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CHARACTERS OF AVERAGED DISCRETE SERIES ON SEMISIMPLE REAL LIE GROUPS

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Let G be a real simple Lie group of classical type having a compact Cartan subgroup. Then G has discrete series representations. The purpose of this paper is to establish explicit formulas for certain sums of discrete series characters. These "averaged" discrete series characters have simple formulas which can be used for certain problems in harmonic analysis on G, for example, for the computation of the Plancherel measure on $\hat{G}$.

1. Introduction. Let G be a connected, acceptable, semisimple real Lie group with finite center. Suppose that G has a compact Cartan subgroup T. Then G has discrete series representations. The characters of these representations were initially described by Harish-Chandra in [2]. The characters have simple formulas on T. On the noncompact Cartan subgroups, the formulas are complicated, and contain certain integer constants which Harish-Chandra did not compute.

Using the procedure described in [2], these constants can be computed if related constants are known for each type of simple root system which is spanned by a strongly orthogonal set of roots. These are the root systems of types A_1, B_n, C_n, $D_{2n} (n \geqq 2)$, E_7, E_8, F_4, and G_2, and they correspond to the complex simple Lie groups for which the split real form has a compact Cartan subgroup, and hence discrete series representations. Partial solutions to the problem of computing these constants have been given in [4, 5, 6, 7, 8, 10, 11, 12]. A complete solution is now available in work of T. Hirai [11]. Hirai's formulas express discrete series constants for groups of arbitrary rank in terms of constants for groups of real rank one and two.

Explicit formulas for discrete series characters, besides being of interest for the representation theory of G, are needed for harmonic analysis on G. However for some of these problems, for example, computation of the Plancherel measure on $\hat{G}$, it is necessary only to have certain sums of discrete series characters.

Let $\mathfrak{g}$ and $\mathfrak{t}$ denote the Lie algebras of G and T respectively, and $\mathfrak{g}_C$ and $\mathfrak{t}_C$ their complexifications. Then the discrete series characters of G are parameterized by regular elements τ in a lattice $L_T \subseteqq \sqrt{-1}\mathfrak{t}^*$. The Weyl group W of the pair $(\mathfrak{g}_C, \mathfrak{t}_C)$ acts on L_T. Instead of the characters $(-1)^q \varepsilon(\tau)\theta_\tau$ defined by Harish-Chandra in [2], we consider the sum of characters

$$(-1)^q \sum_{w \in W} \varepsilon(w\tau)\theta_{w\tau} = (-1)^q \varepsilon(\tau) \sum_{w \in W} \det w\theta_{w\tau} .$$

Here $q = (1/2)\dim(G/K)$, K a maximal compact subgroup of G, and $\varepsilon(\tau) = \pm 1$ satisfies $\varepsilon(w\tau) = \det w\varepsilon(\tau)$. Note that if $\tau \in L_T$ is singular, invariant eigendistributions θ_τ are defined by Harish-Chandra in [3]. However, for singular τ, $\sum_{w \in W} \det w\theta_{w\tau} = 0$.

These averaged discrete series characters are sufficient for the Fourier inversion of stabilized invariant integrals

$$\mathscr{F}_f^T(t_0) = \sum_{w \in W} \det wF_f^T(w^{-1}t_0),\ f \in C_c^\infty(G),\ t_0 \in T' .$$

Here T' is the set of regular elements in T and F_f^T is the invariant integral of f with respect to T defined in [1]. $F_f^T(t_0)$ can be regarded as the integral of f over the orbit of t_0 in G under the adjoint action of G. $\mathscr{F}_f^T(t_0)$ is the integral of f over the orbit in G under the adjoint action of G_C, a complex Lie group with Lie algebra $\mathfrak{g}_C$. Fourier inversion formulas for $\mathscr{F}_f^T(t_0)$ can be used to derive the Plancherel formula for G.

For the Fourier inversion of $\mathscr{F}_f^T$, it is necessary to have explicit formulas for the averaged discrete series characters. These formulas could in theory be obtained by summing the formulas given by Hirai in [11]. However, the formulas for the averaged discrete series for the classical infinite families (B_n, C_n, D_{2n}) having discrete series can be established independently of Hirai's general results. The simplicity of the averaged formulas in these important cases is not obvious from the general treatment in [9].

Thus the purpose of this paper is to establish the formulas for the averaged discrete series for the classical families of real simple Lie groups. These formulas will be used for work to appear on the Fourier inversion of stabilized invariant integrals and Plancherel theorem.

2. Averaged discrete series characters. We first establish some notation. For any reductive group G and Cartan subgroup H, define $W(G, H) = N_G(H)/H$ where $N_G(H)$ is the normalizer of H in G. Let $\Phi(\mathfrak{g}_C, \mathfrak{h}_C)$ denote the root system of the complexified Lie algebras of G and H. Let $W(\Phi)$ denote the Weyl group of the root system Φ. We regard $W(G, H)$ as a subgroup of $W(\Phi(\mathfrak{g}_C, \mathfrak{h}_C))$. For $\lambda \in \mathfrak{h}_C^*$, we define ξ_λ on H_C by $\xi_\lambda(\exp H) = \exp(\lambda(H))$, $H \in \mathfrak{h}_C$, whenever this gives a well-defined character of H_C. Let $\Phi^+(\mathfrak{g}_C, \mathfrak{h}_C)$ denote a set of positive roots for $\Phi(\mathfrak{g}_C, \mathfrak{h}_C)$. Let $\delta = (1/2)\sum \alpha$, $\alpha \in \Phi^+(\mathfrak{g}_C, \mathfrak{h}_C)$. Then if G is acceptable, ξ_δ is well-defined on H, and we define

$$\Delta(h) = \xi_\delta(h) \prod_{\alpha>0} (1 - \xi_\alpha(h)^{-1}) .$$

Let H' denote the set of regular elements in H; that is, $H' = \{h \in H \mid \Delta(h) \neq 0\}$.

Let G be as in §1. Let K be a maximal compact subgroup of G containing T, and denote by θ the Cartan involution of G with fixed point set K. Then $\mathfrak{g}$ has Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ where $\mathfrak{k}$ is the Lie algebra of K. Let H be a θ-stable Cartan subgroup of G with Lie algebra $\mathfrak{h}$. Write $\mathfrak{h}$ and H according to their Cartan decompositions as $\mathfrak{h} = \mathfrak{h}_k + \mathfrak{h}_p$ and $H = H_K H_p$. Let $y \in G_C$ satisfy $\operatorname{ad} y(\mathfrak{t}_C) = \mathfrak{h}_C$. Then y induces a mapping from $\mathfrak{t}_C^*$ to $\mathfrak{h}_C^*$ which we denote by $\tau \to {}^y\tau$.

Let $h \in H'$. Write $h = h_K h_p$ where $h_K \in H_K$, $h_p \in H_p$. Let H_K^+ be the connected component of H_K containing h_K. Assume $H_K^+ \subseteqq T$. Let $\mathfrak{z}$ denote the centralizer in $\mathfrak{g}$ of H_K^+, Z the connected subgroup of G with Lie algebra $\mathfrak{z}$. Let $\Phi = \Phi(\mathfrak{z}_C, \mathfrak{t}_C)$. We consider Φ as a subset of $\Phi(\mathfrak{g}_C, \mathfrak{t}_C)$. Let $\Phi^+ = \{\alpha \in \Phi \mid {}^y\alpha(\log h_p) > 0\}$. Let $\tau \in L_T$ and denote by θ_τ the corresponding invariant eigendistribution defined by Harish-Chandra. Then it follows from [2] that:

$$(2.1) \quad \theta_\tau(h_K h_p) = \Delta(h)^{-1} \sum_{t \in W(Z,T) \backslash W(G,T)} \det t \sum_{S \in W(\Phi)} \det s \, c(s\colon t\tau\colon \Phi^+) \xi_{st\tau}(y^{-1}h) .$$

The $c(s\colon \tau\colon \Phi^+)$ are integers satisfying:

$$(2.2) \qquad c(su\colon \tau\colon \Phi^+) = c(s\colon u\tau\colon \Phi^+) , \quad u \in W(Z, T) .$$

LEMMA 2.3. *Let $W = W(\mathfrak{g}_C, \mathfrak{t}_C)$, other notation as above. Then*

$$\sum_{w \in W} \det w \theta_{w\tau}(h) = [W(G, T)] \Delta(h)^{-1} \sum_{w \in W} \det w \ \bar{c}(w\tau\colon \Phi^+) \xi_{w\tau}(y^{-1}h)$$

where $\bar{c}(\tau\colon \Phi^+) = \sum_{s \in W(\Phi)\backslash W(Z,T)} c(s\colon s^{-1}\tau\colon \Phi^+)$.

Proof. The formula follows directly from (2.1) and (2.2) since $W(G, T)$ and $W(\Phi)$ can both be regarded as subgroups of W.

The constants $\bar{c}(\tau\colon \Phi^+)$ have the following properties which can be deduced from their definition and from the corresponding properties of the constants $c(s\colon \tau\colon \Phi^+)$ proved in [2].

$$(2.4) \qquad \bar{c}(s\tau\colon s\Phi^+) = \bar{c}(\tau\colon \Phi^+) \quad \text{for} \quad s \in W(\Phi) .$$

Let $\{\alpha_1, \cdots, \alpha_l\}$ be a set of simple roots for Φ^+. Let $\Lambda_1, \cdots, \Lambda_l$ in L_T satisfy $\langle \Lambda_i, \alpha_j \rangle = \delta_{ij}$. Then:

$$(2.5) \qquad \bar{c}(\tau\colon \Phi^+) = 0 \quad \text{if} \quad \langle \tau, \Lambda_i \rangle > 0 \quad \text{for any } 1 \leqq i \leqq l .$$

Let $\alpha \in \Phi^+$ be a simple root. Let $\alpha_0 = {}^y\alpha \in \Phi(\mathfrak{z}_C, \mathfrak{h}_C)$. Let X_{α_0} and Y_{α_0} denote the root vectors for α_0 and $-\alpha_0$ in $\mathfrak{g}$ satisfying $[X_{\alpha_0}, Y_{\alpha_0}] = 2H_{\alpha 0}/\langle\alpha_0, \alpha_0\rangle$ where $H_{\alpha_0} \in \mathfrak{h}$ satisfies $\beta(H_{\alpha_0}) = \langle\beta, \alpha_0\rangle$ for all $\beta \in \Phi(\mathfrak{g}_C, \mathfrak{h}_C)$. Let $\nu = \exp((-\pi\sqrt{-1(1)}/4)\,\mathrm{ad}\,(X_{\alpha_0} + Y_{\alpha_0}))$. Then $\mathfrak{i} = \nu(\mathfrak{h}_C) \cap \mathfrak{g}$ is a θ-stable Cartan subalgebra of $\mathfrak{g}$. Let $\Phi_0 = \{\beta \in \Phi \mid \langle\beta, \alpha\rangle = 0\}$, $\Phi_0^+ = \Phi_0 \cap \Phi^+$. Let s denote the reflection in $W(\Phi)$ corresponding to α. Then:

$$\bar{c}(\tau : \Phi^+) + \bar{c}(s\tau : \Phi^+) = \bar{c}(\tau : \Phi_0^+) + \bar{c}(s\tau : \Phi_0^+) = 2\bar{c}(\tau ; \Phi_0^+) \tag{2.6}$$

since $\bar{c}(s\tau : \Phi_0^+) = \bar{c}(\tau : s\Phi_0^+) = \bar{c}(\tau : \Phi_0^+)$ as $s\beta = \beta$ for all $\beta \in \Phi_0$.

Let $\mathscr{F}$ be the real subspace of $\sqrt{-1}\mathfrak{t}^*$ spanned by Φ. For $\tau \in \sqrt{-1}\mathfrak{t}^*$, τ can be written uniquely as $\tau = \tau_I + \tau_0$, where $\tau_0 \in \mathscr{F}$, and ${}^y\tau_I$ takes purely imaginary values of $\mathfrak{h}$. Let $\mathscr{F}' = \{\lambda \in \mathscr{F} \mid \langle\lambda, \alpha\rangle \neq 0, \alpha \in \Phi\}$. Then $\bar{c}(\tau : \Phi^+)$ depends only on the component $\mathscr{F}^+$ of τ_0 in $\mathscr{F}'$. We write

$$\bar{c}(\mathscr{F}^+ : \Phi^+) = \bar{c}(\tau : \Phi^+) \quad \text{if} \quad \tau_0 \in \mathscr{F}^+ . \tag{2.7}$$

If $\Phi = \Phi_1 \cup \cdots \cup \Phi_s$ where the Φ_i, $1 \leqq i \leqq s$, are simple root systems, then $\lambda \in \mathscr{F}^+$ can be written uniquely as $\lambda = \lambda_1 + \cdots + \lambda_s$ where for $1 \leqq i \leqq s$, $\lambda_i \in \mathscr{F}_i$, the real linear span of the elements of Φ_i. Let $\mathscr{F}_i^{\,+}$ be the component of $\mathscr{F}_i' = \{\lambda \in \mathscr{F}_i \mid \langle\alpha, \lambda\rangle \neq 0, \alpha \in \Phi_i\}$ containing λ_i. Then if $\Phi_i^+ = \Phi_i \cap \Phi^+$,

$$\bar{c}(\mathscr{F}^+ : \Phi^+) = \prod_{i=1}^{s} \bar{c}(\mathscr{F}_i^{\,+} : \Phi_i^+) . \tag{2.8}$$

Note that if $\Phi(\mathfrak{g}_C, \mathfrak{t}_C)$ is of classical type, so are all the simple components Φ_i in the decomposition of Φ.

We see that the problem of computing constants for averaged discrete series characters reduces to the problem of computing certain constants $\bar{c}(\mathscr{F}^+ : \Phi^+)$ connected to a simple root system Φ, a choice of positive roots Φ^+, and a component $\mathscr{F}^+ \subseteqq \mathscr{F}'$, the set of regular elements in the underlying real vector space of Φ. We will derive formulas for these constants for the cases $\Phi = B_n$, C_n, or D_n, $n \geqq 1$, where for D_n we assume n is even. (Of course, $B_1 = C_1 = A_1$, and $D_2 = A_1^2$.)

Let

$$\Phi = \begin{cases} \{\pm e_i \pm e_j, \pm e_i \mid 1 \leqq i \neq j \leqq n\} & \text{if} \quad \Phi = B_n ; \\ \{\pm e_i \pm e_j, \pm 2e_i \mid 1 \leqq i \neq j \leqq n\} & \text{if} \quad \Phi = C_n ; \\ \{\pm e_i \pm e_j \mid 1 \leqq i \neq j \leqq n\} & \text{if} \quad \Phi = D_n . \end{cases}$$

Assume

$$\Phi^+ = \begin{cases} \{e_i \pm e_j, e_K \mid 1 \leqq i < j \leqq n, 1 \leqq K \leqq n\} & \text{if} \quad \Phi = B_n , \\ \{e_i \pm e_j, 2e_K \mid 1 \leqq i < j \leqq n, 1 \leqq K \leqq n\} & \text{if} \quad \Phi = C_n , \\ \{e_i \pm e_j \mid 1 \leqq i < j \leqq n\} & \text{if} \quad \Phi = D_n . \end{cases}$$

Then a set S of simple roots for Φ^+ is given by:

$$S = \begin{cases} \{e_1 - e_2, e_2 - e_3, \cdots, e_{n-1} - e_n, e_n\} & \text{if } \Phi = B_n , \\ \{e_1 - e_2, e_2 - e_3, \cdots, e_{n-1} - e_n, 2e_n\} & \text{if } \Phi = C_n , \\ \{e_1 - e_2, e_2 - e_3, \cdots, e_{n-1} - e_n, e_{n-1} + e_n\} & \text{if } \Phi = D_n . \end{cases}$$

In each case an element λ of $\mathscr{F}$ can be written as $\lambda = \sum_{i=1}^n m_i e_i$, $m_i \in \boldsymbol{R}$. (If λ is in the weight lattice for Φ, the m_i's will be integers or half-integers.) In each case the permutation group S_n on n elements acts on Φ and on $\mathscr{F}$ by permuting the indices of the e_i's. With this action, S_n is a subgroup of $W(\Phi)$. Let $S_n^* = \{\sigma \in S_n \mid \sigma(2i-1) < \sigma(2i), 1 \leqq i \leqq [n/2]\}$ and $S_n^{**} = \{\sigma \in S_n^* \mid \sigma(1) < \sigma(3) < \cdots < \sigma(2[n/2]-1)\}$. For $\lambda = \sum_{i=1}^n m_i e_i$, let $\lambda_i = m_{2i-1} e_1 + m_{2i} e_2$, $1 \leqq i \leqq [n/2]$. If n is odd, let $\lambda_n = m_n e_1$. If Φ is of type B_n or C_n, let

$$\bar{c}_2(ne_1 + me_2) = \begin{cases} 4 & \text{if } \ 0 > n > m \ \text{ or } \ 0 > -m > n \\ 0 & \text{otherwise} \end{cases} \tag{2.9}$$

$$\bar{c}_1(ne_1) = \begin{cases} 2 & \text{if } \ n < 0 \\ 0 & \text{if } \ n > 0 . \end{cases} \tag{2.10}$$

If Φ is of type D_n, let

$$\bar{c}_2(ne_1 + me_2) = \begin{cases} 4 & \text{if } \ n < -|m| \\ 0 & \text{otherwise} . \end{cases} \tag{2.11}$$

THEOREM 2.12. *If* $\lambda \in \mathscr{F}^+$, *then* $\bar{c}(\mathscr{F}^+ : \Phi^+) = P(\lambda : \Phi^+)$ *where*

$$P(\lambda : \Phi^+) = \sum_{\sigma \in S_n^{**}} \det \sigma \prod_{i=1}^{n/2} \bar{c}_2((\sigma^{-1}\lambda)_i)$$

if n is even

$$= \sum_{\sigma \in S_n^{**}} \det \sigma \bar{c}_1((\sigma^{-1}\lambda)_n) \prod_{i=1}^{n-1/2} \bar{c}_2((\sigma^{-1}\lambda)_i)$$

if n is odd.

Proof. The theorem is true for $n = 1$ or 2 because it reduces to formulas (2.9), (2.10), and (2.11) which are known from averaging the known discrete series constants for rank one and two groups [3, 7]. Assume that it is true for root systems of rank less than n, $n \geqq 3$. We prove in Lemma 2.14 that for any simple root α, $P(\lambda : \Phi^+) + P(s\lambda : \Phi^+) = 2\bar{c}(\lambda : \Phi_0^+)$ where s is the reflection in $W(\Phi)$ corresponding to α and $\Phi_0 = \{\beta \in \Phi \mid \langle \beta, \alpha \rangle = 0\}$ as in (2.6). Then using (2.6), $P(\lambda : \Phi^+) + P(s\lambda : \Phi^+) = \bar{c}(\mathscr{F}^+ : \Phi^+) + \bar{c}(s\mathscr{F}^+ : \Phi^+)$ for $\lambda \in \mathscr{F}^+$. We show in Lemma 2.13 that $\bar{c}(\mathscr{F}^{\#} : \Phi^+) = P(\lambda : \Phi^+)$, $\lambda \in \mathscr{F}^{\#}$, for one fixed chamber $\mathscr{F}^{\#}$ of $\mathscr{F}'$. Then $\bar{c}(\mathscr{F}^{+}: \Phi^{+}) = P(\lambda: \Phi^{+})$, $\lambda \in \mathscr{F}^{+}$, for all chambers, since any chamber $\mathscr{F}^{+}$ can be reached by applying simple reflections to $\mathscr{F}^{\#}$.

LEMMA 2.13. *Let* $\mathscr{F}^{\#} = \{\lambda \in \mathscr{F}' \mid \langle \lambda, \alpha \rangle > 0$ *for all* $\alpha \in \Phi^{+}\}$. *Then* $\bar{c}(\mathscr{F}^{\#}: \Phi^{+}) = P(\lambda: \Phi^{+})$ *if* $\lambda \in \mathscr{F}^{\#}$.

Proof. It follows from (2.5) that $\bar{c}(\mathscr{F}^{\#}: \Phi^{+}) = 0$. For $\lambda = \sum_{i=1}^{n} m_i e_i$, $\lambda \in \mathscr{F}^{\#}$ implies that $m_i > |m_j|$ for $1 \leqq i < j \leqq n$. For any $\sigma \in S_n^{**}$ and any $1 \leqq i \leqq [n/2]$, $(\sigma^{-1}\lambda)_i = m_{\sigma(2i-1)}e_1 + m_{\sigma(2i)}e_2$ where $\sigma(2i-1) < \sigma(2i)$. Thus $m_{\sigma(2i-1)} > |m_{\sigma(2i)}|$ and using (2.9) or (2.11), $\bar{c}_2((\sigma^{-1}\lambda)_i) = 0$. Thus $P(\lambda: \Phi^{+}) = 0$.

LEMMA 2.14. *Assume that Theorem* 2.12 *is true for root systems of rank less then* n. *Let* α *be a simple root for* Φ^{+}. *Then for* $\lambda \in \mathscr{F}^{+}$, $P(\lambda: \Phi^{+}) + P(s\lambda: \Phi^{+}) = 2\bar{c}(\lambda: \Phi_0^{+})$.

Proof.

Case I. Suppose $\alpha = e_l - e_{l+1}$, $1 \leqq l \leqq n-1$. Let Φ_{n-2} denote the subset of Φ contained in the linear span of $\{e_1, \cdots, e_{l-1}, e_{l+2}, \cdots, e_n\}$, $\Phi_{n-2}^{+} = \Phi_{n-2} \cap \Phi^{+}$. Let A_1 denote the rank one root system with positive root $e_l + e_{l+1}$. Then $\Phi_0^{+} = \Phi_{n-2}^{+} \cup A_1^{+}$. For $\lambda = \sum_{i=1}^{n} m_i e_i$, let $\lambda' = \lambda - m_l e_l - m_{l+1}e_{l+1}$ and $\lambda'' = (m_l + m_{l+1})/2(e_l + e_{l+1})$. Let $\lambda_0 = \lambda' + \lambda''$. Then by (2.8),

$$\bar{c}(\lambda: \Phi_0^{+}) = \bar{c}(\lambda'': A_1^{+})\bar{c}(\lambda': \Phi_{n-2}^{+}) = \bar{c}_1((m_l + m_{l+1})e_1)P(\lambda': \Phi_{n-2}^{+})$$

by the induction hypothesis. In $P(\lambda': \Phi_{n-2}^{+})$ the sum is taken over S_{n-2}^{**} where S_{n-2} is considered as the group of permutations of $\{1, 2, \cdots, l-1, l+2, \cdots, n\}$ and $(\sigma^{-1}\lambda')_i$, $1 \leqq i \leqq [(n-2)/2]$ and $(\sigma^{-1}\lambda')_{n-2}$, $n-2$ odd, have the obvious meaning.

Let $k = [n/2]$ so that $n = 2k$ or $2k+1$. In formulas for $P(\lambda: \Phi^{+})$ the terms $\bar{c}_1((\sigma^{-1}\lambda)_n)$ are included. For the case $n = 2k$ they are understood not to appear. If s is the reflection in $W(\Phi)$ corresponding to $\alpha = e_l - e_{l+1}$, then s is the permutation which interchanges l and $l+1$. $P(\lambda: \Phi^{+}) + P(s\lambda: \Phi^{+}) = 1/k! \sum_{\sigma \in S_n^{*}} \det \sigma[\bar{c}_1((\sigma^{-1}\lambda)_n) \prod_{i=1}^{k} \bar{c}_2((\sigma^{-1}\lambda)_i) + \bar{c}_1(((s\sigma)^{-1}\lambda)_n) \prod_{i=1}^{k} \bar{c}_2(((s\sigma)^{-1}\lambda)_i)]$.

Let $S = \{\sigma \in S_n^{*} \mid s\sigma \in S_n^{*}\}$. Then $sS = S$, and

$$\sum_{\sigma \in S} \det \sigma \bar{c}_1(((s\sigma)^{-1}\lambda)_n) \prod_{i=1}^{k} \bar{c}_2(((s\sigma)^{-1}\lambda)_i) = -\sum_{\sigma \in S} \det \sigma \bar{c}_1((\sigma^{-1}\lambda)_n) \prod_{i=1}^{k} \bar{c}_2((\sigma^{-1}\lambda)_i) .$$

If $\sigma \in S_n^{*}$ and $s\sigma \notin S_n^{*}$, then there is an index j, $1 \leqq j \leqq k$, for which $\sigma(2j-1) = l$, $\sigma(2j) = l+1$. Denote this subset of S_n^{*} by $S(j)$. Then for $\sigma \in S(j)$, $s\sigma(2j-1) = l+1$, $s\sigma(2j) = l$, and for $i \neq 2j-1, 2j$,

$s\sigma(i) = \sigma(i)$. Further, using (2.9), (2.10), and (2.11), $\bar{c}_2((\sigma^{-1}\lambda)_j) + \bar{c}_2(((s\sigma)^{-1}\lambda)_j) = \bar{c}_2(m_l e_1 + m_{l+1}e_2) + \bar{c}_2(m_{l+1}e_1 + m_l e_2) = 2\bar{c}_1((m_l + m_{l+1})e_1)$. Thus

$$P(\lambda\colon \Phi^+) + P(s\lambda\colon \Phi^+) = \frac{1}{k!}\sum_{j=1}^{k}\sum_{\sigma\in S(j)} \det\sigma\bar{c}_1((\sigma^{-1}\lambda)_n) \times \prod_{\substack{i=1\\ i\neq j}}^{k} \bar{c}_2((\sigma^{-1}\lambda)_i)2\bar{c}_1((m_l + m_{l+1})e_1)\ .$$

For each $1 \leqq j \leqq k$,

$$\sum_{\sigma\in S(j)} \det\sigma\bar{c}_1((\sigma^{-1}\lambda)_n)\prod_{\substack{i=1\\ i\neq j}}^{k}\bar{c}_2((\sigma^{-1}\lambda)_i) = \sum_{\sigma\in S^*_{n-2}} \det\sigma\bar{c}_1((\sigma^{-1}\lambda')_{n-2})\prod_{i=1}^{k-1}\bar{c}_2((\sigma^{-1}\lambda')_i) = (k-1)!\,\bar{c}(\lambda'\colon \Phi^+_{n-2})\ .$$

Thus

$$P(\lambda\colon \Phi^+) + P(s\lambda\colon \Phi^+) = 2\bar{c}_1((m_l + m_{l+1})e_1)\bar{c}(\lambda'\colon \Phi^+_{n-2}) = 2\bar{c}(\lambda_0\colon \Phi^+_0)\ .$$

Case II. Suppose $\Phi = B_n$ or C_n and $\alpha = e_n$ or $2e_n$. Then $\Phi_0 = B_{n-1}$ or C_{n-1}. For $\lambda = \sum_{i=1}^{n} m_i e_i$, let $\lambda_0 = \lambda - m_n e_n$. By the induction hypothesis, $\bar{c}(\lambda_0\colon \Phi^+_0) = P(\lambda_0\colon \Phi^+_0)$.

Suppose $n = 2k$ is even. Then $S^*_n = \bigcup_{j=1}^{k} S(j)$ where $S(j) = \{\sigma\in S^*_n \mid \sigma(2j) = n\}$. For $\sigma\in S(j)$, $(\sigma^{-1}\lambda)_i = (\sigma^{-1}s\lambda)_i$ for $i\neq j$, $(\sigma^{-1}\lambda)_j = m_{\sigma(2j-1)}e_1 + m_n e_2$, and $(\sigma^{-1}s\lambda)_j = m_{\sigma(2j-1)}e_1 - m_n e_2$. Using (2.9) and (2.10), $\bar{c}_2(m_{\sigma(2j-1)}e_1 + m_n e_2) + \bar{c}_2(m_{\sigma(2j-1)}e_1 - m_n e_2) = 2\bar{c}_1(m_{\sigma(2j-1)}e_1)$.

$$\begin{aligned} P(\lambda\colon \Phi^+) &+ P(s\lambda\colon \Phi^+) \\ &= \frac{1}{k!}\sum_{j=1}^{k}\sum_{\sigma\in S(j)}\det\sigma 2\bar{c}_1(m_{\sigma(2j-1)}e_1)\prod_{\substack{i=1\\ i\neq j}}^{k}\bar{c}_2((\sigma^{-1}\lambda)_i) \\ &= \frac{2}{(k-1)!}\sum_{\sigma\in S^*_{n-1}}\det\sigma\bar{c}_1(m_{\sigma(n-1)}e_1)\prod_{i=1}^{k-1}\bar{c}_2((\sigma^{-1}\lambda)_i) = 2\bar{c}(\lambda_0\colon \Phi^+_0)\ . \end{aligned}$$

Suppose $n = 2k+1$. Define $S(j)$ as above. Then $S^*_n = \bigcup_{j=1}^{n} S(j) \cup S^*_{n-1}$ where S^*_{n-1} can be identified with $\{\sigma\in S^*_n \mid \sigma(n) = n\}$. For $\sigma\in S^*_{n-1}$, $(\sigma^{-1}\lambda)_i = (\sigma^{-1}s\lambda)_i$, $1\leqq i\leqq k$, and $\bar{c}_1((\sigma^{-1}\lambda)_n) + \bar{c}_1((\sigma^{-1}s\lambda)_n) = \bar{c}_1(m_n e_1) + \bar{c}_1(-m_n e_1) = 2$, using (2.10).

$$\begin{aligned} P(\lambda\colon \Phi^+) &+ P(s\lambda\colon \Phi^+) \\ &= \frac{1}{k!}\sum_{j=1}^{k}\sum_{\sigma\in S(j)}\det\sigma 2\bar{c}_1(m_{\sigma(2j-1)}e_1)\bar{c}_1(m_{\sigma(n)}e_1)\prod_{\substack{i=1\\ i\neq j}}^{k}\bar{c}_2((\sigma^{-1}\lambda)_i) \\ &\quad + \frac{2}{k!}\sum_{\sigma\in S^*_{n-1}}\det\sigma\prod_{i=1}^{k}\bar{c}_2((\sigma^{-1}\lambda)_i)\ . \end{aligned} \tag{2.15}$$

The second term in (2.15) involving the sum over S^*_{n-1} is exactly $2\bar{c}(\lambda_0: \Phi_0^+)$. For each $1 \leqq j \leqq k$, let $S(j)^+ = \{\sigma \in S(j) \mid \sigma(2j-1) < \sigma(n)\}$. Let $S(j)^- = \{\sigma \in S(j) \mid \sigma(2j-1) > \sigma(n)\}$. Let τ denote the permutation $(2j-1\ n)$ which interchanges $2j-1$ and n. Then $S(j)^- = \{\sigma\tau \mid \sigma \in S(j)^+\}$ and

$$\sum_{\sigma \in S(j)-} \det \sigma \bar{c}_1(m_{\sigma(2j-1)}e_1)\bar{c}_1(m_{\sigma(n)}e_1) \prod_{\substack{i=1\\ i\neq j}}^{k} \bar{c}_2((\sigma^{-1}\lambda)_i)$$
$$= -\sum_{\sigma \in S(j)^+} + \det \sigma \bar{c}_1(m_{\sigma(n)}e_1)\bar{c}_1(m_{\sigma(2j-1)}e_1) \prod_{\substack{i=1\\ i\neq j}}^{k} \bar{c}_2((\sigma^{-1}\lambda)_i) .$$

Thus the sum over $S(j)$ in (2.15) is zero for each j, and $P(\lambda: \Phi^+) + P(s\lambda: \Phi^+) = 2\bar{c}(\lambda_0: \Phi^+)$.

Case III. Suppose $\Phi = D_n$ where $n = 2k$ is even, and $\alpha = e_{n-1} + e_n$. Then $\Phi_0^+ = D^+_{n-2} \cup A_1^+$ where A_1 has positive root $e_{n-1} - e_n$. For $\lambda = \sum_{i=1}^n m_i e_i$, let $\lambda' = \lambda - m_{n-1}e_{n-1} - m_n e_n$ and $\lambda'' = (m_{n-1} - m_n/2)(e_{n-1} - e_n)$. Let $\lambda_0 = \lambda' + \lambda''$. For $1 \leqq j \leqq k$, let $S(j) = \{\sigma \in S_n^* \mid \sigma(2j-1) = n-1, \sigma(2j) = n\}$. Note $S(k) = S^*_{n-2}$. For $1 \leqq l \neq j \leqq k$, let $S(l, j) = \{\sigma \in S_n^* \mid \sigma(2l) = n-1, \sigma(2j) = n\}$. Then $S_n^* = \bigcup_{j=1}^k S(j) \cup \bigcup_{1\leqq l\neq j\leqq k} S(l, j)$. For $\sigma \in S(j)$, $(\sigma^{-1}\lambda)_i = (\sigma^{-1}s\lambda)_i$, $i \neq j$, and $\bar{c}_2((\sigma^{-1}\lambda)_j) + \bar{c}_2((\sigma^{-1}s\lambda)_j) = \bar{c}_2(m_{n-1}e_1 + m_n e_2) + \bar{c}_2(-m_n e_1 - m_{n-1}e_2) = 2\bar{c}_1((m_{n-1} - m_n)e_1)$ using (2.10) and (2.11). For $\sigma \in S(l, j)$, $(\sigma^{-1}\lambda)_i = (\sigma^{-1}s\lambda)_i$, $i \neq j$ or l, and

$$\begin{aligned}\bar{c}_2((\sigma^{-1}s\lambda)_j)\bar{c}_2((\sigma^{-1}s\lambda)_l) &= \bar{c}_2(m_{\sigma(2j-1)}e_1 - m_{n-1}e_2)\bar{c}_2(m_{\sigma(2l-1)}e_1 - m_n e_2)\\ &= \bar{c}_2(m_{\sigma(2j-1)}e_1 + m_{n-1}e_2)\cdot\bar{c}_2(m_{\sigma(2l-1)}e_1 + m_n e_2)\end{aligned}$$

using (2.11).

Then

$$\begin{aligned}&P(\lambda: \Phi^+) + P(s\lambda: \Phi^+)\\ &= \frac{1}{k!}\sum_{j=1}^{k}\sum_{\sigma\in S(j)} \det\sigma \prod_{\substack{i=1\\ i\neq j}}^{k} \bar{c}_2((\sigma^{-1}\lambda)_i) 2\bar{c}_1((m_{n-1} - m_n)e_1)\\ &\quad + \frac{1}{k!}\sum_{1\leqq j\neq l\leqq k}\sum_{\sigma\in S(l,j)} \det\sigma \prod_{\substack{i=1\\ i\neq j,l}}^{k} \bar{c}_2((\sigma^{-1}\lambda)_i)\\ &\quad \times[\bar{c}_2(m_{\sigma(2j-1)}e_1 + m_{n-1}e_2)\bar{c}_2(m_{\sigma(2l-1)}e_1 + m_n e_2)\\ &\quad + \bar{c}_2(m_{\sigma(2j-1)}e_1 + m_n e_2)\bar{c}_2(m_{\sigma(2l-1)}e_1 + m_{n-1}e_2)] .\end{aligned} \tag{2.16}$$

For each

$$1 \leqq j \leqq k, \sum_{\sigma\in S(j)} \det\sigma \prod_{\substack{i=1\\ i\neq j}}^{k} \bar{c}_2((\sigma^{-1}\lambda)_i)$$
$$= \sum_{\sigma\in S^*_{n-2}} \det\sigma \prod_{i=1}^{k-1} \bar{c}_2((\sigma^{-1}\lambda)_i) = (k-1)!\,\bar{c}(\lambda': D^+_{n-2}) .$$

For $1 \leqq j \neq \boldsymbol{l} \leqq k$, let $S(\boldsymbol{l}, j)^+ = \{\sigma \in S(\boldsymbol{l}, j) | \sigma(2\boldsymbol{l} - 1) < \sigma(2j - 1)\}$ and $S(\boldsymbol{l}, j)^- = \{\sigma \in S(\boldsymbol{l}, j) | \sigma(2\boldsymbol{l} - 1) > \sigma(2j - 1)\}$. Then $S(\boldsymbol{l}, j)^- = \{\sigma\tau | \sigma \in S(\boldsymbol{l}, j)^+\}$ where $\tau = (2j - 1, 2\boldsymbol{l} - 1)$. Then the sum over $S(\boldsymbol{l}, j)$ in (2.16) is zero since the sum over $S(\boldsymbol{l}, j)^-$ will cancel with the sum over $S(\boldsymbol{l}, j)^+$ for each $\boldsymbol{l}$ and j. Thus

$$P(\lambda : \Phi^+) + P(s\lambda : \Phi^+) = 2\bar{c}_1((m_{n-1} - m_n)e_1)\bar{c}(\lambda' : D_{n-2}^+) = 2\bar{c}(\lambda_0 : \Phi_0^+) .$$

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