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SOLUTION FOR AN INTEGRAL EQUATION WITH CONTINUOUS INTERVAL FUNCTIONS

J. A. CHATFIELD

Suppose R is the set of real numbers and all integrals are of the subdivision-refinement type. Suppose each of G and H is a function from $R \times R$ to R and each of f and h is a function from R to R such that $f(a) = h(a)$, dh is of bounded variation on $[a, x]$, and $\int_a^x H^2 = \int_a^x G^2 = 0$ for $x > a$. The following two statements are equivalent:

(1) If $x > a$, then f is bounded on $[a, x]$, $\int_a^x H$ exists, $\int_a^x G$ exists, $(RL)\int_a^x (fG + fH)$ exists, and

$$f(x) = h(x) + (RL)\int_a^x (fG + fH)\ ;$$

(2) If $a \leqq p < q \leqq x$, then each of ${}_p\Pi^q (1 + H)$ and ${}_p\Pi^q (1 - G)^{-1}$ exists and neither is zero,

$$(R)\int_a^x [{}_t\Pi^x (1 + H)(1 + G)][(1 - G)^{-1}]dh$$

exists, and

$$f(x) = f(a)\, {}_a\Pi^x (1 + H)(1 - G)^{-1} + (R)\int_a^x [{}_t\Pi^x (1 + H)(1 + G)][1 - G)^{-1}]dh\ .$$

Introduction. In a recent paper [4], B. W. Helton solved the equation $f(x) = h(x) + (RL)\int_a^x (fG + fH)$ using product integration. All functions involved were required to be of bounded variation and the existence of various integrals was also required. In a subsequent paper [9], J. C. Helton was able to reduce the conditions placed on h to being a quasicontinuous function although other conditions such as requiring G and H to be of bounded variation were maintained. In still another paper [7], J. C. Helton was able to reduce the restrictions placed on G and H to that of being product bounded but he also used other restrictions not used in [4] or [9] such as requiring h to be a constant function and $G(r, s) = -G(s, r)$, a condition not unlike that of being additive. In this paper we are concerned with obtaining a solution for the equation $f(x) = h(x) + (RL)\int_a^x (fG + fH)$ without requiring either G or H to be of bounded variation or that $G(r, s) = -G(s, r)$ or that h be a constant function. Instead, our major restriction placed on G and

H is that each be continuous (i.e., $\int_a^x G^2 = \int_a^x H^2 = 0$) and in this respect, we note that we have shown in an earlier paper [3] with W. P. Davis that neither of the two statements, (1) $\int_a^b G^2 = 0$ and (2) G is of bounded variation on $[a, b]$, is a consequence of the other. Also, some functions are either required to be product bounded or shown to be product bounded and we note that the set of function having bounded variation on an interval is a proper subset of the set of functions which are product bounded on the same interval.

The reader is also referred to recent studies of D. L. Lovelady [15], [16], J. S. MacNerney [17], J. W. Neuberger [18] and J. C. Helton [8] for related results and to put the present result in perspective. For examples of solutions of integral equations using product integrals and heretofore known results, the reader is referred to [4, page 319–322] and [2].

DEFINITIONS AND NOTATIONS. All functions will be functions from $R \times R$ to R or R to R where R is the set of real numbers. All integrals are of the subdivision-refinement type and we will use upper case (G) for functions from $R \times R$ to R and lower case (g) for functions from R to R. If G is a function from $R \times R$ to R then the statement that G is (a) product bounded, (b) of bounded variation, (c) bounded on $[a, b]$ means there is a number M and a subdivision D of $[a, b]$ such that if $D' = \{x_i\}_{i=1}^n$ is a refinement of D, then

(a) if $0 < p \leqq q \leqq n$, $|\prod_{i=p}^q 1 + G(x_{i-1}, x_i)| < M$.

(b) $\sum_{i=1}^n |G(x_{i-1}, x_i)| < M$.

(c) if $0 < i \leqq n$, then $|G(x_{i-1}, x_i)| < M$, respectively.

The statement that the function G from $R \times R$ to R is (a) product integrable, (b) sum integrable on $[a, b]$ means there is a number A such that if $\varepsilon > 0$ then there is a subdivision D of $[a, b]$ such that if $D' = \{x_i\}_{i=0}^n$ is a refinement of D, then

(a) $|\prod_{i=1}^n [1 + G(x_{i-1}, x_i)] - A| < \varepsilon$.

(b) $|\sum_{i=1}^n G(x_{i-1}, x_i) - A| < \varepsilon$, respectively.

If h is a function from R to R then dh denotes the function G from $R \times R$ to R such that for each $x < y$, $G(x, y) = h(y) - h(x)$. If G is a function from $R \times R$ to R and $G(x, y)$ exists, then x is assumed to be less than y.

The following notations will be used to facilitate writing:

$$\prod_{i=1}^n [1 + G(x_{i-1}, x_i)] = \prod_D (1 + G_i) ,$$

$$\sum_{i=1}^n G(x_{i-1}, x_i) = \sum_D G_i ,$$

$$dh_i = h(x_i) - h(x_{i-1}) ,$$

and

$$f(x_i) = f_i$$

where $D = \{x_i\}_{i=0}^n$ is a subdivision of some interval and $0 < i \leqq n$. Further, left and right integrals are used extensively and the appropriate approximating term is indicated by $\approx$.

$$(LR) \int_a^b (fH + fG) \approx f(x_{i-1})G(x_{i-1}, x_i) + f(x_i)G(x_{i-1}, x_i)$$

$$(R) \int_a^b \prod_t^b (1 + G) df \approx \left[\prod_{x_i}^b (1 + G)\right][f(x_j) - f(x_{i-1})]$$

$$(RR) \int_a^b (fH + fG) \approx f(x_i)G(x_{i-1}, x_i) + f(x_i)G(x_{i-1}, x_i) .$$

THEOREMS. *The following lemmas are needed in the proof of our main results.*

LEMMA 1.1. *If G is a function from $R \times R$ to R, ${}_a\prod^b (1 + G)$ exists and is not zero, and $\int_a^b G$ exists, then G is bounded on $[a, b]$* [12, Theorem 6].

LEMMA 1.2. *If $\int_a^b G^2 = 0$, then the following statements are equivalent:*

(1) *${}_a\prod^b (1 + G)$ exists and is not zero.*

(2) *$\int_a^b G$ exists.*

Furthermore, if either (1) *or* (2) *is true, then* $\ln {}_a\prod^b (1 + G) = \int_a^b G$ [3, Theorem 3].

LEMMA 1.3. *If G is a function from $R \times R$ to R such that $\int_a^b G^2 = 0$, then there is a subdivision D of $[a, b]$ and a number M such that if $D' = \{x_i\}_{i=0}^n$ is a refinement of D and $0 < i \leqq n$, then $(1 - G_i)^{-1}$ exists and $|(1 - G_i)^{-1}| < M$.*

Proof. This lemma follows directly from the fact that $\int_a^b G^2 = 0$.

LEMMA 1.4. *Suppose G is a function from $R \times R$ to R such that $|G| < 1$ on $[a, b]$, ${}_a\prod^b (1 + G)$ exists and is not zero, and there is a subdivision D of $[a, b]$ and a number M such that if D' is a refinement of D then $[\prod_{D'} (1 + G_i)^{-1}]$ and $|\prod_{D'} (1 + G_i)^{-1}| < M$. Then, there is a subdivision D of $[a, b]$ and a number M such that* *if* $D' = \{x_i\}_{i=0}^n$ *is a refinement of* D *and* $0 < p < q \leqq n$, *then* $|\prod_{i=p}^q (1 + G_i)^{-1}| < M$ [13, Lemma 1].

LEMMA 1.5. *If* G *is a function from* $R \times R$ *to* R *such that* $\int_a^b G^2 = 0$ *and* $\int_a^b G$ *exists, then there is a subdivision* D *of* $[a, b]$ *and a number* M *such that if* $D' = \{x_i\}_{i=0}^n$ *is a refinement of* D *and* $0 < p < q \leqq n$, *then* $|\prod_{i=p}^q (1 + G_i)^{-1}| < M$.

Indication of proof. Since $\int_a^b G^2 = 0$ and $\int_a^b G$ exists, then from Lemma 1.2, ${}_a\prod^b (1 + G)$ exists and is not zero. Hence, since $\int_a^b G^2 = 0$ implies that $|G_i| < 1$ for any refinement $D' = \{x_i\}_{i=0}^n$ of an appropriate subdivision of $[a, b]$, then Lemma 1.5 follows from Lemma 1.4.

LEMMA 1.6. *If* G *is a function from* $R \times R$ *to* R *such that* $\int_a^b G$ *exists and for each* $x < y$, $H(x, y) = \left|\int_x^y G - G(x, y)\right|$, *then* $\int_a^b H$ *exists and is* 0.

This lemma is due to A. Kolmogoroff [14]. For related results the reader is referred to W. D. L. Appling [1, Theorem 1.2] and B. W. Helton [4, Theorem 4.1].

LEMMA 1.7. *If* G *is a function from* $R \times R$ *to* R *such that* G *is bounded on* $[a, b]$, ${}_a\prod^b (1 + G)$ *exists and is not zero, and* H *is a function from* $R \times R$ *to* R *such that for each* $a \leqq x < y \leqq b$, $H(x, y) = |1 + G(x, y) - {}_x\prod^y (1 + G)|$, *then* $\int_a^b H$ *exists and is* 0 [6, Lemma 1.4].

LEMMA 1.8. *If each of* H *and* G *is a function from* $R \times R$ *to* R *such that* ${}_a\prod^b (1 + H)$ *exists and* ${}_a\prod^b (1 + G)$ *exists and neither is zero, then* ${}_a\prod^b (1 + H)(1 + G)$ *exists and is not zero.*

Proof. The proof of this lemma is straightforward and we omit it.

LEMMA 1.9. *If* G *is a function from* $R \times R$ *to* R, $\int_a^b G$ *exists, and* G *is bounded on* $[a, b]$ *then there is a subdivision* D *of* $[a, b]$ *and a number* M *such that if* $D' = \{x_i\}_{i=0}^n$ *is a refinement of* D *and* $0 < p < q \leqq n$, *then* $|\sum_{i=p}^q G_i| < M$.

Proof. This lemma follows from Lemma 1.6.

The following algebraic identity is used frequently and it may be established by induction.

LEMMA 1.10. *If each of* $\{a_i\}_{i=1}^n$ *and* $\{b_i\}_{i=1}^n$ *is a sequence of numbers and* $n > 1$, *then*

$$\prod_{i=1}^n a_i - \prod_{i=1}^n b_i = \sum_{i=1}^n \Big(\prod_{j=1}^{i-1} b_j\Big)(a_i - b_i)\Big(\prod_{j=i+1}^n a_j\Big) .$$

THEOREM 1. *Suppose each of* G, H, *and* J *is a function from* $R \times R$ *to* R *such that* J *is of bounded variation on* $[a, b]$, $\int_a^b G^2 = \int_a^b H^2 = 0$, $\int_a^b J$ *exists, for each* $a \leqq x < y \leqq b$, *each of* ${}_x\prod^y (1 + G)$ *and* ${}_x\prod^y (1+H)$ *exists and neither is zero,* $(RR)\int_a^b J[{}_y\prod^t (1+G)][{}_t\prod^y (1+H)]$ *exists, and for each* $a \leqq x < y \leqq b$,

$$K(x, y) = \left|J(x, y) - (RR)\int_x^y J[{}_y\prod^t (1 + G)][{}_t\prod^y (1 + H)]\right| .$$

Then, $\int_a^b K$ *exists and is* 0.

Proof. Let $\varepsilon > 0$. Since $\int_a^b J$ exists and J is of bounded variation on $[a, b]$, then, from Lemma 1.6, there is a subdivision D_1 of $[a, b]$ and a number M_1 such that if $D' = \{x_i\}_{i=0}^n$ is a refinement of D_1, then

$$\sum_{D'} |J_i| < M . \tag{1}$$

and

$$\sum_{D'} \left|\int_{x_{i-1}}^{x_i} J - J_i\right| < \frac{\varepsilon}{4} . \tag{2}$$

Since $\int_a^b G^2 = \int_a^b H^2 = 0$ and each of ${}_a\prod^b (1 + G)$ and ${}_a\prod^b(1 + H)$ exists and neither is zero then, from Lemma 1.2, $\int_a^b G$ and $\int_a^b H$ each exists, and there is a subdivision D_2 of $[a, b]$ and a number M_2 such that if $D' = \{x_i\}_{i=0}^n$ is a refinement of D_2, then

$$\left|\exp \int_{x_{i-1}}^{x_i} G\right| < M_2 , \tag{3}$$

$$\left|\exp \int_{x_{i-1}}^{x_i} H\right| < M_2 , \tag{4}$$

$$|G(x_{i-1}, x_i)| < \frac{1}{6} \ln\Big(1 + \frac{\varepsilon}{4M_1M_2^3}\Big) , \tag{5}$$

and

$$|H(x_{i-1}, x_i)| < \frac{1}{6} \ln\Big(1 + \frac{\varepsilon}{4M_1M_2^3}\Big) . \tag{6}$$

Again, from Lemma 1.6 there is a subdivision D_3 of $[a, b]$ such that if $D' = \{x_i\}_{i=0}^n$ is a refinement of D_3, then

$$(7) \qquad \sum_{D'} \left|(G_i - H_i) - \int_{x_{i-1}}^{x_i} (G - H)\right| < \frac{1}{6} \ln\left(1 + \frac{\varepsilon}{M_1 M_2^3}\right).$$

Let $D = D_1 + D_2 + D_3$ and $D' = \{x_i\}_{i=0}^n$ be a refinement of D. Since, for $0 < i \leqq n$, $(RR)\int_{x_{i-1}}^{x_i} J[{}_{x_i}\Pi^t(1 + G)][{}_t\Pi^{x_i}(1 + H)]$ exists, then there is a subdivision $D_i = \{t_j\}_{j=0}^{k_i}$ of $[x_{i-1}, x_i]$ such that

$$(8) \qquad \begin{aligned} &\Big|\sum_{D_i} [J(t_{j-1}, t_j)][{}_{x_i}\Pi^{t_j}(1 + G)][{}_{t_j}\Pi^{x_i}(1 + H)] \\ &\quad - \int_{x_{i-1}}^{x_i} J[{}_{x_i}\Pi^t(1 + G)][{}_t\Pi^{x_i}(1 + H)]\Big| < \frac{\varepsilon}{4\cdot 2^i}. \end{aligned}$$

Therefore, for $x_{i-1} \leqq t_j \leqq x_i$,

$$(9) \qquad \begin{aligned} &\left|\int_{x_{i-1}}^{x_i} (G - H) - \int_{x_{i-1}}^{t_j} (G - H)\right| \\ &\leqq \left|\int_{x_{i-1}}^{x_i} (G-H) - (G_i - H_i)\right| + \left|G(x_{i-1}, t_j) - H(x_{i-1}, t_j) - \int_{x_{i-1}}^{t_j} (G-H)\right| \\ &\quad + |G(x_{i-1}, t_j)| + |H(x_{i-1}, t_j)| + |G(x_{i-1}, x_i)| + |H(x_{i-1}, x_i)| \\ &< 6\left[\frac{1}{6} \ln\left(1 + \frac{\varepsilon}{4M_1 M_2^3}\right)\right] \qquad (7, 5, 6) \\ &= \ln\left(1 + \frac{\varepsilon}{4M_1 M_2^3}\right). \end{aligned}$$

Hence, from (9) it follows that

$$(10) \qquad \left|\exp \int_{x_{i-1}}^{x_i} (G - H) - \exp \int_{x_{i-1}}^{t_j} (G - H)\right| < \frac{\varepsilon}{4M_2^2 M_1}.$$

Then,

$$\begin{aligned} \sum_{D'} |K_i| &= \sum_{D'} \left|J_i - \int_{x_{i-1}}^{x_i} J[{}_{x_i}\Pi^t(1 + G)][{}_t\Pi^{x_i}(1 + H)]\right| \\ &\leqq \sum_{D'} \left|J_i - \int_{x_{i-1}}^{x_i} J\right| \\ &\quad + \sum_{D'} \left|\int_{x_{i-1}}^{x_i} J - \sum_{D_i} J(t_{j-1}, t_j)\right| \\ &\quad + \sum_{D'} \left|\sum_{D_i} J(t_{j-1}, t_j) - \sum_{D_i} [J(t_{j-1}, t_j)][{}_{x_i}\Pi^{t_j}(1+G)][{}_{t_j}\Pi^{x_i}(1+H)]\right| \\ &\quad + \sum_{D'} \Big|\sum_{D_i} [J(t_{j-1}, t_j)][{}_{x_i}\Pi^{t_j}(1 + G)][{}_{t_j}\Pi^{x_i}(1 + H)] \\ &\quad - \int_{x_{i-1}}^{x_i} J[{}_{x_i}\Pi^t(1 + G)][{}_t\Pi^{x_i}(1 + H)]\Big| \\ &\leqq \frac{\varepsilon}{4} + \sum_{D'} \left|\sum_{D_i} \int_{t_{j-1}}^{t_j} J - \sum_{D_i} J(t_{j-1}, t_j)\right| \end{aligned}$$

$$+ \sum_{D'} \sum_{D_i} |J(t_{j-1}, t_j)| \cdot |1 - {}_{x_i}\prod^{t_j} (1 + G)\, {}_{t_j}\prod^{x_i} (1 + H)|$$

$$+ \sum_{D'} \frac{\varepsilon}{4 \cdot 2^i} \qquad (8, 2)$$

$$\leq \sum_{D'} \sum_{D_i} \left| \int_{t_{j-1}}^{t_j} J - J(t_{j-1}, t_j) \right|$$

$$+ \sum_{D'} \sum_{D_i} |J(t_{j-1}, t_j)| \left| 1 - \left[\exp \int_{x_i}^{t_j} G \right] \left[\exp \int_{t_j}^{x_i} H \right] \right|$$

$$+ \frac{\varepsilon}{2} \qquad \text{(Lemma 1.2)}$$

$$\leq \frac{\varepsilon}{4} + \sum_{D'} \sum_{D_i} |J(t_{j-1}, t_j)| \cdot \left| \exp \int_{x-1}^{x_i} (G - H) \right|$$

$$\times \left| \exp \int_{x_{i-1}}^{x_i} (G - H) - \exp \int_{x_{i-1}}^{t_j} (G - H) \right|$$

$$+ \frac{\varepsilon}{2} \qquad (2)$$

$$\leq M_2^2 \sum_{D'} \sum_{D_i} |J(t_{j-1}, t_j)| \frac{\varepsilon}{4M_1 M_2^2} + \frac{3\varepsilon}{4} \qquad (1, 10)$$

$$< M_1 \cdot \frac{\varepsilon}{4M_1} + \frac{3\varepsilon}{4}$$

$$= \varepsilon .$$

Hence, $\int_a^b K = 0$.

We now state the main result of this paper.

THEOREM 2. *Suppose each of G and H is a function from $R \times R$ to R and each of f and h is a function from R to R such that dh is of bounded variation on $[a, x]$ and $\int_a^x H^2 = \int_a^x G^2 = 0$ for $x > a$.*

The following two statements are equivalent:

(1) *If $x > a$, then f is bounded on $[a, x]$, $\int_a^x H$ exists, $\int_a^x G$ exists, $(RL) \int_a^x (fG + fH)$ exists, and*

$$f(x) = h(x) + (RL) \int_a^x (fG + fH) ;$$

(2) *If $a \leq p < q \leq x$, then each of ${}_p\prod^q (1 + H)$ and ${}_p\prod^q (1 - G)^{-1}$ exists and neither is zero, $(R) \int_a^x [{}_t\prod^x (1 + H)(1 + G)][(1 - G)^{-1}]dh$ exists, and*

$$f(x) = f(a)\, {}_a\prod^x (1 + H)(1 - G)^{-1}$$
$$+ (R) \int_a^x [{}_t\prod^x (1 + H)(1 + G)][(1 - G)^{-1}]dh .$$

Proof. $1 \Rightarrow 2$. Let $a \leq p < q \leq x$. Since $\int_a^x H^2 = \int_a^x G^2 = 0$ and each of $\int_a^x H$ and $\int_a^x G$ exists, then, from Lemma 1.2, ${}_p\prod^q (1 + H)$ and

${}_p\Pi^q (1-G)$ exist and neither is 0, and hence $[{}_p\Pi^q (1-G)]^{-1} = {}_p\Pi^q (1-G)^{-1}$ exists. Since $\int_a^x G^2 = 0$, then, from Lemma 1.3, $(1-G)^{-1}$ is bounded on $[a, x]$ and since dh has bounded variation on $[a, x]$, then $(1-G)^{-1}dh$ has bounded variation on $[a, x]$. Let $\varepsilon > 0$. Since dh is of bounded variation on $[a, x]$ then there is a number M_1 and a subdivision D_1 of $[a, x]$ such that if $D' = \{x_i\}_{i=0}^n$ is a refinement of D_1, then $\sum_{D'} |dh_i| < M_1$. From, Lemma 1.5, there is a number M_2 and a subdivision D_2 of $[a, x]$ such that if $D' = \{x_i\}_{i=0}^n$ is a refinement of D_2 and $0 < p < q \leqq n$, then $|\prod_{i=p}^q (1-G)^{-1}| < M_2$ Since $\int_a^x H$ exists then, from Lemma 1.9, there is a subdivision D_3 of $[a, x]$ and a number M_3 such that if $D' = \{x_i\}_{i=0}^n$ is a refinement of D_3 and $0 < p < q \leqq n$, then $|\sum_{i=p}^q H_i| < M_3$. Since $(RL)\int_a^b (fG + fH)$ exists then there is a subdivision D_4 of $[a, x]$ such that if $D' = \{x_i\}_{i=0}^n$ is a refinement of D_4, then

$$\sum_{D'} \left| (RL) \int_{x_{i-1}}^{x_i} (fG + fH) - (f_i G_i + f_{i-1} H_i) \right| < \frac{\varepsilon}{3M_2^2 \exp M_3} .$$

Also, there exists a subdivision D_5 of $[a, x]$ such that if $D' = \{x_i\}_{i=0}^n$ is a refinement of D_5 then

$$(1) \qquad \left| {}_a\Pi^x (1+H) \, {}_a\Pi^x_{D'} (1-G)^{-1} - \prod_{E'} (1+H_i)(1-G_i)^{-1} \right| < \frac{\varepsilon}{3(|f(a)|+1)}$$

and

if $0 < p < q \leqq n$,

$$(2) \qquad \left| \prod_{j=p}^q (1+H_j)(1-G_j)^{-1} - {}_{x_p}\Pi^{x_q} (1+H)(1-G)^{-1} \right| < \frac{\varepsilon}{3M_1 M_2} .$$

Let $D = D_1 + D_2 + D_3 + D_4 + D_5$ and $D' = \{x_i\}_{i=0}^n$ be a refinement of D, then, from the iterative technique of B. W. Helton [4, page 311], we have that

$$\begin{aligned} f(x) &= f(x_n) \\ &= f_0\left[\prod_{i=1}^n (1+H_i)(1-G_i)^{-1}\right] \\ &\quad + \prod_{i=1}^n \left[\prod_{j=i+1}^n (1+H_j)(1-G_j)^{-1}\right](1-G_i)^{-1} dh_i \\ &\quad + \prod_{i=1}^n (1-G_i)^{-1}\left[\prod_{j=i+1}^n (1+H_j)(1-G_j)^{-1}\right] \\ &\quad \times \left[(RL)\int_{x_{i-1}}^{x_i} (fG + fH) - (f_i G_i + f_{i-1} H_i)\right] . \end{aligned}$$

Hence,

$$|f(x)-f(a)\,{}_a\Pi^x(1+H)\,{}_a\Pi^x(1-G)^{-1}-\sum_{D'}[{}_{x_i}\Pi^x(1+H)(1+G)][(1-G_i^{-1})]dh_i|$$
$$\leqq |f(a)[\prod_{D'}(1+H_i)][(1-G_i)^{-1}]-f(a)\,{}_a\Pi^x(1+H)\,{}_a\Pi^x(1-G)^{-1}|$$
$$+\left|\sum_{D'}\left[\prod_{j=i+1}^{n}(1+H_i)\right]\left[\prod_{j=i+1}^{n}(1-G_i)^{-1}\right][(1-G_i)^{-1}]dh_i\right.$$
$$\left.-\sum_{D'}[{}_{x_i}\Pi^x(1+H)(1+G)^{-1}][(1-G_i)^{-1}]dh_i\right|$$
$$+\sum_{D'}\left|\left[\prod_{j=i+1}^{n}(1+H_j)\right]\left[\prod_{j=i+1}^{n}(1-G_j)^{-1}\right][(1-G_i)^{-1}]\right|$$
$$\times\left|(RL)\int_{x_{i-1}}^{x_i}(fG+fH)-(f_iG_i+f_{i-1}H_i)\right|$$
$$\leqq |f(a)|\cdot\frac{\varepsilon}{3[|f(a)|+1]}$$
$$+\sum_{D'}|(1-G_i)^{-1}|\cdot|dh_i|\cdot\left|\prod_{j=i+1}^{n}(1+H_i)(1-G_i)^{-1}\right.$$
$$\left.-[{}_{x_i}\Pi^x(1+H)(1-G)^{-1}]\right|$$
$$+\sum_{D'}\left|\exp\sum_{j=i+1}^{n}H_j\right|\cdot M_2\cdot M_2\cdot\left|(RL)\int_{x_{i-1}}^{x_i}(fG+fH)-(f_iG_i+f_{i-1}H_i)\right|$$
$$<\frac{\varepsilon}{3}+M_2\cdot\frac{\varepsilon}{3M_1M_2}\sum_{D'}|dh_i|+M_2^2\exp M_3\left[\frac{\varepsilon}{3M_2^2\exp M_3}\right]$$
$$<\frac{\varepsilon}{3}+\frac{\varepsilon}{3M_1}\cdot M_1+\frac{\varepsilon}{3}$$
$$=\varepsilon\,.$$

Hence, $(R)\int_a^x[{}_t\Pi^x(1+H)(1+G)][(1-G)^{-1}]dh$ exists and

$$f(x)=f(a)\,{}_a\Pi^x(1+H)(1-G)^{-1}+(R)\int_a^x[{}_t\Pi^x(1+H)(1+G)][(1-G)^{-1}]dh\,.$$

$2\Rightarrow 1$. Suppose $x>a$ and $\varepsilon>0$. Since each of ${}_a\Pi^x(1+H)$ and ${}_a\Pi^x(1-G)^{-1}$ exists and neither is 0, then, from Lemma 1.2, $\int_a^x H$ exists, $\int_a^x G$ exists, and from Lemma 1.1, each of H and G is bounded on $[a, x]$. From Lemma 1.5, $(1-G)^{-1}$ is bounded on $[a, x]$ and since dh is of bounded variation $[a, x]$, then dh is bounded on $[a, x]$. Therefore, it follows from the boundedness of the functions involved that f is bounded on $[a, x]$. Hence, there is a number M and a subdivision D_1 of $[a, x]$ such that if $D'=\{x_i\}_{i=0}^n$ is a refinement of D_1 and $0<i\leqq n$, then (1) $|f_{i-1}|<M$ and (2) $|1-G_i|<M$.

Since $(1-G)^{-1}$ is bounded on $[a, x]$ and hence $[(1-G)^{-1}]dh$ is of bounded variation on $[a, x]$, then from Theorem 1, there is a subdivision D_2 of $[a, x]$ such that if $D'=\{x_i\}_{i=0}^n$ is a refinement of D_2, then

$$\sum_{D'}\left|(R)\int_{x_{i-1}}^{x_i}[{}_t\prod^{x_i}(1+H)(1-G)^{-1}][(1-G)^{-1}]dh - [(1-G_i)^{-1}]dh_i\right| < \frac{\varepsilon}{3M}.$$

Furthermore, since ${}_a\prod^x(1+H)(1-G)^{-1}$ exists and is not zero, then from Lemma 1.7, there is a subdivision D_3 of $[a, x]$ such that if $D' = \{x_i\}_{i=0}^n$ is a refinement of D_3, then

$$\sum_{D'}|{}_{x_{i-1}}\prod^{x_i}(1+H)(1-G)^{-1} - (1+H_i)(1-G_i)^{-1}| < \frac{\varepsilon}{3M^2}.$$

If $D = D_1 + D_2 + D_3$ and $D' = \{x_i\}_{i=0}^n$ is a refinement of D, then, again using the iterative technique employed by B. W. Helton in [4, page 312], we have, for $0 < i \leqq n$,

$$\begin{aligned} f_i &= f_{i-1}\,{}_{x_{i-1}}\prod^{x_i}(1+H)(1-G)^{-1} + (R)\int_{x_{i-1}}^{x_i}[{}_t\prod^{x_i}(1+H)(1-G)^{-1}][(1-G)^{-1}]dh \\ &= f_{i-1}(1+H_i)(1-G_i)^{-1} + f_{i-1}[{}_{x_{i-1}}\prod^{x_i}(1+H)(1-G)^{-1} - (1+H_i)(1-G_i)^{-1}] \\ &\quad + dh_i(1-G_i)^{-1} \\ &\quad + (R)\int_{x_{i-1}}^{x_i}[{}_t\prod^{x_i}(1+H)(1-G)^{-1}](1-G)^{-1}dh - dh_i(1-G_i)^{-1}. \end{aligned}$$

By multiplying both sides of the preceding equation by $(1-G_i)$ and then rearranging terms, we have

$$\begin{aligned} f_i - f_{i-1} &= f_iG_i + f_{i-1}H_i + dh_i \\ &\quad + f_{i-1}[{}_{x_{i-1}}\prod^{x_i}(1+H)(1-G)^{-1} - (1+H_i)(1-G_i)^{-1}][1-G_i] \\ &\quad + (1-G_i)\Big[(R)\int_{x_{i-1}}^{x_i}[{}_t\prod^{x_i}(1+H)(1-G)^{-1}][(1-G)^{-1}]dh \\ &\quad - [(1-G)^{-1}]dh_i\Big]. \end{aligned} \tag{1}$$

Therefore,

$$\begin{aligned} &|f(x) - h(x) - \sum_{D'}(f_iG_i + f_{i-1}H_i)| \\ &\quad = |f(x) - f(a) + h(a) - h(x) - \sum_{D'}(f_iG_i + f_{i-1}H_i)| \\ &\quad = |\sum_{D'}(f_i - f_{i-1}) - \sum_{D'}dh_i - \sum_{D'}(f_iG_i + f_{i-1}H_i)| \\ &\quad \leqq |\sum_{D'}dh_i - \sum_{D'}dh_i| \\ &\qquad + \sum_{D'}|f_{i-1}|\cdot|1-G_i|\cdot|{}_{x_{i-1}}\prod^{x_i}(1+H)(1-G)^{-1} - (1+H_i)(1-G_i)^{-1}| \\ &\qquad + \sum_{D'}|1-G_i|\cdot\left|(R)\int_{x_{i-1}}^{x_i}[{}_t\prod^{x_i}(1+H)(1-G)^{-1}][(1-G)^{-1}]dh \right. \\ &\qquad \left. - (1-G_i)^{-1}dh_i\right| \qquad (1) \\ &\quad < 0 + M^2\sum_{D'}|{}_{x_{i-1}}\prod^{x_i}(1+H)(1-G)^{-1} - (1+H_i)(1-G_i)^{-1}| \end{aligned}$$

$$+ M \sum_{D'} \left| (R) \int_{x_{i-1}}^{x_i} [{}_t\prod^{x_i} (1+H)(1-G)^{-1}][(1-G)^{-1}]dh - (1-G)^{-1}dh_i \right|$$

$$< M^2 \frac{\varepsilon}{3M^2} + M \frac{\varepsilon}{3M}$$

$$< \varepsilon .$$

Hence, $(RL) \int_a^x (fG+fH)$ exists and $f(x) = h(x) + (RL) \int_a^x (fG+fH)$.

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SOUTHWEST TEXAS STATE UNIVERSITY
SAN MARCOS, TX 78666