PACIFIC JOURNAL OF MATHEMATICS
Vol. 80, No. 2, 1979

ISOMORPHISMS AND SIMULTANEOUS EXTENSIONS IN $C(S)$

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Let h map a subspace A continuously into the completely regular space S so that A and $h[A]$ are completely separated in S, and let Q be the quotient space of S gotten by identifying p with $h(p)$ for all p in A. If there exists a simultaneous extension from $C(A)$ into $C(S)$, then there exists an isomorphism of $C(S)$ onto itself, taking $C(Q)$ onto $C(S \| A)$, which is the identity on $C(S \| h[A])$ (whence $C(Q)$ is complemented in $C(S)$). The converse holds providing A and $h[A]$ are normally embedded in S and h is a homeomorphism.

Introduction. Our main results, stated above, are Theorems 2.3 and 2.5 of §2. In §1 we establish these results in the special case when S is the topological sum of two disjoint spaces S_1 and S_2, where $A \subset S_1$ and $h[A] \subset S_2$. We also show, in Theorem 1.3, that isomorphisms of the above type always exist whenever h maps all of S_1 into S_2.

In §2 the results of §1 are extended to the general case by means of Lemma 2.1, which enables us to recover S in a natural way as a quotient space of a topological sum of two disjoint spaces.

Background. $C(S)$ denotes the Banach space of all bounded continuous real or complex valued functions on S with supremum norm.

Let A be a subspace of S. A is *normally embedded* in S if every element of $C(A)$ has a continuous extension to S. $C(S \| A)$ (resp. $C(S, A)$) denotes the subspace of all functions in $C(S)$ that are zero (resp. constant) on A and $\Lambda(A, S)$ denotes the set of all *simultaneous extensions* (bounded linear operators that extend functions) from $C(A)$ into $C(S)$. For further information on simultaneous extensions, see [2], [3] and the references therein.

Two Banach spaces X and Y are *isomorphic* (write $X \sim Y$) if there exists a one-to-one bicontinuous linear operator from X onto Y. A subspace Z of X is *complemented* in X if there exists a projection of X onto Z in the sense of [6, p. 480].

βN denotes the Stone-Čech compactification of the integers.

1. Throughout this section $S_1 \oplus S_2$ denotes the topological sum (free union) of the disjoint completely regular spaces S_1 and S_2, A a subspace of S_1, h a continuous function from A into S_2, and Q the

quotient space of $S_1 \oplus S_2$ gotten by identifying p with $h(p)$ for each p in A. $C(Q)$ is identified with the space

$$\{f_1 \oplus f_2 \in C(S_1 \oplus S_2): f_1(p) = f_2(h(p)) \forall p \in A\} .$$

We state first an elementary result which serves as motivation for what follows.

PROPOSITION 1.1. *If A is normally embedded in S_1, then*

$$\frac{C(S_1 \oplus S_2)}{C(Q)} \sim C(A) .$$

Now, as is well known, it is also true that

$$\frac{C(S_1 \oplus S_2)}{C(S_1 \oplus S_2 || A)} \sim C(A)$$

when A is normally embedded in S_1. Hence the two quotient spaces are isomorphic and it is natural to ask whether there exists an isomorphism of $C(S_1 \oplus S_2)$ onto itself which takes $C(Q)$ onto $C(S_1 \oplus S_2 || A)$ (a stronger statement). As the next example shows, this is false in general. Theorems 1.3 and 1.6 below, however, give sufficient conditions for the existence of such an isomorphism. Note that whenever this isomorphism does exist it follows that $C(Q)$ is complemented in $C(S_1 \oplus S_2)$ if and only if the same is true for $C(S_1 \oplus S_2 || A)$.

EXAMPLE 1.2. *If $S_1 = \beta N$, $A = \beta N - N$, S_2 is a disjoint copy of $\beta N - N$, and h any homeomorphism from A onto S_2, then*
(a) $\Lambda(A, S_1) = \varnothing$
(b) $C(Q)$ is complemented in $C(S_1 \oplus S_2)$
(c) $C(Q) \not\sim C(S_1 \oplus S_2 || A)$.

Proof. (b) is obvious. Suppose

$$C(Q) \sim C(S_1 \oplus S_2 || A) .$$

Then

$$\begin{aligned} C(\beta N) = C(Q) &\sim C(S_1 \oplus S_2 || A) \\ &\sim C(\beta N || \beta N - N) \times C(\beta N - N) , \end{aligned}$$

contradicting a classical result of Phillips [7]. Hence (c) holds.

As is well known, (a) also follows from the aforementioned theorem of Phillips.

THEOREM 1.3. *If h maps all of S_1 continuously into S_2, then* *there exists an isomorphism of* $C(S_1 \oplus S_2)$ *onto itself taking* $C(Q)$ *onto* $C(S_1 \oplus S_2 \| A)$.

Proof. It can be verified in a straightforward manner that the formula

$$T(f_1 \oplus f_2) = (f_1 - f_2 \circ h) \oplus f_2 ,$$
$$f_1 \oplus f_2 \in C(S_1 \oplus S_2) ,$$

defines the required isomorphism.

Note that A need not be normally embedded in S_1 in the above. When it is, however, we can state the following:

COROLLARY 1.4. *If h maps S_1 continuously into S_2 and A is normally embedded in S_1, then the following are equivalent*:
 (a) $\Lambda(A, S_1) \neq \emptyset$
 (b) $\Lambda(A, S_1 \oplus S_2) \neq \emptyset$
 (c) $C(S_1 \| A)$ *is complemented in* $C(S_1)$
 (d) $C(S_1 \oplus S_2 \| A)$ *is complemented in* $C(S_1 \oplus S_2)$
 (e) $C(Q)$ *is complemented in* $C(S_1 \oplus S_2)$.

Proof. The equivalence of (a) with (c) and (b) with (d) follows from [4]. The equivalence of (a) with (b) is obvious and that of (d) with (e) is implied by Theorem 1.3.

In the special case where S_2 is a one point set the quotient space Q can be alternately constructed from S_1 alone by squeezing A to a point [5, p. 128] and $C(Q)$ can be identified with $C(S_1, A)$. Since $C(S_1 \| A)$ is a complemented subspace of $C(S_1, A)$, the following is an immediate consequence of Corollary 1.4.

COROLLARY 1.5. *Let A be normally embedded in S and Q the quotient space of S gotten by squeezing A to a point. Then $C(Q)$ is complemented in $C(S)$ if and only if $\Lambda(A, S) \neq \emptyset$.*

THEOREM 1.6. *If $\Lambda(A, S_1) \neq \emptyset$, then there exists an isomorphism of $C(S_1 \oplus S_2)$ onto itself taking $C(Q)$ onto $C(S_1 \oplus S_2 \| A)$.*

Proof. If $E \in \Lambda(A, S_1)$ it is easily verified that the formula

$$T(f_1 \oplus f_2) = [f_1 - E(f_2 \circ h)] \oplus f_2 ,$$
$$f_1 \oplus f_2 \in C(S_1 \oplus S_2) ,$$

defines the required isomorphism.

Note, as Example 1.2 shows, that the hypothesis $\Lambda(A, S_1) \neq \emptyset$ is necessary in the above.

It follows from Theorem 1.6 that $C(Q)$ is complemented in $C(S_1 \oplus S_2)$ whenever $\Lambda(A, S_1) \neq \emptyset$. This was essentially proven by D. Amir in [1].

As the next example shows, the converse of Theorem 1.6 is false even when h is a homeomorphism.

EXAMPLE 1.7. Let S_1 and S_2 be disjoint copies of βN, h any homeomorphism of S_1 onto S_2, $A = \beta N - N$. Then

(a) $\Lambda(A, S_1) = \emptyset$

(b) there exists an isomorphism of $C(S_1 \oplus S_2)$ onto itself taking $C(Q)$ onto $C(S_1 \oplus S_2 || A)$

(c) $C(Q)$ is not completed in $C(S_1 \oplus S_2)$.

Proof. (b) follows from Theorem 1.3 and (c) from Corollary 1.4.

Under the additional assumption that h is a homeomorphism we are able to characterize the condition $\Lambda(A, S_1) \neq \emptyset$ in terms of the existence of isomorphisms of the above type. We shall use the following:

LEMMA 1.8. *Let A and $h[A]$ be normally embedded in S_1 and S_2, respectively. If h is a homeomorphism, then $\Lambda(A, S_1) \neq \emptyset$ whenever there exists a bounded linear operator from $C(Q)$ into $C(S_1 \oplus S_2 || h[A])$ which is the identity on $C(Q) \cap C(S_1 \oplus C(S_2 || h[A])$.*

Proof. For any $f \in C(A)$, let f_1 be an extension of f to $C(S_1)$ and f_2 an extension of $f \circ h^{-1}$ to $C(S_2)$. Clearly, $f_1 \oplus f_2 \in C(Q)$. If T is a linear operator satisfying the hypothesis, define E by the formula

$$Ef = f_1 - T(f_1 \oplus f_2) | S_1 .$$

We show first that Ef is independent of the choice of f_1 and f_2. If g_1 and g_2 are any corresponding extensions, then $f_2 - g_2 \in C(S_2 || h[A])$. Hence by hypothesis,

$$\begin{aligned} f_1 - g_1 &= T[(f_1 - g_1) \oplus (f_2 - g_2)] | S_1 \\ &= T(f_1 \oplus f_2) | S_1 - T(g_1 \oplus g_2) | S_1 , \end{aligned}$$

so E is well defined.

Now, E is clearly a bounded linear operator from $C(A)$ into $C(S)$ and $Ef | A = f$ since $T(f_1 \oplus f_2) \in C(S_1 \oplus S_2 || A)$. Therefore, $E \in \Lambda(A, S_1)$.

THEOREM 1.9. *Let A and $h[A]$ be normally embedded in S_1 and S_2, respectively, and h a homeomorphism. Then $\Lambda(A, S_1) \neq \emptyset$ if and only if there exists an isomorphism of $C(S_1 \oplus S_2)$ onto itself, taking $C(Q)$ onto $C(S_1 \oplus S_2 \| A)$, which is the identity on $C(S_1 \oplus S_2 \| h[A])$.*

Proof. In light of Lemma 1.8, it remains to show that if $E \in \Lambda(A, S_1)$, then the isomorphism T defined in Theorem 1.6 is the identity on $C(S_1 \oplus S_2 \| h[A])$. Suppose $f_2 \in C(S_2 \| h[A])$. Then $f_2 \circ h \in C(S_1 \| A)$ and $E(f_2 \circ h) = 0$. Hence, for any $f_1 \in C(S_1)$,

$$\begin{aligned} T(f_1 \oplus f_2) &= [f_1 - E(f_2 \circ h)] \oplus f_2 \\ &= f_1 \oplus f_2 . \end{aligned}$$

2. Throughout this section A denotes a subspace of the completely regular space S, h is a continuous function from A into $S - A$, and Q is the quotient space of S gotten by identifying p with $h(p)$ for each $p \in A$. We identify $C(Q)$ as a subspace of $C(S)$ in the natural way.

LEMMA 2.1. *Let $S = S_1 \cup S_2$, where S_1 and S_2 need not be disjoint. If S_1 and S_2 have their relative topologies, S' is the quotient space of $S_1 \oplus S_2$ gotten by identifying each point of $S_1 \cap S_2$ in S_1 with the corresponding point in S_2, and f is the natural mapping which takes S one-to-one and onto S', then*

(a) *f is open. (Whence f is a homeomorphism if S' is compact.)*

(b) *f is a homeomorphism whenever S_1 and S_2 are both closed (or both open) in S.*

Proof. For any subset $B \subset S_1 \cup S_2$, define $r[B] \subset S_1 \oplus S_2$ by $r[B] = (B \cap S_1) \oplus (B \cap S_2)$. Note that $r[B]$ is always q-saturated (i.e., $q^{-1}[q[r[B]]] = r[B]$), where q is the quotient mapping of $S_1 \oplus S_2$ onto S'.

Now, if U is any open subset of $S_1 \cup S_2$, let $V_i = U \cap S_i$, $i = 1, 2$. Then $r[U] = V_1 \oplus V_2$, which is an open subset of $S_1 \oplus S_2$, and $f[U] = q[r[U]] = q[V_1 \oplus V_2]$. Therefore, $q^{-1}[f[U]] = V_1 \oplus V_2$, and it follows that $f[U]$ is open. Hence (a) holds.

To prove (b), let C be any closed (resp. open) subset of Q and define $C_i = S_i \cap q^{-1}[C]$, $i = 1, 2$. The C_i are closed (resp. open) in S_i and $q^{-1}[C] = C_1 \oplus C_2$, which is q-saturated. It follows that $f^{-1}[C] = C_1 \cup C_2$, and since the S_i are closed (resp. open) in $S_1 \cup S_2$, $C_1 \cup C_2$ is closed (resp. open) in $S_1 \cup S_2$. Therefore, f is continuous.

THEOREM 2.2. *Suppose $S = S_1 \cup S_2$, where S_1 and S_2 are both closed* *(or both open) in S, $A \subset S_1$, $h[A] \subset S_2$, and A and $S_1 \cap S_2$ are completely separated in S_1. If $\Lambda(A, S) \neq \varnothing$, then there exists an isomorphism of $C(S)$ onto itself taking $C(Q)$ onto $C(S||A)$.*

Proof. By Lemma 2.1, we may identify $C(S)$ with $\{f_1 \oplus f_2 \in C(S_1 + S_2): f_1(p) = f_2(p)$, for and $p \in S_1 \cap S_2\}$, and $C(Q)$ with $\{f_1 \oplus f_2 \in C(S): f_1(a) = f_2(h(a))$, for all $a \in A\}$.

Since A and $S_1 \cap S_2$ are completely separated in S_1 and $\Lambda(A, S_1) \neq \varnothing$, there exists $E \in \Lambda(A, S_1)$ which takes $C(A)$ into $C(S_1 || S_1 \cap S_2)$. By Theorem 1.6, the operator T defined by

$$T(f_1 \oplus f_2) = [f_1 - E(f_2 \circ h)] \oplus f_2 ,$$

$$f_1 \oplus f_2 \in C(S_1 \oplus S_2) ,$$

is an isomorphism of $C(S_1 \oplus S_2)$ onto itself, taking $X = \{f_1 \oplus f_2 \in C(S_1 \oplus S_2): f_1(a) = f_2(h(a))$, for all $a \in A\}$ onto $C(S_1 \oplus S_2 || A)$. Since it is easily verified that T takes $C(S)$ onto itself, it follows that T takes $X \cap C(S) = C(Q)$ onto $C(S_1 \oplus S_2 || A) \cap C(S) = C(S||A)$, and the theorem is proved.

THEOREM 2.3. *If A and $h[A]$ are completely separated in S and $\Lambda(A, S) \neq \varnothing$, then there exists an isomorphism of $C(S)$ onto itself taking $C(Q)$ onto $C(S||A)$. (Whence $C(Q)$ is complemented in $C(S)$.)*

Proof. By hypothesis, there exists $g \in C(S)$ such that $g = 0$ on A and $g = 1$ on $h[A]$. Define

$$S_1 = \{p \in S: g(p) \leqq 1/2\} ,$$

$$S_2 = \{p \in S: g(p) \geqq 1/2\} .$$

S_1 and S_2 are closed subspaces of S satisfying all the hypotheses of Theorem 2.2, from which the result follows.

When h is a homeomorphism, we can state the following generalization of Lemma 1.8.

LEMMA 2.4. *Suppose h is a homeomorphism and A and $h[A]$ are normally embedded and completely separated in S. Then $\Lambda(A, S) \neq \varnothing$ whenever there exists a bounded linear operator from $C(Q)$ into $C(S||A)$ which is the identity on $C(Q) \cap C(S||h[A])$.*

Proof. Let $f \in C(A)$. Then $f \circ h^{-1} \in C(h[A])$. Since $A \cup h[A]$ is normally embedded in S, we may choose $g \in C(S)$ such that $g|A = f$ and $g|h[A] = f \circ h^{-1}$. Clearly, $g \in C(Q)$.

If T is the hypothesized linear operator and E is defined by the formula

$$Ef = g - Tg , \qquad f \in C(A) ,$$

it can be shown as in Lemma 1.8 that $E \in \Lambda(A, S)$

THEOREM 2.5. *Suppose h is a homeomorphism and $h[A]$ and A are normally embedded and completely separated in S. Then $\Lambda(A, S) \neq \emptyset$ if and only if there exists an isomorphism of $C(S)$ onto itself, taking $C(Q)$ onto $C(S \| A)$, which is the identity on $C(S \| h[A])$.*

Proof. If $\Lambda(A, S) \neq \emptyset$, it is easily verified that the isomorphism of Theorem 2.3 is the identity on $C(S \| h[A])$. The converse follows directly from Lemma 2.4.

Note that the above hypothesis is satisfied whenever S is compact and A is closed in S (and h is a homeomorphism).

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Received December 15, 1976.

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