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# OPERATORS OVER REGULAR MAPS

STEPHEN E. WILSON

**In this paper, we define certain operators, each of which transforms one regular map into another. These operator are based on the notions of Petrie path and *j*th order "hole" introduced by Coxeter. Together with the usual dual operator, they are a powerful tool for the analysis and taxonomy of regular maps. We produce, as an example, 18 distinct maps from the icosahedron, including six of Brahana and Coble's eight pentagonal dodecahedra.**

DEFINITIONS. A *map* is a division of a compact 2-manifold into simply connected regions called the *faces* of the map by an embedded graph or multigraph. A *flag* in a map $M$ is a mutual incidence of a face, an edge and a vertex. A *symmetry* or *automorphism* of $M$ is a permutation of its parts which preserves kind and incidence. The map $M$ is to be called *regular* provided that its group of symmetries, $G(M)$, acts transitively on its flags. Consider Fig. 1: a regular map $M$ must possess a symmetry $\alpha$ which interchanges the flags (A 1 V) and (A 1 U), another, $\beta$, which interchanges (A 1 V) and (B 1 V), and a third, $X$, which interchanges (A 1 V) and (A 2 V). These three symmetries generate $G(M)$, and we may think of them as reflections about the appropriate axes. The map also has rotational symmetries: $R = \alpha X$ (meaning first apply $\alpha$ and then $X$) is rotation one step counterclockwise about face $A$. $S = \beta X$ is rotation one step clockwise about $V$, and $\gamma = \alpha\beta$ is rotation 180° around edge 1.

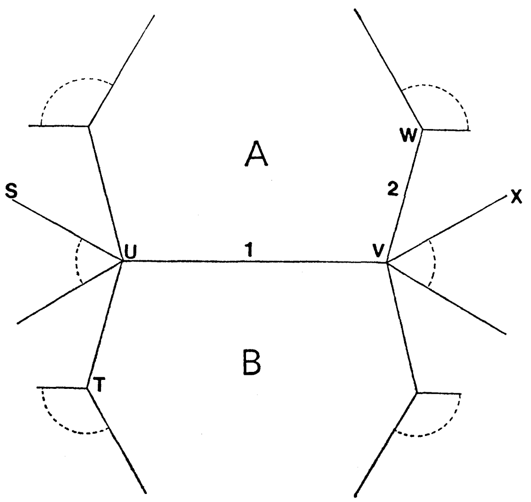


FIGURE 1. Flags in a regular map

*Other forms of regularity*:

This definition of regularity is a little stronger than is standard and it will be worth our while to look at others.

A few authors have only required that the vertices all have the same valence and that all the faces have the same co-valence. This is the weakest form of regularity, and we will call such a map *uniform*. Many authors have required that the map only need possess rotational symmetries like $R$, $S$, and $\gamma$. Such a map we will call *rotary*. Every regular map is rotary, and a map which is rotary but not regular we will call *chiral* (lit. "handed"). There are no chiral nonorientable maps [4].

Much of the literature deals with rotary maps. We prefer the present definition because it is in some sense more natural to require that the map have all its possible symmetries than just half, because it is easier to generalize to higher-dimensional complexes, and, primarily, because it is more powerful — one of our most important tools, the operator $P$, will be seen to be closed over regular maps, but not over rotary ones.

*Holes and Petrie paths*:

A *jth order hole* is a cyclic sequence of edges, each two consecutive sharing a vertex, so that at each vertex, the adjacent edges subtend $j$ faces on one side, either the right or the left but consistently throughout.

*A jth order Petrie path* is a similar sequence of edges, but at each vertex, $j$ faces are enclosed on the right and on the left alternately. A first-order Petrie path will be called simply a Petrie path and a first-order hole is just a face.

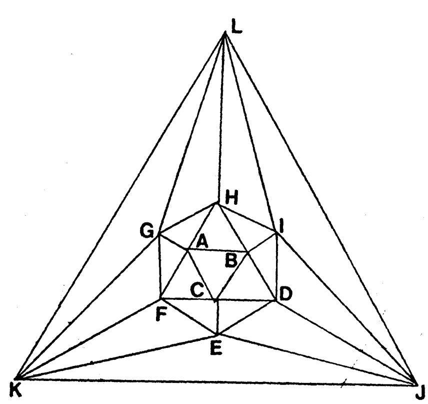


FIGURE. 2 The icosahedron

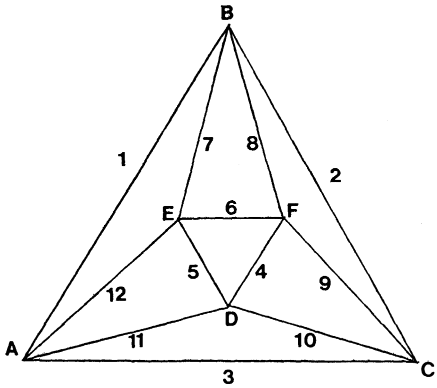


FIGURE 3. The octahedron

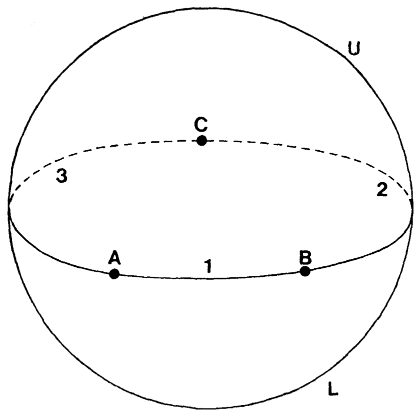


FIGURE 4. $\varepsilon_3 = \{3, 2\}$

Consider Fig. 2, the icosahedron. Here, *AHDIC* is a second- and third-order hole; *AHBIDJEKFG* is a Petrie path and *DEFGHI* is a second-order Petrie path.

In these definitions we intend enclosing on the right to be distinct from enclosing on the left. To clarify that distinction and its consequences, consider Fig. 3, the octahedron. Edge 3 in this figure belongs to two different second-order holes, both consisting of the edges 12, 6, 9, 3 in that order, but in one, the angle 9C3 contains the faces CDF and CDA (on the right) and in the other, 9C3 contains CFB and CAB (on the left). Also examine the map of Fig. 4, $\varepsilon_3 = \{3, 2\}$. A Petrie path in this map has length 6: For instance, starting at 1B2 enclosing the upper hemisphere, face $U$, the one and only Petrie path in this map is 1B2($U$), 2C3($L$), 3A1($U$), 1B2($L$), 2C3($U$), 3C1($L$).

The symmetry $T = \gamma X$ is easily seen to move the map in a glide reflection one step along the Petrie path *TUVW* in Fig. 1. Motion one step along a $j$th order hole is $RS^{j-1} = \gamma S^j$. For instance in Fig. 1, $RS = \gamma S^2$ is one step along the second-order hole *SUVX*.

*The Operators D, P, opp and $H_j$.*

As is well-known, for any map $M$ on a surface, a new map, called the *dual* of $M$, $D(M)$, can be formed on the same surface in the following way: take for vertices all the face-centers of $M$, and for each edge of $M$, draw a new edge across it joining the centers of the adjacent faces. The new edges are in one-to-one correspondence with the old edges they cross, and, in fact, we would like to think of $D(M)$ as being made up of the edges of $M$ differently arranged. Thus a set of edges which forms a face in $M$ forms a vertex in $D(M)$ and vice versa; further, a set of edges which forms a Petrie path in $M$ also forms a Petrie path in $D(M)$. Indeed, one possible definition for a Petrie path is to require that it be a set of edges which is a cycle both in $M$ and in $D(M)$. $D$ is defined over all maps, and $D(M)$ is regular (rotary, uniform) iff $M$ is regular (rotary, uniform).

We can also form from $M$ a new map called the Petrie of $M$, $P(M)$, as follows: dissolve the faces of $M$ and span by a membrane each cycle of edges which forms a Petrie path in $M$. The resulting figure is a map on a surface, in general a different surface than that of $M$. A set of edges which forms a face in $M$ forms a Petrie path in $P(M)$, while vertices in $M$ are also vertices of $P(M)$. $P(M)$ is regular iff $M$ is regular; however, if $M$ is chiral, $PM$ must be uniform but not necessarily even rotary.

Since $D$ transposes faces and vertices, leaving Petrie paths fixed, and $P$ transposes faces and Petrie paths, leaving vertices alone, we can see that the operators $P$ and $D$ satisfy $I = P^2 = D^2 = (PD)^3$. Thus, $P$ and $D$ generate a copy of $S_3$, the group of permutations on three objects. The third involution of the group, $PDP = DPD$, we call the *opposite* operator; $\text{opp}\,(M) = PDP(M) = DPD(M)$ can be formed directly from $M$ in the following interesting way: Label each edge with a number and an arrow running along it on both sides. Cut the map apart along the edges and then glue it back together again so that all the numbers match but none of the arrows do. The resulting map is $\text{opp}\,(M)$. Its faces are the faces of $M$, but all of the joinings have been reversed.

Let us prove that with a picture. Fig. 5a shows the neighborhood of an edge in $M$ (the edge numbered 2 here); if we apply the operator $P$ to $M$, the neighborhood of edge 2 in $PM$ will be as in Fig. 5b:

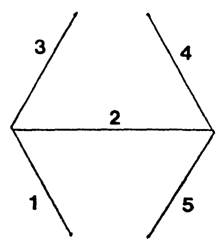


FIGURE 5a. $M$

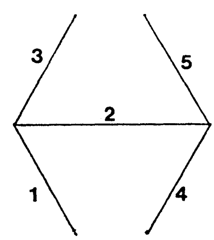


FIGURE 5b. $P$(M)

If we now apply $D$ to $P(M)$ and $P$ to that, the neighborhood of 2 will be as in Figs. 5c and 5d, respectively:

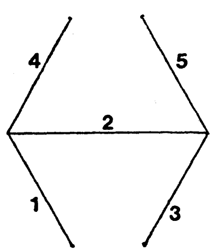


FIGURE 5c. $DP(M)$

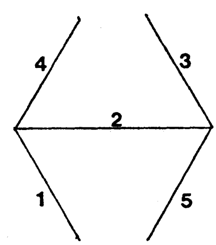


FIGURE 5d. $\text{opp}\,(M) = PDP(M)$

As predicted, the faces (..125..) and (..324..), which meet at 2 in $M$, exist and meet at 2 in $\mathrm{opp}(M) = PDP(M)$, but are oppositely matched.

We would like to construct another operator, $H_j$, using $j$th order holes as the operator $P$ used Petrie paths. However, some care must be used. Consider the object we get by dissolving the faces of $M$ and spanning by a membrane each cycle of edges which is a $j$th

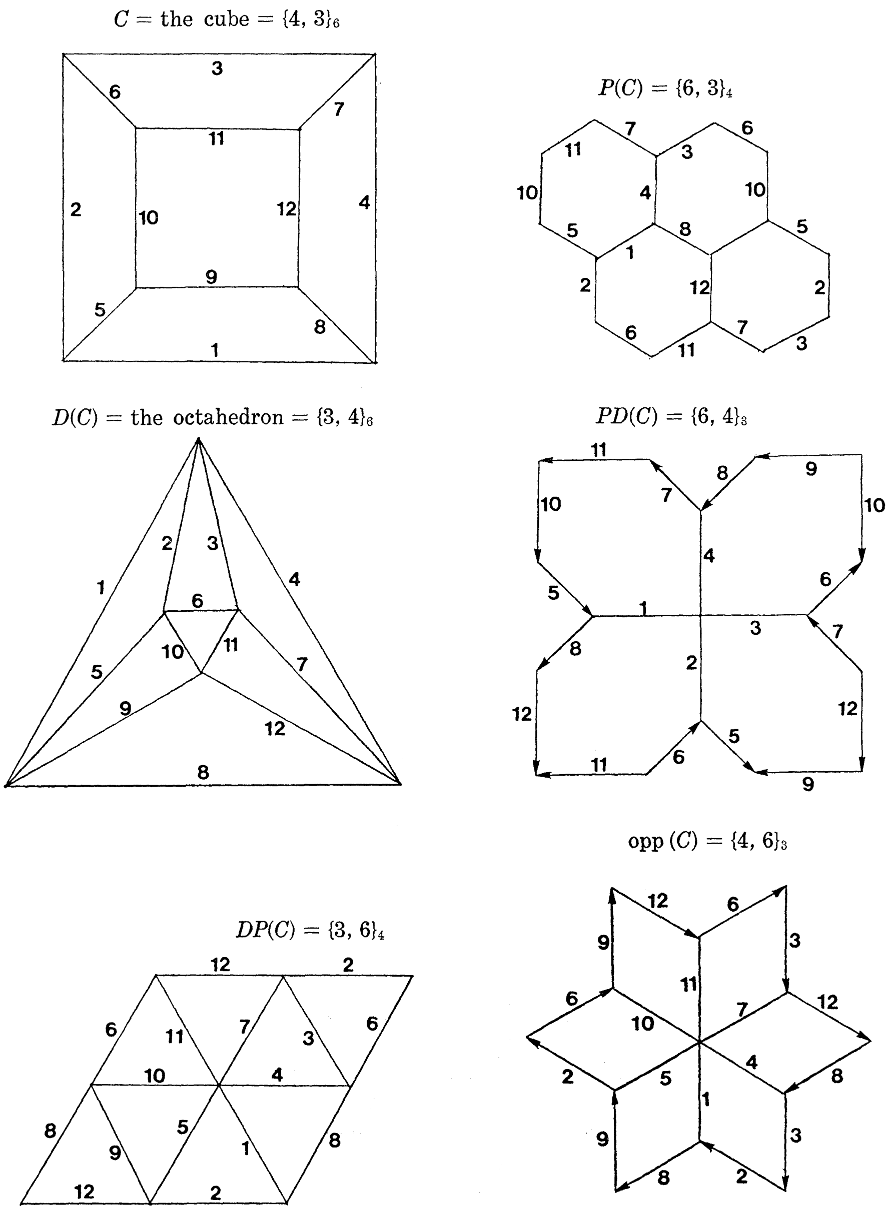


FIGURE 6. The direct derivater of the cube

order hole in $M$. If $j$ is relatively prime to the valence $q$ of a vertex, this will be a map on a manifold which we can call $H_j(M)$. If however, $d = (j, q)$ is not 1, the $j$th order holes meeting at a given vertex will resolve themselves into $d$ cycles of $q/d$ holes apiece, none from one cycle meeting one from any other (at that vertex). Thus our putative $H_j(M)$ will look like a manifold except that at each vertex, $d$ sheets will be pinched together. In this case, we separate each vertex into $(j, q)$ vertices, one on each sheet. The result of this surgery is a manifold which is either one connected map or the union of a number of identical connected components. $H_j(M)$ is one connected component of this manifold. Clearly $H_iH_j = H_{ij}$, and $PH_i = H_iP$.

*Derivates and Direct Derivates*:

The maps derivable from $M$ under $D, P, H_j$ and their products we call the *derivates* of $M$; under $D, P$ and their products only, the *direct derivates* of $M$, numbering at most six. Consider, for example, the cube; its direct derivates are shown in Fig. 6. In Fig. 6, the edges are numbered consistently throughout, so that the edges of a Petrie path in $C$ form a face of $P(C)$, a vertex in opp $(C)$, etc. Follow, for instance, the fate of the cycle 1234, which is a face in $C$ and opp $C$, a vertex in $DC$ and $PDC$, and a Petrie path in $PC$ and $DPC$. Note that $PDC$ and opp $C$ are nonorientable.

Considering the action of the operators $H_j$ on these maps, first let us realize that in any map $M$, $H_j(M) = H_{q-j}(M)$, so that we only need consider values of $j$ between 2 and $q/2$, inclusive. Thus we need only consider $H_2$ on $DC$ and $PDC$, $H_2$ and $H_3$ on $DPC$ and opp $C$, and none at all on $C$ and $PC$. $H_2DC$ is a map on the sphere consisting of two 4-gonal faces sharing four vertices and edges along the equator, $\varepsilon_4 = \{4, 2\}$. Since $PH_j = H_jP$, it follows that $H_2PDC$ is $P\varepsilon_4$, which is $\varepsilon_4$ itself. It is clear that $H_2(DPC)$ consists of 6-gons meeting three at a vertex (i.e., it is of type $\{6, 3\}$), and tracing out all the second-order holes, we get Fig. 7, which is identical to $PC$. Thus $H_2PDP(C) = PH_2DP(C) = PP(C) = C$! This demonstrates that while $H_j$ of an

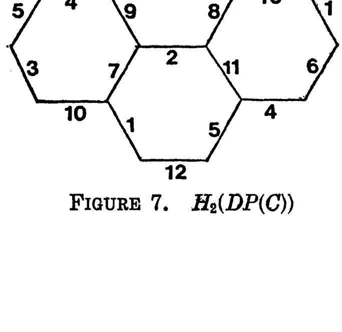


FIGURE 7. $H_2(DP(C))$

orientable map must be orientable, $H_j$ of nonorientable map may be orientable or not. Finally, $H_3DPC$ is $\varepsilon_2$, and so is $H_3 \operatorname{opp} C$.

The various operators may degenerate slightly in their action on a given map. There would seem to be four cases possible:

I. $M$ has six different direct derivates.

II. $M$ is self-dual (or self-Petrie or self-opposite) but no other degeneracies occur, so that $M$ has three direct derivates, $M = DM$, $PM = PDM$, $DPM = \operatorname{opp} M$.

III. $M = PDM = DPM \neq DM = PM = \operatorname{opp} M$, so that $M$ has two direct derivates.

IV. $M = PM = DM$, so $M$ is self-everything and has only one direct derivate.

Glancing through the catalogue of maps produced in [7], we see that there are about equal occurences of cases I and II, very few cases IV, and absolutely *no* examples of case III. This is curious. There seems to be no obvious reason that no case III map should exist, and yet they seem reluctant to appear. The author, in fact, conjectured for a period of two years before he found one that no such map existed. Let $M$ be the regular map $\{3, 7\}_9$. This map has 126 edges ([4], p. 139), and $\operatorname{opp} M = \{3, 9\}_7$. Then $N = H_2 \operatorname{opp} M$ is a regular map of type $\{9, 9\}_9$. Working from a diagram of the map or from a representation of its generators as permutations of edges, it is ersy to check that $N$ satisfies the following relations:

$$\begin{aligned} I = S^9 &= (RS^3)^3 = (TS^3)^7 \\ = T^9 &= (ST^3)^3 = (RT^3)^7 \\ = R^9 &= (TR^3)^3 = (SR^3)^7 , \end{aligned}$$

and is defined by them. From these, it is clear that the operator $PD$, which permutes $R$, $S$, $T$ cyclically, sends the map to itself, while $D$, which interchanges $R$ and $S$, does not.

*History of the Operators*:

The notion of duality, of course, is as ancient as Greek mathematics, as is the idea of using some standard collection of operations (duality, truncation, stellation, etc.) to get a new polyhedron from an old one.

Petrie paths were first noticed as being useful by J. F. Petrie and were first mentioned in print by Coxeter in [2]. Coxeter thereafter used Petrie paths repeatedly and successfully to illuminate many geometric and group-theoretic ideas. He introduced the notation $\{p, q\}_r$ for the (possibly infinite, possibly collapsing) map formed from the spherical, Euclidean or hyperbolic tessellation $\{p, q\}$ by identifying points at distance $r$ steps along a Petrie path. In [4], he and Moser list the known finite $\{p, q\}_r$'s. In article 8.6 of [4], they explicitly formulate a process equivalent to the operator $P$, and so construct the six direct derivates of $\{p, q\}_r$, but curiously, they do not apply these operators to any other regular map.

In [3], Coxeter introduced $j$th order holes and enumerated regular maps $\{p, q \mid h_2\}$ determined by $p$, $q$, and $h_2$ and $\{p, q \mid, h_3\}$ determined by $p, q$, and $h_3$. No one seems previously to have constructed an operation like the $H_j$'s, but this idea also is not new: the Great Dodecahedron is formed precisely by making $H_2$ of the icosahedron. Further, the Great Icosahedron may be seen to be $H_2$ of the Small Stellated Dodecahedron.

In [1], Brahana and Coble produce eight pentagonal dodecahedra in pairs. In each case, they have the group expressed as permutations on the faces and they have two involutions, one of which can be interpreted as a turn around an edge and the other a turn over the same edge (p. 5), the choice being arbitrary. Making one choice gives them a map $M$, and making the other gives opp $M$; i.e., the opposite operator interchanges the roles of $\beta$ and $\gamma$ in the group of the map. (There is a small flaw in Figure VII of [1]: the edge joining face $\beta$ with face $\varepsilon$ should have one of its arrows reversed.)

*An Example*:

As an example of the operators, we list in Table 1 the derivates of the icosahedron with 30 edges, and the effect of the operators $D$, $P$, opp, $H_2$, and $H_3$ on each. Only items 1, 3, and 7 are orientable. Notation for Table 1 is as follows:

$F =$ the number of faces

$V =$ the number of vertices

$-\chi = 30 - F - V$, the negative of the Euler characteristic

$q =$ the valence of a vertex $=$ the order of $S$ in the group

$p =$ the co-valence of a face $=$ the order of $R$

$r =$ the length of a Petrie path $=$ the order of $T$.

Under opp, $D$, $P$, $H_2$, $H_3$ for each map is listed the number of the item in this list that is produced by applying the operator to the map. Under $H_j$, if $j > q/2$, only a dash is shown. The symbol * there means that $H_j(M)$ is $\varepsilon_2 = \{2, 2\}$ the regular map on a sphere with two edges, two faces and two vertices.

Among the items on this list are six of Brahana and Coble's eight dodecahedra, which neatly demonstrates their contention that these six maps have identical groups, though it does *not* show that the other two (formed from $\{5, 4\}_6$ and $\{5, 6\}_4$ by identifying antipodal points) do not have this group. For each of these the table shows the number of the corresponding figure in [1] under *BC*.

TABLE 1. The derivates of the icosahedron

| opp | $D$ | $P$ | $H_2$ | $H_3$ | Item | $-\chi$ | $F$ | $V$ | $q$ | $p$ | $r$ | BC |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 6 | 3 | 2 | 7 | — | 1. Icosahedron | −2 | 20 | 12 | 5 | 3 | 10 | |
| 4 | 5 | 1 | 8 | — | 2. $P(1)$ | 12 | 6 | 12 | 5 | 10 | 3 | |
| 5 | 1 | 4 | 3 | — | 3. Dodecahedron | −2 | 12 | 20 | 3 | 5 | 10 | VI |
| 2 | 6 | 3 | 4 | — | 4. $PD(1)$ | 4 | 6 | 20 | 3 | 10 | 5 | |
| 3 | 2 | 6 | 14 | 6 | 5. $DP(1)$ | 12 | 12 | 6 | 10 | 5 | 3 | VIII |
| 1 | 4 | 5 | 13 | 5 | 6. opp (1) | 4 | 20 | 6 | 10 | 3 | 5 | |
| 9 | 7 | 8 | 1 | — | 7. Great Dodecahedron | 6 | 12 | 12 | 5 | 5 | 6 | VII |
| 8 | 9 | 7 | 2 | — | 8. $P(7)$ | 8 | 20 | 12 | 5 | 6 | 5 | |
| 7 | 8 | 9 | 10 | * | 9. $DP(7)$ | 8 | 12 | 20 | 6 | 5 | 5 | IX |
| 12 | 11 | 10 | 10 | — | 10. $H_2(9)$ | 4 | 6 | 20 | 3 | 10 | 10 | |
| 11 | 10 | 12 | 2 | 15 | 11. $D(10)$ | 4 | 20 | 6 | 10 | 3 | 10 | |
| 10 | 12 | 11 | 1 | 16 | 12. $PD(10)$ | 18 | 6 | 6 | 10 | 10 | 3 | |
| 18 | 15 | 14 | 14 | — | 13. $H_2(6)$ | 12 | 6 | 12 | 5 | 10 | 6 | |
| 16 | 17 | 13 | 13 | — | 14. $P(13)$ | 8 | 10 | 12 | 5 | 6 | 10 | |
| 17 | 13 | 16 | 8 | 11 | 15. $D(13)$ | 12 | 12 | 6 | 10 | 5 | 6 | IV |
| 14 | 18 | 15 | 7 | 12 | 16. $PD(13)$ | 14 | 10 | 6 | 10 | 6 | 5 | |
| 15 | 14 | 18 | 4 | * | 17. $DP(13)$ | 8 | 12 | 10 | 6 | 5 | 10 | V |
| 13 | 16 | 17 | 3 | * | 18. opp (13) | 14 | 6 | 10 | 6 | 10 | 5 | |

*Other Uses of the Operators*:

Besides their use in constructing new regular maps from old ones, the operators can be used as a compact language for describing the properties of a given map or for talking about the relationship of one map to another. For instance, if $M$ is a large 6-valent regular map, it helps one to understand the structure of $M$ to notice that $H_2(M)$ is the cube (if that is the case). We close by stating, without proof, three theorems that employ the operators in this way.

Let $M'_{k,i}$ for $i^2 \equiv 1 (\mathrm{mod}\, k)$ be the map with two faces shown in Fig. 8.

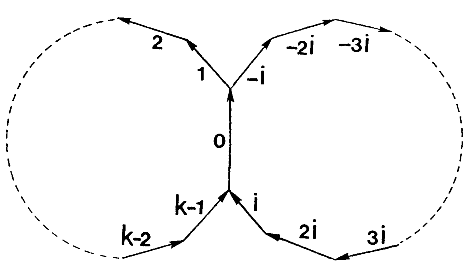


FIGURE 8. $M'_{k,i}$

THEOREM 1. *$M'_{k,i}$ is self-Petrie iff*

(1) *$k$ is even and $i \equiv -1 (\mathrm{mod}\, k)$*

*or* (2) *$k = 8h$ and $i = 4h - 1$ for some positive integer $h$.*

THEOREM 2. *If $M$ is an orientable regular map, then its underlying graph is bipartite iff $P(M)$ is also orientable.*

Note that in a regular map, if some pair of vertices share exactly $k$ edges, then so does every pair of adjacent vertices, and that in general, $k$ could be any positive integer. However, we can prove this:

THEOREM 3. *If the regular map $M$ is self-Petrie, then each pair of adjacent vertices in $M$ share at most two edges.*

## REFERENCES

1. H. R. Brahana and A. B. Coble, *Maps of twelve countries with five sides with a group of order* 120 *containing an ikosahedral subgroup*, Amer. J. Math., **48**, 1-20.

2. H. S. M. Coxeter, *The densities of the regular polytopes*, Proc. Camb. Phil. Soc., **27** (1931), 201-211.

3. ———, *Regular skew polyhedra in three and four dimensions, and their topological analogues*, Proc. Lond. Math. Soc., (2), **43** (1937), 33-62.

4. H. S. M. Coxeter and W. O. J. Moser, *Generators and Relations for Discrete Groups*, Berlin-New York, 1972, Springer-Verlag.

5. D. Garbe, *Über die regulären Zerlegungen geschlossener orientierbarer Flächen*, J. Reine. Angw. Math., **237** (1969), 39-55.

6. F. A. Sherk, *The regular maps on a surface of genus three*, Canad. J. Math., **11** (1959), 452-480.

7. S. E. Wilson, *New Techniques for the Construction of Regular Maps*, Doctoral Dissertation, Univ. of Washington, 1976.

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MICHIGAN STATE UNIVERSITY
EAST LANSING, MI 48824