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SEMIGROUPS OF CONTINUOUS TRANSFORMATIONS AND GENERATING INVERSE LIMIT SEQUENCES

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Suppose that T denotes a strongly continuous semigroup of continuous transformations on a closed subset C of a complete metric space. For arbitrary decreasing sequences $\{\delta_n\}_{n=1}^{\infty}$ and $\{\alpha_n\}_{n=1}^{\infty}$ of positive numbers converging to 0, the inverse limit spaces generated by $\{T(\delta_n)(C), T(\delta_n-\delta_{n+1})\}_{n=1}^{\infty}$ and $\{T(\alpha_n)(C), T(\alpha_n-\alpha_{n+1})\}_{n=1}^{\infty}$ are homeomorphic and contain a dense one-to-one continuous image of C. Conversely, given an inverse limit system with bonding maps $\{f_n\}_{n=1}^{\infty}$ so that (i) $f_n: C \to C$, (ii) if x is in $C, \lim_{n\to\infty} f_n(x) = x$, and (iii) $f_{n+1}\circ f_{n+1} = f_n$, conditions are given under which a semigroup,and consequently a family of homeomorphic inverse limits, can be recovered.

Examples are given which illustrate analytical applications and topological implications.

1. Introduction. This paper deals with the generation of strongly continuous semigroups of continuous transformations. Historically the notion of generation of a strongly continuous semigroups has been the identification of it with the differential equation its trajectories satisfy. Work in [2] shows that for semigroups of nonlinear transformations, this association is not always possible. Recent work by Kobayashi [4], Kobayasi [5], and the author [6] has shown that under various conditions, given the existence of a strongly continuous semigroup, approximating semigroups which must be associated with a differential equation can be constructed. The purpose of this paper is to show that the problem of initial construction of a semigroup (thus establishing the hypothesis in the work mentioned above) is that of constructing a special sequence of functions which in turn generates a family of homeomorphic inverse limit sequences.

Theorem 1 establishes a basic topological structure inherent in semigroups and relates that structure to the set of initial conditions. Theorem 2 is a partial converse to Theorem 1. From six conditions (conditions (i) and (ii) reflect the inverse limit structure demonstrated in Theorem 1; condition (iii) reflects the algebraic structure of semigroups; conditions (iv)-(vi) are not present in all semigroups but do occur in many examples which have been studied) a strongly continuous semigroup on $[0, \infty)$ is recovered from a sequence of functions. Four lemmas provide the proof of Theorem 2.

2. Definitions and theorems.

DEFINITION 1. Suppose that C is a subset of a topological space. The statement that T is a strongly continuous semigroup of continuous transformations on C means that T is a function with domain $[0, \infty)$ and range in the continuous functions with domain C and range contained in C so that

(i) $T(0) = I$, the identity function on C;

(ii) if each of δ and α is a nonnegative number, $T(\delta + \alpha) = T(\delta)\circ T(\alpha)$; and

(iii) if x is in C, the trajectory of T from x, $g_x = \{(\delta, T(\delta)(x))$: δ is a nonnegative number$\}$, is a continuous function.

DEFINITION 2. Suppose that $(X_1, \mathscr{T}_1), (X_2, \mathscr{T}_2), \cdots$ is a sequence of topologies and that $C_1, C_2, \cdots$ is a sequence of point sets so that C_n is a subset of X_n. Suppose also that $f_1, f_2, \cdots$ is a sequence of continuous functions so that f_n has domain C_{n+1} and range C_n. The statement that p is a point of X means that p is a sequence so that

(i) $p(n)$ is an element of C_n; and

(ii) $(p(n + 1), p(n))$ is an element of f_n.

The statement that R is a region in $\mathscr{T}$ means that there is a natural number n and a region U in $\mathscr{T}_n$ so that $R = \{p$ in X: $p(n)$ is in $U\}$. The statement that J is the inverse limit space determined by $\{(C_n, f_n)\}_{n=1}^{\infty}$ means that J is the topological space determined by X and $\mathscr{T}$.

THEOREM 1. *Suppose that T is a strongly continuous semigroup of continuous transformations on C and that each of $\{\delta_i\}_{i=1}^{\infty}$ and $\{\alpha_i\}_{i=1}^{\infty}$ is a decreasing sequence of positive numbers converging to* 0. *Denote by $\mathscr{I}_\delta$ the inverse limit space determined by $\{(R_{T(\delta_n)}, f_n))\}_{n=1}^{\infty}$ where f_n is the restriction of $T(\delta_n - \delta_{n+1})$ to $R_{T(\delta_{n+1})}$; and by I_α the corresponding inverse limit space determined by $\{(R_{T(\alpha_n)}, T(\alpha_n - \alpha_{n+1})\}_{n=1}^{\infty}$. Then*

(i) *if g_x is a trajectory of T, there is exactly one point p in $\mathscr{I}_\delta$ so that $\{(\delta_n, p(n))\}_{n=1}^{\infty}$ is a subset of g_x;*

(ii) *$Q_\delta = \{p$: there is x in C so that $\{(\delta_n, p(n))\}_{n=1}^{\infty} \subset g_x\}$ is dense in $\mathscr{I}_\delta$; and*

(iii) *$\mathscr{I}_\delta$ and $\mathscr{I}_\alpha$ are homeomorphic.*

DEFINITION 3. Suppose that C is a subset of a topological space and $<$ is a partial order on C. The statement that $<$ agrees with the topology on C means that if $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$ are sequences which converge to x and $\{z_i\}_{i=1}^{\infty}$ is a sequence so that $x_i < z_i < y_i$ for each index i, then $\{z_i\}_{i=1}^{\infty}$ converges to x.

A partial order such as that in Definition 3 is imposed by any semigroup in which $t > 0$ implies $T(t)(x) \neq x$ by defining $x < y$ means there is $s > 0$ so that $T(s)(x) = y$. However, the motivation for its inclusion in the hypothesis to the following theorem is that a pre-existing order structure on C might facilitate selection of a sequence of square roots which converges pointwise to the identity.

THEOREM 2. *Suppose that $<$ is a partial order on the closed subset C of the complete metric space X which agrees with the topology on C. Suppose further that $f_1, f_2, \cdots$ is a sequence of functions so that if n is a natural number*

(i) *f_n is continuous with domain C and range contained in C;*

(ii) *if x is an element of $C, f_1(x), f_2(x), \cdots$ converges to x;*

(iii) *$f_{n+1} \circ f_{n+1} = f_n$;*

(iv) *if x is in $C, x \leqq f_n(x)$;*

(v) *if $x \leqq y, f_n(x) \leqq f_n(y)$; and*

(vi) *if x and y are in $C, d(f_n(x), f_n(y)) \leqq d(x, y)$.*

Then there is a strongly continuous semigroup T of nonexpansive tranformations on C so that $T(1/2^{n-1}) = f_n$.

3. Proofs of the Theorems.

Proof of Theorem 1.

(i) Suppose that g_x is the trajectory of T from x and consider $p_x = \{T(\delta_n)(x)\}_{n=1}^{\infty}$. Since for a given natural number k,

$$T(\delta_k - \delta_{k+1})(T(\delta_{k+1})(x)) = T(\delta_k)(x) ,$$

p_x is an element of $\mathscr{I}_\delta$. Hence, $\{(\delta_n, p_x(n))\}_{n=1}^{\infty} \subset g_x$. Suppose p is a point of $\mathscr{I}_\delta$ so that $\{(\delta_n, p(n))\}_{n=1}^{\infty} \subset g_x$. Then $g_x(\delta_n) = p(n) = T(\delta_n)(x)$ and $p = p_x$. Since T is strongly continuous, if $x \neq y$, $p_x \neq p_y$.

(ii) Suppose p is a point of $\mathscr{I}_\delta$ and $\mathscr{O}$ is a region in C so that $p(n)$ is an element of $\mathscr{O} \cap R_{T(\delta_n)}$. Since $p(n)$ is in $R_{T(\delta_n)}$, there is a point x of C so that $T(\delta_n)(x) = p(n)$. Thus $\{T(\delta_k)(x)\}_{k=1}^{\infty}$ is an element of Q_δ in the region of $\mathscr{I}_\delta$ determined by $\mathscr{O}$, and Q_δ is dense in $\mathscr{I}_\delta$.

(iii) Consider the sequence $\{\rho_n\}_{n=1}^{\infty}$ defined by $\rho_1 = \max(\{\delta_n\}_{n=1}^{\infty} \cup \{\alpha_n\}_{n=1}^{\infty})$ and $\rho_k = \max(\{\delta_n\}_{n=1}^{\infty} \cup \{\alpha_n\}_{n=1}^{\infty}) - (\{\rho_n\}_{n=1}^{k-1})$. Associated with $\{\rho_n\}_{n=1}^{\infty}$ is the inverse limit space $\mathscr{I}_\rho$ determined by $\{(R_{T(\rho_n)}, h_n)\}_{n=1}^{\infty}$ where h_n is the restriction of $T(\rho_n - \rho_{n+1})$ to $R_{T(\rho_{n+1})}$. Let $\{n_i\}_{i=1}^{\infty}$ be that sequence of indices so that $\rho_{n_i} = \delta_i$ and define F with domain $\mathscr{I}_\rho$ and range contained in $\mathscr{I}_\delta$ by $F(\{x_n\}_{n=1}^{\infty}) = \{x_{n_i}\}_{i=1}^{\infty}$. Note that since $\rho_{n_i} = \delta_i$, given $\{x_n\}_{n=1}^{\infty}$ in $\mathscr{I}_\rho$, that is having (x_{n+1}, x_n) in $T(\rho_n - \rho_{n+1})$ implies that $(x_{n_{i+1}}, x_{n_i})$ is in $T(\rho_{n_i} - \rho_{n_{i+1}}) = T(\delta_i - \delta_{i+1})$. This holds since (x_{n_i+1}, x_{n_i}) is in $T(\rho_{n_i} - \rho_{n_i+1}), \cdots, (x_{n_{i+1}}, x_{n_{i+1}-1})$ is in $T(\rho_{n_{i+1}-2} - \rho_{n_{i+1}})$ and $T(\rho_{n_i} - \rho_{n_{i+1}}) = \pi_{j=1}^{n_{i+1}-n_i} T(\rho_{n_i+j-1} - \rho_{n_i+j})$ by the semigroup property.

Suppose that $\{x_n\}_{n=1}^{\infty} \neq \{y_n\}_{n=1}^{\infty}$. Then there is an index k so that $x_k \neq y_k$. Let $j > 0$ and consider x_{k+j} and y_{k+j}.

$$\{(x_{k+j}, x_{k+j-1}), (y_{k+j}, y_{k+j-1})\} \subset T(\rho_{k+j-1} - \rho_{k+j}), \cdots,$$
$$\{(x_{k+1}, x_k), (y_{k+1}, y_k)\} \subset T(\rho_k - \rho_{k+1})$$

and thus by the considerations in the preceding argument,

$$\{(x_{k+j}, x_k), (y_{k+j}, y_k)\} \subset T(\rho_k - \rho_{k+j}) .$$

This means that $x_{k+j} \neq y_{k+j}$. But since $\{x_{n_i}\}_{i=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=1}^{\infty}$ and $\{y_{n_i}\}_{i=1}^{\infty}$ is a subsequence of $\{y_n\}_{n=1}^{\infty}$, $F(\{x_n\}_{n=1}^{\infty}) \neq F(\{y_n\}_{n=1}^{\infty})$ and F is $1-1$.

Suppose that $\{w_n\}_{n=1}^{\infty}$ is a point of $\mathscr{S}_\delta$ and that $n_{i+1} = n_i + k_i$. (u_{i+1}, u_i) is in $T(\delta_i - \delta_{i+1})$ and $T(\delta_i - \delta_{i+1}) = \pi_{j=1}^{k_i} T(\rho_{n_i+j-1} - \rho_{n_i+j})$. Thus the sequence $T(\rho_1 - \rho_{n_1})(w_1)$, $T(\rho_2 - \rho_{n_1})(w_1), \cdots, w_1 = T(\rho_{n_1} - \rho_{n_2})(w_2)$, $T(\rho_{n_1+1} - \rho_{n_2})(w_2), \cdots, T(\rho_{n_2-1} - \rho_{n_2}((w_2), w_2 = T(\rho_{n_2} - \rho_{n_3})(w_3), \cdots$ is a point p of $\mathscr{S}_\rho$ so that $F(p) = \{w_n\}_{n=1}^{\infty}$ and F is onto $\mathscr{S}_\delta$.

Suppose that $\{x_n\}_{n=1}^{\infty}$ is a point of $\mathscr{S}_\rho$ and that R is a region in $\mathscr{S}_\delta$ containing $F(\{x_n\}_{n=1}^{\infty}) = \{x_{n_i}\}_{i=1}^{\infty}$. Then there is an index n_i and a region R' of C so that R is the set to which $\{w_n\}_{n=1}^{\infty}$ belongs provided that w_{n_i} is in R'. $T(\rho_{n_i} - \rho_{n_i+1})$ is continuous, thus there is a region U of C so that if x is in U, then $T(\rho_{n_i} - \rho_{n_i+1})(x)$ is in R'. Thus if p is a point of $\mathscr{S}_\rho$ so that $p(n_i + 1)$ is in U, then $p(n_i)$ is in R'; that is, if $\{x_n\}_{n=1}^{\infty}$ is in the region of $\mathscr{S}_\rho$ determined by $U \cap R_{T(\rho_{n_i+1})}$, then $F(\{x_n\}_{n=1}^{\infty}) = \{x_{n_i}\}_{i=1}^{\infty}$ is in the region of $\mathscr{S}_\delta$ determined by R'. Thus F is continuous. For the continuity of F^{-1} note that the continuity of $T(\rho_n - \rho_{n_i})$ guarantees that an argument similar to the one above can be applied. Thus F is $1-1$, onto, continuous, and reversibly continuous, and $\mathscr{S}_\delta$ and $\mathscr{S}_\rho$ are homeomorphic by F.

Let $\{n_k\}_{k=1}^{\infty}$ be that sequence of integers so that $\alpha_k = \rho_{n_k}$. By defining $G\colon \mathscr{S}_\rho$ onto $\mathscr{S}_\alpha$ by $G(\{x_k\}_{k=1}^{\infty}) = \{x_{n_k}\}_{k=1}^{\infty}$, the arguments for F can be applied to G. Hence $F \circ G^{-1}$ is a homeomorphism of $\mathscr{S}_\alpha$ onto $\mathscr{S}_\delta$.

Proof of Theorem 2.

LEMMA 1. *$\{f_n\}_{n=1}^{\infty}$ is a commuting family.*

Proof of Lemma 1. Suppose m and n are natural numbers. By (iii), $f_{n+1} \circ f_{n+1} = f_n$; so $f_m \circ f_{m+n} = f_{m+n}^{2^n} \circ f_{m+n} = f_{m+n} \circ f_{m+n}^{2^n} = f_{m+n} \circ f_m$.

LEMMA 2. *If x is a point of C and $\delta > 0$, there is a natural number N so that if $n_1, n_2, \cdots$ is an increasing sequence of natural numbers so that $n_1 > N$, then $d(\pi_{k=1}^{m} f_{n_k}(x), x) < \delta$.*

Proof of Lemma 2. Suppose that x is in C. By (iv) in the hypothesis, if n is a natural number, $x \leqq f_n(x)$. For an increasing sequence of natural numbers $n_1, n_2, \cdots$ consider x and $f_{n_1}(x)$. Since $x \leqq f_{n_1}(x), f_{n_2}(x) \leqq f_{n_1}\circ f_{n_2}(x)$ by (v). But $x \leqq f_{n_2}(x)$, so $x \leqq f_{n_1}\circ f_{n_2}(x)$. By induction, $x \leqq \pi_{k=1}^{m} f_{n_k}(x)$. Consider, for a given m,

$$(\pi_{j\neq n_k, k<m} f_j)\circ f_{n_m}(x) .$$

From above, $x \leqq (\pi_{j\neq n_k, k<m} f_j)\circ f_{n_m}(x)$. Hence $f_{n_1}(x) \leqq f_{n_1}\circ(\pi_{j\neq n_k,k<m} f_j)\circ f_{n_m}(x), \cdots, \pi_{k=1}^{m} f_{n_k}(x) \leqq (\pi_{k=n_1}^{n_m} f_k)\circ f_{n_m}(x)$. But from (iii), $f_{n_m}\circ f_{n_m} = f_{n_m-1}$, and thus the righthand member of the inequality collapses to $f_{n_1-1}(x)$. Thus $x \leqq \pi_{k=1}^{m} f_{n_k}(x) \leqq f_{n_1-1}(x)$.

Suppose that there were $\delta > 0$ so that for each natural number $n > 2$, there was an increasing sequence ω_n of natural numbers with $\omega_n(1) > n$ and a natural number m_n so that $d(\pi_{k=1}^{m_n} f_{\omega_n(k)}(x), x) \geqq \delta$. From above, $x \leqq \pi_{k=1}^{m_n} f_{\omega_n(k)}(x) \leqq f_{\omega_n(1)-1}(x)$. But by hypothesis $<$ agrees with the topology on C, and since each of $\{f_{\omega_n(1)-1}(x)\}_{n=1}^{\infty}$ and $x, x, x, \cdots$ converges to x, $\{\pi_{k=1}^{m_n} f_{\omega_n(k)}(x)\}_{n=3}^{\infty}$ converges to x. But this contradicts that each term is at least distance δ from x, and the lemma is established.

LEMMA 3. *If* $\{n_k\}_{k=1}^{\infty}$ *is an increasing sequence of natural numbers and* x *is in* C, *then* $\{\pi_{k=1}^{m} f_{n_k}(x)\}_{m=1}^{\infty}$ *converges.*

Proof of Lemma 3. Suppose $\{n_k\}_{k=1}^{\infty}$ is an increasing sequence of natural numbers, x is a point of C, and $\delta > 0$. By Lemma 2, there is a natural number N so that if $j > N$, $d(\pi_{k=N}^{j} f_{n_k}(x), x) < \delta$. By (vi), f_k is nonexpansive for each index n. Hence, if

$$j, t > N,\ d(\pi_{k=1}^{j} f_{n_k}(x), \pi_{k=1}^{t} f_{n_k}(x)) \leqq d(\pi_{k=\max(j,t)-\min(j,t)}^{\max(j,t)} f_{n_k}(x), x) < \delta .$$

Thus $\{\pi_{k=1}^{m} f_{n_k}(x)\}_{m=1}^{\infty}$ is Cauchy and must have a sequential limit point in the closed set C.

LEMMA 4. $\{\pi_{n=k}^{m} f_{n+1}(x)\}_{m=k}^{\infty}$ *converges to* $f_k(x)$ *for each point* x *of* C.

Proof of Lemma 4. Let $\delta > 0$.

$$d(f_k(x), \pi_{n=k}^{m} f_{n+1}(x)) = d((\pi_{n=k}^{m} f_{n+1})\circ f_{m+1}(x), \pi_{n=k}^{m} f_{n+1}(x)) \leqq d(f_{m+1}(x), x) .$$

By (ii), $\{f_n(x)\}_{n=1}^{\infty}$ converges to x so N can be chosen so that if $m > N$, $d(f_{m+1}(x), x) < \delta$. Thus $\{\pi_{n=k}^{m} f_{n+1}(x)\}_{m=k}^{\infty}$ converges to $f_k(x)$.

Let δ be a number in $(0, 1]$ and associate with δ the increasing sequence of natural numbers $n_1, n_2, \cdots$ with the property that n_1 is the least positive integer so that $1/2^{n_1-1} < \delta$ and n_k is the least positive integer so that $1/2^{n_k-1} < \delta - \Sigma_{j=1}^{k-1} 1/2^{n_j-1}$. ($\Sigma_{k=1}^{\infty} 1/2^{n_k-1}$ will be called the base 2 expansion for δ.) Define $T(\delta)(x)$ to be the sequential limit point of $\{\pi_{k=1}^{m} f_{n_k}(x)\}_{m=1}^{\infty}$. If $\delta = 0$, define $T(\delta)(x) = x$. If $\delta > 1$ and n is the greatest integer in δ, define $T(\delta)(x) = f_1^n \circ T(\delta - n)(x)$.

Let δ be a number in $(0, 1)$ and p be a natural number. Let $\Sigma_{k=1}^{\infty} 1/2^{n_k-1}$ denote the base 2 expansion for δ. Let $\alpha > 0$. For $\alpha/2$, there is N so that if

$$m > N,\ d(f_1^p \circ \pi_{k=1}^{m} f_{n_k}(x),\ T(\delta + p)(x)) < \alpha/2$$

and $d(f_1^p \circ \pi_{k=1}^{m} f_{n_k}(y),\ T(\delta + p)(y)) < \alpha/2$. Thus if

$$\begin{aligned} m > N,\ d(T(\delta + p)(x),\ T(\delta + p)(y)) &\leqq d(T(\delta + p)(x),\ f_1^p \circ \pi_{k=1}^{m} f_{n_k}(x)) \\ &\quad + d(f_1^p \circ \pi_{k=1}^{m} f_{n_k}(x),\ f_1^p \circ \pi_{k=1}^{m} f_{n_k}(y)) + d(f_1^p \circ \pi_{k=1}^{m} f_{n_k}(y),\ T(\delta + p)(y)) \\ &< \alpha/2 + d(x, y) + \alpha/2 = \alpha + d(x, y). \end{aligned}$$

Thus $T(\delta + p)$ is a nonexpansive function.

Suppose that δ and α are numbers in $[0, \infty)$ and x is a point of C. Let a denote the greatest integer in δ, b denote the greatest integer in α, and c denote the greatest integer in $\delta + \alpha$. Let $\Sigma_{k=1}^{\infty} 1/2^{n_k-1}$ denote the base 2 expansion for $\delta - a$ and $\Sigma_{k=1}^{\infty} 1/2^{j_k-1}$ denote the base 2 expansion for $\alpha - b$. Then

$$T(\delta)(T(\alpha)(x)) = \lim_{p\to\infty} f_1^a \circ \pi_{k=1}^{p} f_{n_k}\left(\lim_{m\to\infty} f_1^b \circ \pi_{k=1}^{m} f_{j_k}(x)\right).$$

$\delta + \alpha = a + b + \Sigma_{k=1}^{\infty} 1/2^{n_k-1} + \Sigma_{k=1}^{\infty} 1/2^{j_k-1}$. Note that in writing the base 2 expansion for $\delta + \alpha - c$, that a given term may be either $2 \cdot 1/2^{n_k-1}$ for some index n_k or a term of exactly one of the base 2 expansions for $\delta - a$ and $\alpha - b$. Since $f_{n_k} \circ f_{n_k} = f_{n_k-1}$, the terms of the sequence which defines $T(\delta + \alpha)(x)$ can be rewritten so that (1) $T(\delta + \alpha)(x) = \lim_{m\to\infty} f_1^a \circ f_1^b \circ (\pi_{k=1}^{p_m} f_{n_k}) \circ (\pi_{k=1}^{q_m} f_{j_k})(x)$ where p_m is the greatest integer so that $n_{p_m} \leqq m$ and q_m is the greatest integer so that $j_{q_m} \leqq m$ if $\delta + \alpha < 1$, or so that (2) $T(\delta + \alpha)(x) = \lim_{m\to\infty} f_1^a \circ f_1^b \circ f_1 \circ \pi_{i=1}^{m} g_i(x)$ where g_1 is the element of least index in $\{f_{n_k}\}_{k=1}^{\infty}$ so that g_1 is in $\{f_{j_k}\}_{k=1}^{\infty}, \cdots,$ and g_n is the element of least index in $\{f_{n_k}\}_{k=1}^{\infty} - \{g_i\}_{i=1}^{n-1}$ so that g_n is in $\{f_{j_k}\}_{k=1}^{\infty}$ if $\delta + \alpha \geqq 1$. Thus to show here that $T(\delta)(T(\alpha)(x)) = T(\delta + \alpha)(x)$, what must be shown is that

$T(\delta)(T(\alpha)(x)) = \lim_{m\to\infty} f_1^a \circ f_1^b \circ \pi_{k=1}^{p_m} f_{n_k} \circ \pi_{k=1}^{q_m} \circ f_{j_k}(x)$ when case 1 applies or that $T(\delta)(T(\alpha)(x)) = \lim_{m\to\infty} f_1^a \circ f_1^b \circ f_1 \circ \pi_{k=1}^{m} g_k(x)$ when case 2 applies.

For case 1,

$$\begin{aligned} &d(T(\delta)(T(\alpha)(x)),\ f_1^a \circ f_1^b \circ \pi_{k=1}^{p_m} f_{n_k} \circ \pi_{k=1}^{q_m} f_{j_k}(x)) \leqq d(T(\delta)(T(\alpha)(x))\ , \\ &f_1^a \circ \pi_{k=1}^{p_m} f_{n_k} \circ T(\alpha)(x)) + d(f_1^a \circ \pi_{k=1}^{p_m} f_{n_k} \circ T(\alpha)(x),\ f_1^a \circ \pi_{k=1}^{p_m} f_{n_k} \circ f_1^b \pi_{k=1}^{q_m} f_{j_k}(x)) \\ &\qquad \leqq d(T(\delta)(T(\alpha)x)),\ f_1^a \circ \pi_{k=1}^{q_m} f_{n_k}(T(\alpha)(x)) + d(T(\alpha)(x),\ f_1^b \circ \pi_{k=1}^{q_m} f_{j_k}(x))\ . \end{aligned}$$

But $\{f_1^a \circ \pi_{k=1}^{p_m} f_{n_k}(T(\alpha)(x))\}_{m=1}^{\infty}$ converges to $T(\delta)(T(\alpha)(x))$ and $\{f_1^b \circ \pi_{k=1}^{q_m} f_{j_k}(x)\}_{m=1}^{\infty}$ converges to $T(\alpha)(x)$. Thus $\{f_1^a \circ f_1^b \circ \pi_{k=1}^{p_m} f_{n_k} \circ \pi_{k=1}^{q_m} f_{j_k}(x)\}_{m=1}^{\infty}$ converges to $T(\delta)(T(\alpha)(x))$. For case 2,

$$\begin{aligned} & d(T(\delta)(T(\alpha)(x)), f_1^a \circ f_1^b \circ f_1 \circ \pi_{k=1}^{m} g_k(x)) \\ & \quad \leqq d(T(\delta)(T(\alpha)(x)), f_1^a \circ f_1^b \circ \pi_{k=1}^{m} g_k \circ \pi_{k=2}^{m} f_k(x)) \\ & \qquad + d(f_1^a \circ f_1^b \circ \pi_{k=1}^{m} g_k \circ \pi_{k=2}^{m} f_k(x), f_1^a \circ f_1^b \circ \pi_{k=1}^{m} g_k \circ f_1(x)) \\ & \quad \leqq d(T(\delta)(T(\alpha)(x)), f_1^a \circ f_1^b \circ \pi_{k=1}^{p_m} f_{n_k} \circ \pi_{k=1}^{q_m} f_{j_k}(x)) + d(\pi_{k=2}^{m} f_k(x), f_1(x)) . \end{aligned}$$

For m large, the distance described in the first term is small by the argument advanced in case 1; the second term is small by Lemma 4. Thus $T(\delta + \alpha)(x) = T(\delta)(T(\alpha)(x))$ and T is a semigroup of nonexpansive transformations on C so that $f_n = T(1/2^{n-1})$.

Suppose that x is in C and $\{\delta_n\}_{n=1}^{\infty}$ converges to δ in $[0, 1)$. Let $\Sigma_{k=1}^{\infty} 1/2^{j_k - 1}$ denote the base 2 expansion for δ and $\Sigma_{k=1}^{\infty} 1/2^{n_k - 1}$ denote the base 2 expansion for δ_n. By Lemmas 2 and 4, there is a number M so that if $n_1, n_2, \cdots$ is an increasing sequence of natural numbers so that $n_1 \geqq M$, then $d(\pi_{k=M}^{p} f_k(x), x) < \alpha/3$ and $d(\pi_{k=1}^{M} f_{j_k}(x), T(\delta)(x)) < \alpha/3$ for predetermined $\alpha > 0$. Given this number M, there is a natural number N so that if $n \geqq N$, and $\delta_n \leqq \delta$ or δ is not an integral power of 1/2, then the first M terms of the base 2 expansion for δ_n are the same as the first M terms of the base 2 expansion for δ. If δ is an integral power of 1/2, there is N so that if $n \geqq N$ and $\delta_n > \delta$, then the second term of the base 2 expansion for δ_n is less than $1/2^M$, and the first term is δ.

For

$$\begin{aligned} n \geqq N,\ & d(T(\delta_n)(x), T(\delta)(x)) \leqq d(T(\delta_n)(x), \pi_{k=1}^{M+p} f_{n_k}(x)) \\ & \quad + d(\pi_{k=1}^{M+p} f_{n_k}(x), \pi_{k=1}^{M} f_{j_k}(x)) + d(\pi_{k=1}^{M} f_{j_k}(x), T(\delta)(x)) \\ & \leqq d(T(\delta_n)(x), \pi_{k=1}^{M+p} f_{n_k}(x)) + d(\pi_{k=M+1}^{M+p} f_{n_k}(x), x) \\ & \quad + d(\pi_{k=1}^{M} f_{j_k}(x), T(\delta)(x)) < d(T(\delta_n)(x), \pi_{k=1}^{M+p} f_{n_k}(x)) + 2\alpha/3 . \end{aligned}$$

Since the choice of p is independent of the choice of M and N and $\{\pi_{k=1}^{M+p} f_{n_k}(x)\}_{p=1}^{\infty}$ converges to δ_n for each natural number n, this means that $d(T(\delta_n)(x), T(\delta)(x)) < \alpha$ for each natural number $n > N$. Thus $\{T(\delta_n)(x)\}_{n=1}^{\infty}$ converges to $T(\delta_n)(x)$. The continuity on $(k, k+1)$ follows from the continuity of f_1 and continuity at the integers from Lemmas 4 and 2. Thus T is strongly continuous.

4. Examples.

Although the assumption of nonexpansiveness is quite restrictive, the result of Theorem 2 suggests an alternative approach to the generation of semigroups: describe those continuous functions which have the property that they have continuous square roots, 4th roots, $\cdots$ which converge pointwise to the identity function.

EXAMPLE 1. $C = [0, 1]$, $f_1(x) = x^2$. Purely algebraic considerations suggest that f_n be defined by $f_n(x) = x^\lambda$ where λ is the real $(2^n - 1)$-th root of 2. If $0 \leqq x \leqq 1$ and $n_1, n_2, \cdots$ is an increasing sequence of natural numbers, then $\lim_{m\to\infty} \pi_{k=1}^{m} f_{n_k}(x)$ exists and for increasing sequences $n_1, n_2, \cdots$ and $j_1, j_2, \cdots$, $\lim_{m\to\infty} \pi_{k=1}^{m} f_{n_k}(\lim_{p\to\infty} \pi_{k=1}^{p} f_{j_k}(x)) = \lim_{m\to\infty} \pi_{n_k<m} f_{n_k} \circ \pi_{j_k<m} f_{j_k}(x)$. With the machinery of the proof of Theorem 2 thus established, for $\delta > 0$, a semigroup is generated so that $T(\delta) = f_1, \cdots, T(\delta/2^{n-1}) = f_n, \cdots$. In this case the square root process gives rise to the family of infinitesimal generators $\{A_\delta : A_\delta(x) = \log 2/\delta \cdot x \cdot \log x$ if x is in $(0, 1]$, $A_\delta(0) = 0\}$.

EXAMPLE 2. Let $\begin{bmatrix} a & b \\ c & a \end{bmatrix}$ be an element of $\mathscr{L}(\mathscr{R}^2, \mathscr{R}^2)$ so that $a \geqq 1$ and b and c are positive. Computation shows that if A is a member of $\mathscr{L}(\mathscr{R}^2, \mathscr{R}^2)$ so that $A^2 = \begin{bmatrix} a & b \\ c & a \end{bmatrix}$, A is of the form $\begin{bmatrix} w & x \\ y & w \end{bmatrix}$. Furthermore if $bc > 4a^2$, $\begin{bmatrix} a & b \\ c & a \end{bmatrix}$ has no matrix square root. Hence, under these conditions, no semigroup of continuous linear transformations on $\mathscr{R}^2$ can contain $\begin{bmatrix} a & b \\ c & a \end{bmatrix}$ in its range. However, if $\begin{bmatrix} a & b \\ c & a \end{bmatrix}$ has the property that $(a + \sqrt{a^2 - bc})/2 - bc > 1$, then a sequence of continuous linear transformations $f_1 = \begin{bmatrix} a & b \\ c & a \end{bmatrix}$, $f_2, \cdots$ can be constructed satisfying (i), (ii), and (iii) of the hypothesis to Theorem 2. Considerations in [3] indicate that $f_1, f_2, \cdots$ can be extended to a semigroup.

Brown [1] established that if $\{X, f_n\}_{n=1}^{\infty}$ and $\{X, g_n\}_{n=1}^{\infty}$ are so that X is compact and metric and g_n and f_n are near homeomorphisms for each n, then the inverse limit spaces are homeomorphic. The vehicle of the proof is to show that each is homeomorphic to X. Topologically, consideration of Theorem 1 shows that the algebraic properties of the semigroup replace the assumptions on the bonding maps and the bonded sets. In particular, compactness is not an issue.

EXAMPLE 3. Let $C = \{(x, y): x > 0$ and $0 \leqq y \leqq 1/x$ or $x = 0$ and $y \geqq 0\}$. If (x, y) is in C, define, for $\delta > 0$, $T(\delta)$ by

$$T(\delta)((x, y)) = \begin{cases} (x, y + \delta) \text{ if } x > 0 \text{ and } y + \delta < 1/x \\ (x + y + \delta - (1/x), 1/(x + y + \delta - (1/x))) \text{ if } x > 0 \\ \quad \text{and } y + \delta \geqq 1/x \\ (x + \delta, 1/(x + \delta)) \text{ if } y = 1/x \\ (x, y + \delta) \text{ if } x = 0 . \end{cases}$$

In this example, notice that if $\{\delta_n\}_{n=1}^{\infty}$ is a decreasing sequence of positive numbers converging to 0, then $\{(\delta_n, 1/\delta_n)\}_{n=1}^{\infty}$ is a point of the inverse limit space determined by $\{(R_{T(\delta_n)}, T(\delta_n - \delta_{n+1}))\}_{n=1}^{\infty}$ so that $\{(\delta_n, (\delta_n, 1/\delta_{n+1}))\}_{n=1}^{\infty}$ is a subset of no trajectory of T. Also, in light of the proof of Brown's theorem, one should note the apparent structural differences between $R_{T(\delta)}$ and the inverse limit which must contain a dense continuous one-to-one copy of C.

EXAMPLE 4. Let $C = (0, 1] \times [0, 1] \cup \{(0, 1)\}$. Define T by

$$T(\delta)((x, y)) = \begin{cases} (x, (\delta + xy)/x) & \text{if} \quad x > 0 \quad \text{and} \quad \delta + xy < x \\ (x, 1) & \text{if} \quad \delta + xy \geqq x \quad \text{or} \quad x = 0 . \end{cases}$$

In this example, the function pairing a point of C with the point in the inverse limit which contains the trajectory from that point is not a homeomorphism. In addition, for $\delta > 0$, $T(\delta)$ extends continuously to $\bar{C}$. However, the inverse limit built from T is compact. Thus if points of C are thought of as elements of the inverse limit space, this extension which produces a non strongly continuous semigroup seems less natural.

5. Some questions. At least two problems are suggested by the theorems in this paper. In Theorem 1 a map is established from C into $\mathscr{I}_\delta$ which is always one-to-one and continuous, but which by Example 4 need not be a homeomorphism. What conditions on T can force this map to be a homeomorphism? In Theorem 2, only conditions (i)-(iii) reflect essential semigroup structure. Can conditions (iv)-(vi) be replaced, weakened, or eliminated? An affirmative answer to this second question would produce additional information toward a generation theory for semigroups based on square root processes.

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