PACIFIC IOURNAL OF MATHEMATICS
Vol. 81, No. 1, 1979

A CYCLIC INEQUALITY AND A RELATED EIGENVALUE PROBLEM

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A cyclic sum $S(\underline{x})=\Sigma x_i/(x_{i+1}+x_{i+2})$ is formed with the N components of a vector $\underline{x}$, where $x_{N+1}=x_1$, $x_{N+2}=x_2$, and *where all denominators are positive and all numerators non-negative.* It is known that the inequality $S(\underline{x})\geqq N/2$ does not hold for even $N\geqq 14$; this result is derived in a uniform manner by considering a related algebraic eigenvalue problem. Numerical evidence is presented for the conjecture that this cyclic inequality is true for even $N\leqq 12$ and odd $N\leqq 23$.

The corresponding cyclic inequality, namely the question for what value of N

$$S(\underline{x}) \geqq N/2$$

holds, has been investigated by many mathematicians (cf. Mitrinović [7] and the references given there). In §1 we prove in a unified manner that the inequality does not hold for even $N \geqq 14$. The method is based on the idea used first by Lighthill for $N = 20$ [4] and then by several other authors. The argument indicates why the case $N = 12$ remains still unresolved. Some properties of this type of solution are described in §2. Section 3 deals with numerical results that strongly suggest that the inequality is valid for $N = 12$ and, if N is odd, for $N = 23$. These numerical results definitely represent stationary values of the cyclic sum, and we are inclined to believe that they are indeed global minima. A connection between the inequality above and a related inequality with indices reversed is considered in the last section. In the Appendix some examples are listed for $N = 14, 25$ and 27.

1. *The linear cyclic inequality.* By considering the cyclic sum $S(\underline{x})$ it is obvious that for any N there exists a vector for which

$$S(\underline{x}) = N/2$$

holds, namely $x_i = 1$ for $i = 1, 2, \cdots, N$. If N is even, there exists also a wider class of "nominal" vectors,

$$x_i^0 = \begin{cases}(1+\alpha)/2 & \text{for } i \text{ odd} \\ (1-\alpha)/2 & \text{for } i \text{ even}\end{cases} \quad 0 \leqq \alpha \leqq 1\,, \tag{1.1}$$

for which $S(\underline{x}^0) = N/2$. Vectors of this type seem to form the basis in the reported solutions for even N where the inequality does not hold, in particular, in Zulauf's solution [7, p. 133] for the important case $N = 14$.

If N is odd, the situation is much more difficult to understand. Indeed, while only $N = 12$ is unresolved for even N, for odd N the answer is still unknown for $N = 11, 13, \cdots, 23$. A simple nominal vector of the form (1.1) exists for odd N only if $\alpha = 0$.

We now show in a uniform manner that the cyclic inequality is violated for even $N \geqq 14$. (In the remainder of this section, N is understood to be even.) We proceed by writing the vector $\underline{x}$ as $\underline{x} = \underline{x}^0 + \underline{e}$ and expanding the cyclic sum $S(\underline{x})$ in terms of the components of the vector $\underline{e}$. If S can be made smaller than $N/2$ for small $\underline{e}$, the inequality is clearly violated.

By including quadratic terms in the expansion—the contribution of the linear terms vanishes—we obtain

$$S^* = N/2 + \sum e_k^2 - e_k e_{k+2} + (-1)^k \alpha e_k e_{k+1} = N/2 + \underline{e}^T A \underline{e}/2$$

where again $e_{N+1} = e_1$, $e_{N+2} = e_2$ and where A is the symmetric matrix

$$A = \begin{pmatrix} 2 & -\alpha & -1 & & & & & -1 & \alpha \\ -\alpha & 2 & \alpha & -1 & & & & & -1 \\ -1 & \alpha & 2 & -\alpha & -1 & & & & \\ & & & - - - & & & & & \\ & & & & -1 & -\alpha & 2 & \alpha & -1 \\ -1 & & & & & -1 & \alpha & 2 & -\alpha \\ \alpha & -1 & & & & & -1 & -\alpha & 2 \end{pmatrix}.$$

In order to minimize S^* we must minimize $\underline{e}^T A \underline{e}$ with $\underline{e}^T \underline{e}$ kept constant. The corresponding eigenvalue problem $(A - \lambda I)\underline{e} = \underline{0}$ has the known solution, which can be easily verified,

$$e_k = \begin{cases} a \sin t_k & \text{for } k \text{ odd} \\ -a \cos t_k & \text{for } k \text{ even} \end{cases} \tag{1.2}$$

where $t_k = t_0 + (k-1)h$; the amplitude $a > 0$ and the phase t_0 are arbitrary, and

$$h = 2\pi j/N\,, \qquad j = 1, 2, \cdots, N\,.$$

The N corresponding eigenvalues are

$$\lambda = 2 \sin h\,(2 \sin h - \alpha)\ ;$$

they are, with the exception of at most two of them, all double eigenvalues. We may choose $t_0 = 0$ so that the $\underset{\sim}{e}$-vector becomes

$$\underset{\sim}{e} = a(0, -\cos h, \sin 2h, -\cos 3h, \cdots, \sin(N-2)h, -\cos(N-1)h) .$$

Now, at the stationary values of S^* we have

$$S^* = N/2 + \lambda \underset{\sim}{e}^T \underset{\sim}{e}/2 .$$

Hence, S^* is smaller than $N/2$ if there exists at least one negative eigenvalue λ. This means that we must require that $0 < 2\sin h < \alpha < 1$, i.e., $0 < \sin(2\pi j/N) < 1/2$, $2\pi j/N < \pi/6$, or finally $N > 12j$. The case where $5\pi/6 < 2\pi j/N < \pi$ can be excluded since it leads to the indentical result for $\underset{\sim}{x}$ and S^*. For $N > 12$, the condition $N > 12j$ can indeed always be satisfied. We conclude that vectors of this kind with $S^* < N/2$, and therefore also for the full cyclic inequality with $S < N/2$, are always possible for $N \geqq 14$, but not possible for $N \leqq 12$ (cf. also [10]). This concludes the main argument.

However, these considerations do not resolve the open case $N = 12$. The inequality holds in the neighborhood of a nominal vector $\underset{\sim}{x}_0$. Consequently, if a vector $\underset{\sim}{x}$ exists that violates the inequality, then it cannot be obtained by a perturbation of a nominal vector $\underset{\sim}{x}^0$.

2. **The minimum of the linear cyclic sum.** It seems worthwhile to elaborate on the vectors formed with (1.2) and add a few remarks.

First, we note that $\lambda = 4\sin^2 h \geqq 0$ for $\alpha = 0$. This means that for odd N, where the only simple nominal vector $\underset{\sim}{x}^0$ is furnished by $\alpha = 0$, the eigenvalues are all nonnegative, so that the argument given above cannot be applied to odd N. Furthermore, higher order terms in the $\underset{\sim}{e}$-expansion do not alter this conclusion.

For $N \geqq 14$ there exists a negative eigenvalue, namely exactly one for $14 \leqq N \leqq 24$. If $24 < N \leqq 36$ both $j = 1$ and $j = 2$ furnish negative eigenvalues, and similarly for larger N values, where for each increase of N by 12 a "higher harmonic" is added. The Figure 1 shows the eigenvectors for $N = 26$, $j = 1$ and $j = 2$. The values of the full (i.e., not linearized) cyclic sum for these vectors are $S = 13\text{-}0.01913$ and $S = 13\text{-}0.0000787$.

Since all x_k are required to be nonnegative, the amplitude a must be chosen sufficiently small, namely

$$a \leqq (1 - \alpha)/2 . \tag{1.3}$$

In some cases, a can be chosen slightly larger, e.g., for $N = 14$

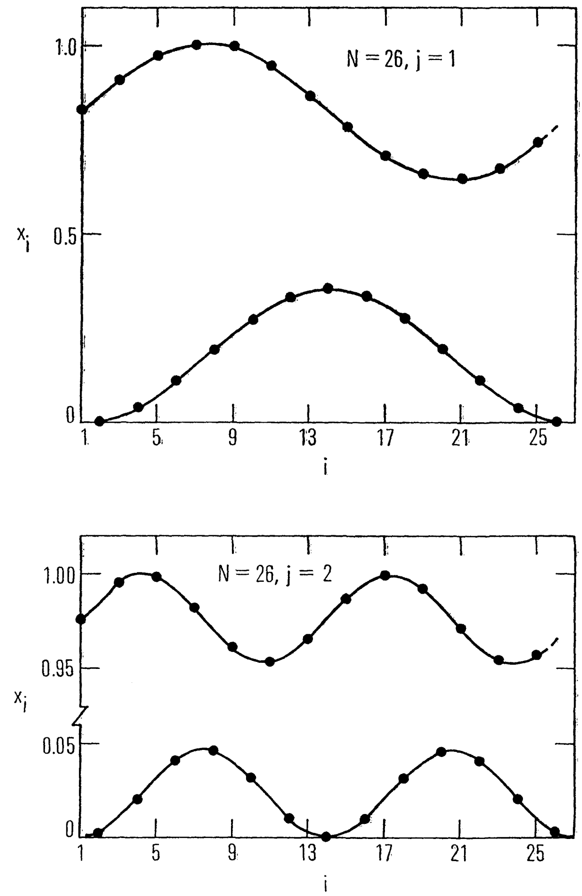

FIGURE 1. Eigenvectors for N=26, j=1,2.

and $j = 1$,

$$a \leqq (1 - \alpha)/2 \cos h \ , \tag{1.4}$$

since the trigonometric functions in (1.2) are evaluated only at discrete points.

The sum S^* is computable in closed form and gives, for the cases of interest,

$$S^* = N(2 + \lambda a^2)/4$$

or, using the (nearly) largest admissible a,

$$S^*(\alpha) = N\Big(2 - \frac{1}{2}(1 - \alpha)^2 \sin h(\alpha - 2 \sin h)\Big)\Big/4 \ .$$

For $\alpha = 1$ and $\alpha = 2 \sin h$, we obtain $S^* = N/2$, and S^* attains its minimum value (for either (1.3) or (1.4)) at

$$\alpha_0 = (1 + 4 \sin h)/3 \ ,$$

namely

$$S^* = N\left(1 - \frac{1}{27}\sin h(1 - 2\sin h)^3\right)\Big/2 . \tag{1.5}$$

The linearized sum S^* has of course a different minimum than the full cyclic sum. As an example, we choose $N = 14$, $j = 1$. From (1.5) we obtain for $a = (1 - \alpha)/2$

$$S^* = 7 - 0.000260 ,$$

and it can be shown that for $a = (1 - \alpha)/2 \cos h$ (1.5) gives

$$S^* = 7 - 0.000320 ,$$

while the full cyclic sum for this vector is

$$S = 7 - 0.000323 .$$

On the other hand, a numerical minimization of the full cyclic sum furnishes

$$S = 7 - 0.000347 .$$

It is not difficult to include the cubic terms in the $\underset{\sim}{e}$-expansion. It turns out that in order to obtain this sum, let us call it S^{**}, one only needs to increase the amplitude a. However, the amplitude is in general restricted to $a \leqq (1 - \alpha)/2$. Hence, it seems reasonable to increase a, except that those x_k which would become negative are replaced by zero. A computation then leads to the result

$$S^{**} = 7 - 0.000331 .$$

One might expect that for large N where more than one negative eigenvalue occurs, the eigenvalue for $j = 1$ would give the smallest sum S^*. However, (1.5) shows that for $N \geqq 74$ this is not the case.

3. **The cases $N = 12$ and $N = 23$.** By considering the numerical minimization for $N \geqq 14$ (cf. Figure 2 and Table 1) we are led to the conjecture that for the still open case $N = 12$ the inequality is indeed satisfied. But it should be kept in mind that these numerical results have not been shown to be global minima.

Similarly, for N odd and larger than 23, the numerical results indicate that the inequality is valid for $N = 23$. Here the solution for $N = 23$ which is similar in structure to the solutions for $N \geqq 25$ is also listed, although in this case the vector $x_k = 1$, for all k, furnishes the lower value $N/2$. The same conclusion has been reached by Malcolm [6] who solved the problem for $N = 25$ by

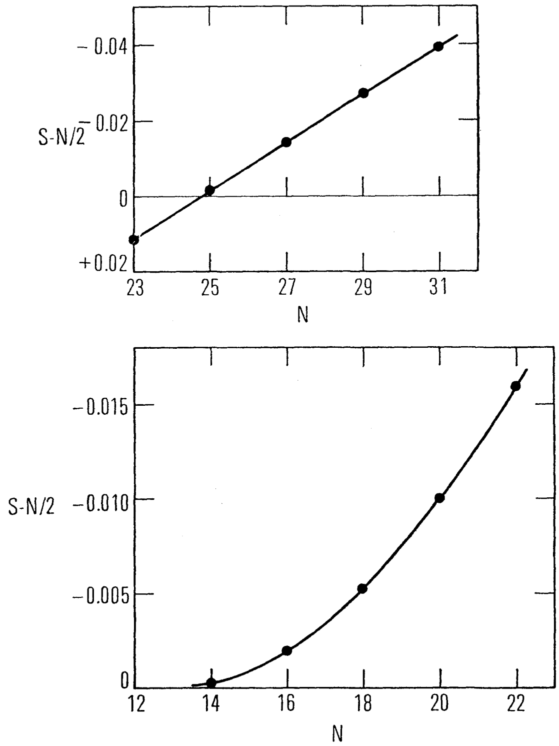

FIGURE 2. Extrapolation of the minimum cyclic sum to $N=12$ and $N=23$.

TABLE 1
Extrapolation of the minimum of the cyclic sum S to $N=12$ and $N=23$.

N	$S-N/2$	N	$S-N/2$
14	−.000347303	23	+.011689438
16	−.002004523	25	−.001514765
18	−.005287982	27	−.014469580
20	−.010062465	29	−.027056111
22	−.015979281	31	−.039127154

convincing numerical minimization and by Daykin [1] who also lists a solution in integer values for the x_i.

Additional numerical results are discussed in the Appendix.

4. The cyclic inequality with indices reversed. The solutions listed above exhibit an interesting general property. We define a vector $\underline{b}$ by setting

(4.1a) $$b_i = x_i/(x_{i+1} + x_{i+2})^2$$

and introduce also

(4.2a) $$r_i = b_i/(b_{i-1} + b_{i-2})$$

as a counterpart to

$$s_i = x_i/(x_{i+1} + x_{i+2}) \; . \tag{4.2b}$$

At the stationary values of $S(\underset{\sim}{x})$ for admissible vectors $\underset{\sim}{x}$, either $x_i = 0$ or $\partial S/\partial x_i = 0$. This leads readily to the relations that either

$$(x_{i+1} + x_{i+2})(b_{i-1} + b_{i-2}) = 1 \text{ or } x_i = b_i = 0 \; ,$$

and hence,

$$x_i = b_i/(b_{i-1} + b_{i-2})^2 \; , \tag{4.1b}$$

$$r_i = b_i(x_{i+1} + x_{i+2}) = x_i(b_{i-1} + b_{i-2}) = s_i$$

and

$$x_i b_i = s_i^2 = r_i^2$$

for all i.

Clearly then, for any stationary solution $\underset{\sim}{x}^{(1)}$ another stationary solution $\underset{\sim}{x}^{(2)}$ can be formed, namely the vector $\underset{\sim}{b}$ read in reverse order. Both solutions lead to the same stationary sum $S = \Sigma s_i = \Sigma r_i$. Therefore, if the minimum of S is unique, the two vectors must be equivalent, i.e., $\underset{\sim}{x}^{(2)}$ must be constant multiple of $\underset{\sim}{x}^{(1)}$. The computation of many minima for both even and odd N showed that in all cases indeed, $\underset{\sim}{x}^{(2)} = c\underset{\sim}{x}^{(1)}$. As an example we list in the Appendix, Table 4, the results for $N = 25$ where $\underset{\sim}{x}^{(1)}$ has been normalized so that $c = 1$, i.e., $b_i = x_{N+2-i}$ and $s_i = s_{N+2-i}$.

This means that for all computed minima (including the result in [6]) the vector $\underset{\sim}{s}$ exhibits a symmetry, and it might be of interest to prove this property, if indeed it holds in general.

Since the difficult cases where the cyclic inequality holds, namely $N = 8$ [3] and $N = 10$ [8], have been proved by discussing all relevant possibilities in turn, the symmetry in $\underset{\sim}{s}$ might just restrict the number of cases sufficiently to make $N = 12$ amenable to a proof.

Appendix. Miscellaneous numerical results. In this appendix we present examples and computational results for the cyclic inequality.

The approach described in § 1 enables us to obtain vectors $\underset{\sim}{x}$ for which $S(\underset{\sim}{x}) < N/2$ without requiring an extensive search on a computer. In Table 2 we present the results for the vector $\underset{\sim}{x}_Z$ [7, p. 133], $\underset{\sim}{x}_H$ [5], and the vector $\underset{\sim}{x}$ suggested by (1.2). For the expansion for small e, one obtains $S(\underset{\sim}{x}) = N/2 - qe^2 + 0(e^3)$. The minimum of the cyclic sum for these vectors is also listed; the comparison

TABLE 2
Vectors $\underline{x}$ with $S(\underline{x})<N/2$ for small e. $N=14$.

$\underline{x}_Z=$	$(1+7e,\ 7e,\ 1+4e,\ 6e,\ 1+e,\ 5e,\ 1,\ 2e,\ 1+e,\ 0,\ 1+4e,\ e,\ 1+6e,\ 4e)$
$\underline{x}_H=$	$(1+10e,\ 7e,\ 1+8e,\ 10e,\ 1+3e,\ 10e,\ 1-2e,\ 5e,\ 1-2e,\ 0,\ 1,\ 0,\ 1+8e,\ 3e)$
$\underline{x}=$	$(1+11e,\ 8e,\ 1+8e,\ 10e,\ 1+3e,\ 8e,\ 1,\ 3e,\ 1+2e,\ 0,\ 1+6e,\ 0,\ 1+10e,\ 4e)$

vector	q	minimum of $S-N/2$	at $e=$
$\underline{x}_Z$	2	-0.0000215	0.0059
$\underline{x}_H$	3	-0.0000028	0.0017
$\underline{x}$	11	-0.0002661	0.0093

between $\underline{x}_Z$ and $\underline{x}_H$ shows that a larger q need not lead to a smaller minimum.

The expansion in small e is not available for odd N. Convincing examples for $S(\underline{x}) < N/2$ are then furnished by vectors with nonnegative integers as components. Table 3 lists examples for $N = 14, 25, 27$. Clearly, there is a limit on how small the largest integer component can be chosen. We believe that the examples are quite close to optimal in this respect. The vector $\underline{x}_D$ for $N =$

TABLE 3
Vectors $\underline{x}$ with integer components and $S(\underline{x})<N/2$.

$\underline{x}_1=$	(0, 42, 2, 42, 4, 41, 5, 39, 4, 38, 2, 38, 0, 40)
$\underline{x}_2=$	(0, 44, 2, 44, 4, 43, 5, 41, 4, 40, 2, 40, 0, 42)
$\underline{x}_D=$	(3, 6, 2, 6, 1, 6, 0, 7, 0, 8, 0, 9, 0, 10, 0, 11, 1, 12, 3, 11, 5, 9, 6, 7, 6, 5, 6)
$\underline{x}_3=$	(3, 5, 2, 5, 1, 5, 0, 6, 0, 7, 0, 8, 0, 9, 0, 10, 1, 11, 3, 10, 5, 8, 5, 6, 5, 4, 5)

vector	N	Largest x_i	$S-N/2$
$\underline{x}_1$	14	42	$-151/28938140=-0.00000522$
$\underline{x}_2$	14	44	$-217/\ 4280760=-0.00005069$
Table 4, $\underline{x}_{\text{int}}$	25	35	$=-0.00013752$
$\underline{x}_{\text{int}}^*$	25	35	$-691\ /80013480=-0.00000863$
$\underline{x}_D$	27	12	$-53/\ 55440\ =-0.00095599$
$\underline{x}_3$	27	11	$-8/\ 3465=-0.00230880$
$\underline{x}_3^*$	27	11	$-1/\ 126=-0.00079365$

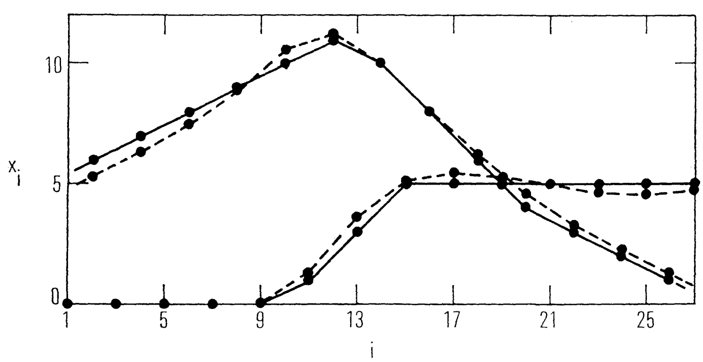

FIGURE 3. The numerical minimization of S. - - - -, and an example with integer components x_i ●—● for $N=27$.

TABLE 4
The numerical minimization of $S(x)$ for $N=25$ and a case $\underset{\sim}{x}_{\text{int}}$ with integer components.

	s	x_{int}
$x_1 = b_1 = .8448196$	.8448196	25
$x_2 = b_{25} = .0$	.0	0
$x_3 = b_{24} = 1.0$	.8448196	29
$x_4 = b_{23} = .0$	.0	0
$x_5 = b_{22} = 1.1836847$	.8448196	34
$x_6 = b_{21} = .1924932$	.1160666	5
$x_7 = b_{20} = 1.2086162$	.8133369	35
$x_8 = b_{19} = .4498554$	.2777040	13
$x_9 = b_{18} = 1.0361416$	.7447432	30
$x_{10} = b_{17} = .5837685$	.4125654	17
$x_{11} = b_{16} = .8075051$	.6676996	24
$x_{12} = b_{15} = .6074671$	.5125019	18
$x_{13} = b_{14} = .6019168$	.5925761	18
$x_{14} = b_{13} = .5833803$	.5925761	17
$x_{15} = b_{12} = .4323827$	.5125019	13
$x_{16} = b_{11} = .5520990$	.6676996	16
$x_{17} = b_{10} = .2915714$	.4125654	9
$x_{18} = b_9 = .5352959$	.7447432	16
$x_{19} = b_8 = .1714317$	.2777040	5
$x_{20} = b_7 = .5473341$	.8133369	16
$x_{21} = b_6 = .0699841$	.1160666	2
$x_{22} = b_5 = .6029648$	.8448196	18
$x_{23} = b_4 = .0$	.0	0
$x_{24} = b_3 = .7137202$	.8448196	21
$x_{25} = b_2 = .0$	.0	0

$S(x) = 12.498485$

27 is published in [2], and the vector $\underset{\sim}{x}_{\text{int}}$ is a slight modification of the vector given in [9] (the authors were unaware of the results in [1] and [6]) and is listed in Table 4. The vector $\underset{\sim}{x}_3$ for $n = 27$ is strongly suggested by the numerical minimization as Figure 3 shows, so that only a very limited search is required. We have also added vectors with the most pleasing fractions for $S - N/2$, namely $\underset{\sim}{x}^*_{\text{int}}$ obtained from $\underset{\sim}{x}_{\text{int}}$ by changing x_9 to 31, and x_3^* by changing the first 10 in $\underset{\sim}{x}_3$ to an 11.

Table 4 lists the results of the numerical minimization and exhibits to high accuracy the relations conjectured in § 4.

ACKNOWLEDGMENT. The authors wish to thank P. H. Diananda for valuable information and suggestions.

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Received October 24, 1977 and in revised form October 24, 1978.

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