bounded solutions of equation* (1) *are oscillatory.*

Proof. Suppose there exists a bounded solution $y(t)$ of equation (1) such that $y(t) > 0$ on $[T, \infty)(T > a)$. Integrating equation (3′) from T_n to t for some $T_n > T$, we have

$$\phi^2(t)z'(t) = \phi^2(T_n)z'(T_n) + \int_{T_n}^{t} f(s)\phi(s)ds . \tag{$*$}$$

$\phi^2(T_n)z'(T_n)$ is greater than 0, for each n, for if $\phi^2(T_n)z'(T_n) = 0$, a second integration yields

$$z(t) = z(T_n) + \int_{T_n}^{t} 1/\phi^2(s)\int_{T_n}^{s} f(r)\phi(r)drds \text{ and by (ii)}$$

$\underline{\lim}\, z(t) = -\infty$, a contradiction. If $\phi^2(T_n)z'(T_n)$ is negative, then choose $\varepsilon > 0$ such that $\phi^2(T_n)z'(T_n) + \varepsilon < 0$. By (ii), it is true for $t > T'$ for some $T' > T_n$ that $\int_{T_n}^{t} f(s)\phi(s)ds < \varepsilon$ and from (*) $z'(t) < \phi^2(T_n)z'(T_n) + \varepsilon/\phi^2(t)$, for $t \geqq T'$. Integrating the above inequality from T' to t gives $z(t) < (\phi^2(T_n)z'(T_n) + \varepsilon)\int_{T_n}^{t} ds/\phi^2(s) + z(T')$. Applying (i), it can be seen that $z(t)$ will eventually be negative.

Now, integrating (*) from T_n to t and multiplying by $\phi(t)$ gives

$$y(t) = \phi(t)z(t) = \phi(t)z(T_n) + \phi^2(T_n)z'(T_n)\phi(t)\int_{T_n}^{t} ds/\phi^2(s) + \phi(t)\int_{T_n}^{t} 1/\phi^2(s)\int_{T_n}^{s} f(r)\phi(r)drds .$$

The left side of the above equality remains bounded while the right side approaches infinity by (i), (ii), and the fact that $\phi^2(T_n)z'(T_n)>0$; the theorem is proved.

It is an easy exercise to see that $w(t) = y_1(t) - y_2(t)$ is a solution of equation (2) whenever $y_1(t)$ and $y_2(t)$ are solutions of equation (1). Thus if equation (2) is nonoscillatory, there are at most a finite number of points $t_1 \cdots t_n$ such that $y_1(t_i) = y_2(t_i)$ for $i = 1, 2, \cdots, n$. Let us further assume that $y_1(t)$ and $y_2(t)$ have no double zeros for large t and that for sufficiently large a, b, $y_1(a) = y_1(b) = 0$ with $y_1 \neq 0$ on (a, b). Then if $y_2(t_0) = 0$ for some $t_0 \in (a, b)$, the solution $y_2(t)$ of (1) has an even number of zeros in (a, b).

To obtain asymptotic results for nonoscillatory solutions of equation (1), equation (3) is considered once more where $\phi(t)$ is not necessarily a solution of equation (2). The following results of Hammett [5] and Graef and Spikes [3] for the differential equation

$$(4) \qquad (r(t)v')' + p(t)v = f(t)$$

will be useful.

THEOREM 4. [Hammett, 5]. *If*
(i) $r(t) > k > 0$ *on* $[0, \infty)$ *and* $\int_0^\infty dt/r(t) = \infty$,
(ii) $p(t) > k > 0$
(iii) $f \in L(0, \infty)$
then all nonoscillatory solutions $v(t)$ *of* (4) *satisfy* $\lim v(t) = 0$.

THEOREM 5. [Graef and Spikes, 3]. *If*
(i) $r(t) > 0$ *on* $[0, \infty)$ *and* $\int_0^\infty dt/r(t) = \infty$,
(ii) $p(t) > 0$ *and* $\int_0^\infty p(s)ds = \infty$,
(iii) $\int_0^\infty \left(\int_0^w ds/r(s)\right)|f(w)|\,dw < \infty$,
then all nonoscillatory solutions $v(t)$ *of* (4) *satisfy* $\lim_{t\to\infty} v(t) = 0$.

THEOREM 6. *If there exists a positive function* $\phi(t)$ *such that* $\phi(t)f(t) \in L(0, \infty)$, $\phi(\phi''(t) + p(t)\phi(t)) > K_1\ \phi^2(t) > K_1$ *for some* $K_1 > 0$ *and* $\int^\infty ds/\phi^2(s) = \infty$, *then every nonoscillatory solution of equation* (1) *satisfies* $\lim_{t\to\infty} y(t)\phi(t) = 0$.

Proof. By Theorem 4 and the hypothesis, each nonoscillatory solution $z(t)$ of equation (3) satisfies $\lim_{t\to\infty} z(t) = 0$.

EXAMPLE 5. For the equation

$$(5) \qquad y'' + t^{-1}y = 2t^{-3} + t^{-2}$$

let $\phi(t) = t^{1/2}$ and the conditions of the theorem are satisfied. Notice that equation (5) does not satisfy all of Hammett's hypothesis.

THEOREM 7. *If* $\int_b^\infty \left(\int_b^s dw/\phi^2(w)\right)|\phi(s)f(s)|\,ds < \infty$ *where* $\phi(t) > 0$, $\int^\infty dw/\phi^2(w) = \infty$, $\int^\infty [\phi''(t)\phi(t) + p(t)\phi^2(t)]dt = \infty$, *and* $\phi''\phi + p(t)\phi^2 > 0$ *then all nonoscillatory solutions of equation* (1) *satisfy* $\lim y(t)/\phi(t) = 0$.

Proof. Equation (3) now satisfies the hypothesis of Theorem 5 and so $\lim_{t\to\infty} z(t) = 0$ for each nonoscillatory solution $z(t)$ of equation (3).

EXAMPLE 6. The following equation is more general than equation (1) but illustrates the usefulness of the transformation $y(t) = \phi(t)z(t)$:

$$(6) \qquad (ty')' + t^{-1/2}y = t^{-2} + t^{-3/2} .$$

Equation (6) does not satisfy condition (iii) of Theorem 5. However, using the above transformation with $\phi(t) = t^{-1/4}$, all conditions of Graef and Spikes' theorem are satisfied for the equation

$$(t^{1/2}z')' + (5/16\, t^{-10/4} + t^{-1})z = t^{-9/4} + t^{-7/4}$$

and so for all nonscillatory solutions $z(t)$, $\lim_{t\to\infty} z(t) = 0$. Since $y(t) = t^{-1/4}z(t)$, all nonoscillatory solutions $y(t)$ of equation (6) satisfy $\lim_{t\to\infty} t^{1/4}y(t) = 0$.

REMARK. The transformation $y(t) = \phi(t)z(t)$ maks it possible not to require $p(t)$ to be positive as required in [3] and [5].

ACKNOWLEDGMENT. The author wishes to thank the referee for his many helpful comments and suggestions.

REFERENCES

1. S. Abramovich, *On the behavior of the solutions of $y''+p(t)=f(t)$*, J. Math. Anal. Appl., **52** (1975), 465-470.
2. R. Grimmer and W. Patula, *Nonoscillatory solutions of forced second order linear equations*, J. Math. Anal. Appl., **56** (1976), 452-459.
3. J. Graef and P. Spikes, *Asymptotic behavior of solutions of a second order nonlinear differential equation*, J. Differential Equations, **17** (1975), 461-476.
4. ———, *Sufficient conditions for nonoscillation of a second order nonlinear differential equation*, Proc. Amer. Math. Soc., **50** (1975), 289-292.
5. M. Hammett, *Nonoscillation properties of a nonlinear differential equation*, Proc. Amer. Math. Soc., **30** (1971), 92-96.
6. G. Jones and S. Rankin, *Oscillation of a forced non-linear second order differential equation*, in Accademia Nazionale Dei Lencei, **59** (1975), 702-705.
7. V. Komkov, *A technique for the detection of oscillation of second order ordinary differential equations*, Pacific J. Math., **42** (1972), 105-115.
8. D. Lovelady, *Oscillations induced by forcing functions*, to appear Accademia Nazionale dei Lencei.
9. ———, *Oscillation and a class of forced even order differential equations*, to appear.
10. J. Macki, *Application of change of variables in the qualitative theory of second order linear ordinary differential equations*, SIAM Review, **18** (1976), 269-274.
11. S. Rankin, *Oscillation for second order nonhomogeneous linear differential equations*, J. Math. Anal. Appl., **53** (1976).
12. ———, *Oscillation of a forced second order nonlinear differential equation*, Proc. Amer. Math. Soc., **58** (1976).
13. B. Singh, *Comparative study of asymptotic nonoscillation and quick oscillation of second order linear differential equations*, J. Math. and Phy. Sci., **8** (1974), 363-376.

14. A. Skidmore and J. Bowers, *Oscillatory behavior of solutions of* $y''+p(t)y=f(t)$, J. Math. Anal. Appl., **49** (1975), 317-323.
15. S. Tefteller, *Oscillation of second order nonhomogeneous linear differential equations*, SIAM J. Applied Math., **31** (1976), 461-467.
16. T. Wallgren, *Oscillation of solutions of the differential equation* $y''+p(t)y=f(t)$, SIAM J. Math. Anal., **7** (1976), 848-857.

Received June 9, 1976 and in revised form October 18, 1978.

MURRAY STATE UNIVERSITY
MURRAY, KY 42071

Current Address: Department of Mathematics, West Virginia University, Morgantown, West Virginia 26506