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ON THE GENERALIZED CALKIN ALGEBRA

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A bounded linear operation $T: X \to Y$ between Banach spaces is said to be weakly compact if it takes bounded sequences onto sequences which have a weakly convergent subsequence. Let $W[X, Y]$ denote the weakly compact operators from X to Y, $B[X, Y]$, the bounded operators and $K[X, Y]$, the compact operators. Now $W[X, Y]$ forms a closed subalgebra of $B[X, Y]$ and for $X = Y$, $W[X, X]$ is a closed (in the uniform topology) two-sided ideal of $B[X,X]$. The purpose of this note is to construct a faithful representation of the Generalized Calkin Algebra $B[X, X]/K[X, X]$, which parallels a similar representation of $B[X, X]/K[X, X]$ in Buoni, Harte and Wickstead, "Upper and lower Fredholm spectra".

This construction in obtained is § 1 and some consequences in § 2 with regards to operators $T \in B[X, Y]$ with a reflexive null space, $N(T)$, and closed range, $R(T)$. Operators of this type have been studied by Yang. Throughout this note, the weak closure of a set S in X will be denoted by $\bar{S}^w$.

1. If X is a complex Banach space then let $l_\infty(X)$ denote the Banach space obtained from the space of all bounded sequences $x = (x_n)$ in X by imposing term-by-term linear combinations and the supremum norm $||x||_\infty = \sup_n ||x_n||$.

DEFINITION 1. If X is a Banach space then denote $m(X) = \{(x_n) \in l_\infty(X) \mid \overline{(x_n)}^w \text{ is weak-compact in } X\}$.

LEMMA 1. *If X is a Banach space then the following hold.*

(1) *$m(X)$ is a subspace of $l_\infty(X)$.*

(2) *a sequence $x = (x_n)$ is in $m(X)$ iff every subsequence of (x_n) has a weak convergent subsequence.*

Proof. (1) is clear and (2) is an immediate application of the Eberlein—Smulian theorem [2, p. 430] which states that for a subset A of X then $\bar{A}^w$ is weak-compact iff every sequence in A has a weakly convergent subsequence.

Let $\overline{m(X)}$ denote the norm closure of $m(X)$ in $l_\infty(X)$.

LEMMA 2. *Every subsequence of an element in $m(X)$ $(\overline{m(X)})$ is* *also in* $m(X)$ $(\overline{m(X)})$.

Proof. This follows immediately from Lemma 1 part 2.

THEOREM 3. *If* X *is a Banach space then* $m(X)$ *is a closed subspace of* $l_\infty(X)$.

Proof. Let $x = (x_n) \in \overline{m(X)}$, i.e., the closure of $m(X)$ in $l_\infty(X)$. It shall first be shown that (x_n) has a weak-Cauchy subsequence and then that this sequence converges to an element in X. Thus there exists $y_1 = (y_{1,n}) \in m(X)$ and $(x_{1,n})$, a subsequence of x such that $y_{1,n} \xrightarrow{w} y_1$ (converges weakly to y_1) and $\|(x_{1,n}) - (y_{1,n})\|_\infty < 1$. Now assume for $1 \leqq l \leqq j-1$, that we have $(x_{l,n})$ and $(y_{l,n})$ which satisfy the following:

$$\text{(1.1)} \quad \begin{aligned} &(1)\ (x_{l,n}) \text{ is a subsequence of } (x_{l-1,n}),\\ &(2)\ y_{l,n} \xrightarrow{w} y_l,\\ &(3)\ \|(x_{l,n}) - (y_{l,n})\|_\infty < 1/l. \end{aligned}$$

Then since $(x_{j-1,n}) \in m(X)$, there exists a subsequence $(x_{j,n})$ of $(x_{j-1,n})$ and there exists $(y_{j,n}) \in \overline{m(X)}$ such that $y_{j,n} \xrightarrow{w} y_j$ and $\|(x_{j,n}) - (y_{j,n})\|_\infty < 1/j$. So by induction, for all j, there exist sequences satisfying (1.1). Now fix j and $f \in X^*$ (the conjugate of X). We claim that there exists M such that for all n and $m \geqq M$ that

$$\text{(1.2)} \qquad |f(x_{j,n}) - f(x_{j,m})| \leqq 4\|f\|/j\ .$$

To see this, recall that $y_{j,n} \xrightarrow{w} y_j$, then there exists M such that

$$\text{(1.3)} \qquad |f(y_{j,n}) - f(y_j)| \leqq \|f\|/j \text{ for all } n > M\ .$$

Now for all $n, m \geqq M$,

$$\text{(1.4)} \quad \begin{aligned} |f(x_{j,n}) - f(x_{j,m})| &\leqq |f(x_{j,n}) - f(y_{j,n})| + |f(y_{j,n}) - f(y_j)| \\ &+ |f(y_j) - f(y_{j,m})| + |f(y_{j,m}) - f(x_{j,m})|\ . \end{aligned}$$

Now by applying (1.1) and (1.3) to (1.4) yields (1.2). We shall now show that $(x_{i,i})$ is a weak-Cauchy sequence. Given $f \varepsilon X^*$ and $\varepsilon > 0$, select j such that $4\|f\|/j < \varepsilon$. Then by (1.2) there exists M_0 such that for all m and $n \geqq M_0$

$$\text{(1.5)} \qquad |f(x_{j,n}) - f(x_{j,m})| \leqq 4\|f\|/j\ .$$

Set $M = \max(j, M_0)$. For m and $n \geqq M$, because $(x_{n,k})$ and $(x_{m,k})$ are subsequences of $(x_{j,k})$, then $|f(x_{n,n}) - f(x_{m,m})| \leqq 4\|f\|/j < \varepsilon$.

It remains to show that any weakly-Cauchy subsequence of

$(x_n) \in \overline{m(X)}$ converges weakly. To this end, let $(x_n) \in \overline{m(X)}$ be a weakly-Cauchy sequence.

Define $F: X^* \to \boldsymbol{C}$ by $F(f) = \lim_{n\to\infty} f(x_n)$. Since $|F(f)| \leqq \|f\| \sup_n \|x_n\|$, then $F \in X^{**}$. Now let $\varepsilon > 0$, it shall be shown that there exists $y \in X$ such that $\|F - y^{**}\| < \varepsilon$ where y^{**} is the canonical image of y in X^{**}. To see this, select $(y_n) \in m(X)$ such that $\|(x_n) - (y_n)\| < \varepsilon/3$ and select a subsequence (y_{n_k}) of (y_n) such that $y_{n_k} \xrightarrow{w} y$. Select $f \in X^*$ such that $\|f\| \leqq 1$. For k sufficiently large, $|f(y_{n_k}) - f(y)| < \varepsilon/3$ and $|f(x_{n_k}) - F(f)| < \varepsilon/3$. Thus, for k sufficiently large,

$$|F(f) - y^{**}(f)| \leqq |F(f) - f(x_{n_k})| + |f(x_{n_k}) - f(y_{n_k})| + |f(y_{n_k}) - f(y)| \leqq \frac{\varepsilon}{3} + \|f\|\,\|x_{n_k} - y_{n_k}\| + \frac{\varepsilon}{3} < \varepsilon .$$

Thus, F is in the norm closure of the canonical image of X in X^{**}. This image is norm closed; therefore, there exists $x \in X$ such that F is the canonical image of x. Thus, $m(X) = \overline{m(X)}$ which proves our theorem.

2. Now for $T \in B[X, Y]$ we have

LEMMA 4. (1) *If $T \in B[X, Y]$, then T sends $m(X)$ to $m(Y)$.*
(2) *T is weakly-compact iff T maps $l_\infty(X)$ into $m(Y)$.*

Proof. Clear.

Now for $T \in B[X, Y]$, let $P(T)$ be the induced operator from $l_\infty(X)/m(X) \to l_\infty(Y)/m(Y)$. Denote by $\mathscr{P}(X)$ the quotient $l_\infty(X)/m(X)$. Then $W \in W[X, Y]$ iff $P(W) = 0$. Therefore we have the following theorem.

THEOREM 5. *$B[\mathscr{P}(X), \mathscr{P}(X)]$ contains a faithful representation of $B[X, X]/W[X, X]$.*

THEOREM 6. *Let $T \in B[X, Y]$.*
(1) *If $N(T)$ is a reflexive subspace and is complemented in X and if $R(T)$ is closed then $P(T)$ is one-to-one.*
(2) *If $P(T)$ is one-to-one, then $N(T)$ is a reflexive subspace of X.*

Proof. To see (1) let $N(T)$ be a complemented reflexive subspace, then there exists a closed subspace M such that $X = N(T) \oplus M$. Since $R(T)$ is closed, then $T|M$(T restricted to M) is an isomorphism. Now let us assume that there exists a sequence (x_n) in $l_\infty(X)$ such that $P(T)(x_n + m(X)) = (Tx_n) + m(Y) = m(Y)$.

Let $x_n = k_n + z_n$ where $x_n \in N(T)$ and $z_n \in M$. Since there exist bounded projections onto $N(T)$ and M then (k_n) and (z_n) are in $l_\infty(X)$. Now (Tx_n) has a weakly-convergent subsequence, say (Tx_{n_j}). Thus (Tz_{n_j}) converges weakly and since $R(T) = T(M)$ is closed then $Tz_{n_j} \xrightarrow{w} Tz$ for some $z \in X$. Since T is invertible when restricted to M, thus, $z_{n_j} \xrightarrow{w} z$. Since $N(T)$ is a reflexive subspace, some subsequence of (k_{n_j}) converges weakly; (x_n) has a weakly convergent subsequence and $(x_n) \in m(X)$.

To see (2), we assume that $N(T)$ is not reflexive, then there exists a bounded sequence (x_n) in $N(T)$ with no convergent subsequence. Hence, $(x_n) \notin m(X)$ while $(Tx_n) \in m(Y)$; contradicting that $P(T)$ is one-to-one.

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