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# ON SELF-ADJOINT DERIVATION RANGES

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**The properties of those operators on a Hilbert space which induce a derivation whose range after closure is self-adjoint are studied. Such operators are termed $D$-symmetric. A characterization of compact $D$-symmetric operators is given. Normal derivations are considered, and an example of an irreducible, not essentially normal, $D$-symmetric operator is presented.**

Let $\mathscr{L}(\mathscr{H})$ denote the bounded linear operator on a Hilbert space $\mathscr{H}$. For $A \in \mathscr{L}(\mathscr{H})$ define a linear operator

$$\Delta_A: \mathscr{L}(\mathscr{H}) \longrightarrow \mathscr{L}(\mathscr{H})$$

as follows

$$\Delta_A: X \longrightarrow AX - XA$$

for all $X \in \mathscr{L}(\mathscr{H})$. Then $\Delta_A$ is an inner derivation on $\mathscr{L}(\mathscr{H})$ and remarkably enough all (linear) derivations on $\mathscr{L}(\mathscr{H})$ are of this form (see [11], [12] and [18]). The properties of inner derivations, their spectrum [13], norm [20] and ranges [2], [10], [21], [23] have been scrutinized carefully in recent years. In the paper we wish to consider the class of operators which have self-adjoint derivation ranges, at least after one closes in the norm topology.

DEFINITION. A operator $A \in \mathscr{L}(\mathscr{H})$ is $D$-symmetric if $(\text{range } \Delta_A)^- = (\text{range } \Delta_{A^*})^-$ (the $-$ indicates closure in the norm topology). We denote range $\Delta_A$ by $\mathscr{R}(\Delta_A)$. We denote the class of $D$-symmetric operotors by $\mathscr{S}$. Obviously $A$ is $D$-symmetric if and only if $\mathscr{R}(\Delta_A)^-$ is a self-adjoint subspace of $\mathscr{L}(\mathscr{H})$. The concept of $D$-symmetric was introduced by Bunce and Williams.

Another paper [1] on this topic appeared at the same time as this one, and we have modified our terminology in accordance with theirs. On one occasion a more general result appears in [1] and in that instance (Theorem 3) we have merely stated our result, which is needed elsewhere, while omitting the proof.

The paper has been expanded to include an example of an irreducible $D$-symmetric operator which is not essentially normal.

1. **General considerations.** We would like to explore the class $\mathscr{S}$ in this paper. We begin by proving a very simple yet often-times useful lemma concerning membership in $\mathscr{S}$.

LEMMA 1. *Let* $A \in \mathscr{L}(\mathscr{H})$. *If there exist nonzero vectors* $f, g \in \mathscr{H}$ *such that*

(1) $Af = \lambda f$, $A^*f \neq \bar{\lambda} f$ *and*

(2) $A^*g = \bar{\lambda} g$.

*Then* $A$ *is not D-symmetric.*

*Proof.* We must show that $\mathscr{R}(\Delta_A)^- \neq \mathscr{R}(\Delta_{A^*})^-$. Since $\mathscr{R} = (\Delta_A) = \mathscr{R}(\Delta_{A-\lambda})$ we may assume without loss of generality that $\lambda = 0$. Note that $A^*f = w \neq 0$ where $w \perp f$. Define an operator $X \in \mathscr{L}(\mathscr{H})$ as follows.

$$Xw = g \quad \text{and} \quad X = 0 \quad \text{on} \quad \{w\}^\perp .$$

Then $((A^*X - XA^*)f, g) = -(g, g) \neq 0$. But for any $Y \in \mathscr{H}$,

$$((AY - YA)f, g) = 0 .$$

Thus dist $[A^*X - XA^*, \mathscr{R}(\Delta_A)] > 0$ which completes the proof.

The last lemma has a sequential analogue which is sometimes useful.

LEMMA 2. *Let* $A \in \mathscr{L}(\mathscr{H})$. *Assume* $\lim_{n\to\infty} \|(A - \lambda)f_n\| = 0$ *and* $\limsup \|(A - \lambda)^* f_n\| \geqq c > 0$ *where* $\{f_n\}$ *is an orthonormal sequence. Assume* $\|(A - \lambda)^* g_n\| \to 0$ *where* $\{g_n\}$ *is an orthonormal sequence. Then* $A$ *is not D-symmetric. Conversely, if* $A$ *is D-symmetric then* $A$ *has an infinite dim. direct summand modulo the compacts.*

*Proof.* We may and do assume $\lambda = 0$. We may also assume $\|A^*f_n\| \geqq c > 0$ for all $n$ by considering a subsequence if necessary. Set $A^*f_n = \alpha_n f_n + w_n$ where $w_n \perp f_n$. Then $|\alpha_n| \to 0$. We claim $w_n \to 0$ weakly. Indeed, for any $h \in \mathscr{H}$,

$$\begin{aligned}(w_n, h) &= (A^*f_n - \alpha_n f_n, h) \\ &= (f_n, Ah) - (\alpha_n f_n, h) \longrightarrow 0 .\end{aligned}$$

By choosing a subsequence of the $\{f_n\}$'s and perturbing slightly if necessary, we can arrive at sequences $\{f_n'\}\{w_n'\}$ such that $\|A^*f_n' - w_n'\| \to 0$ and $\{f_n'\}$, $\{w_n'\}$ are mutually orthogonal i.e., $(f_n', w_m') = 0$ for all $n, m$. (Since the last argument is standard by now we forgo any presentation.) We now define $X \in \mathscr{L}(\mathscr{H})$ as follows

$$\begin{aligned} Xf_n' &= 0 \quad \text{for} \quad n = 1, 2, \cdots \\ Xw_n' &= g_n \quad \text{for} \quad n = 1, 2, \cdots \\ X &= 0 \quad \text{on} \quad \{w_n'\}^\perp . \end{aligned}$$

Then for any $Y \in \mathscr{L}(\mathscr{H})$;

$$((AY - YA)f_n', g_n) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \alpha .$$

On the other hand

$$\begin{aligned} &|((A^*X - XA^*)f_n', g_n)| \\ &\quad = |(XA^*f_n', g_n)| \geqq |(Xw_n', g_n)| - \|X\| \, \|(A^*f_n' - w_n')\| \\ &\quad \geqq \|g_n\|^2 - \varepsilon_n \quad \text{where} \quad \varepsilon_n \longrightarrow 0 . \end{aligned}$$

Thus $\operatorname{dist}[A^*X - XA^*, \mathscr{R}(\Delta_A)] \geqq 1$ and so $A$ is clearly not $D$-symmetric.

REMARK. One can of course replace the orthonormal sequences of the lemma by sequences which converge weakly to zero. One might suggest that no conditions at all are required. After all $\{f_n\}$ has a weakly convergent subsequence; if it converges to zero fine, if it does not converge to zero then $A$ has on eigen vector. Be that as it may we wish to point out that the proof does not go through under these slightly more general conditions.

More precisely, there exists a $D$-symmetric operator $A$ with the following properties.

(1) $Af = 0$ but $A^*f \neq 0$ for some $f \in \mathscr{H}$, $f \neq 0$.

(2) $A^*g_n \to 0$ for orthonormal sequence $\{g_n\}$.

EXAMPLE 1. We define our operator $A$ as follows. Let $\{h_n\}$ be an orthonormal basis for $\mathscr{H}$.

Set

$$A^*h_n = a_n h_{n+1} \quad \text{for} \quad n = 1, 2, \cdots ,$$

where

$$a_n = 1 \quad \text{and} \quad a_n = \frac{1}{\log n} \quad \text{for} \quad n \geqq 2 .$$

Then $A$ satisfies the conditions above (just set $f = h_1$ and $g_n = h_n$). Moreover $A$ is $D$-symmetric since $\mathscr{R}(\Delta_A)^- = \mathscr{K}$, the ideal of compact operators as was proved in [21]. We remark that we will have occasion to use this operator in other parts of the paper. Since a slightly stronger form of the next theorem appears in [1] we state the result without proof.

THEOREM 3. *Let $A \in \mathscr{L}(\mathscr{H})$ be essentially normal. Then $A$ is $D$-symmetric if and only if $A \sim T$ implies $A^* \sim T$ for all $T \in \mathscr{T}$ (trace class).*

DEFINITION. An operator $A \in \mathscr{L}(\mathscr{H})$ is subnormal if there exists

a larger Hilbert space $\mathscr{H}_1 \supset \mathscr{H}$ and a normal operator $N \in \mathscr{L}(\mathscr{H}_1)$ such that $Af = Nf$ for all $f \in \mathscr{H}$.

COROLLARY 4. *Let $A \in \mathscr{L}(\mathscr{H})$ be a subnormal operator with a cyclic vector and no point spectrum. Then $A$ is D-symmetric.*

*Proof.* Since $A$ is subnormal with a cyclic vector, $A$ is essentially normal by a result of Berger and Shaw [4]. To complete the proof we will show that $A$ commutes with no trace class operator $T$ (other than 0). Indeed if $T$ commutes with $A$, then $T$ is subnormal by Yoshino's theorem. But any compact subnormal operator is normal. Since the eigenspaces for $T$ reduce $A$, and $A$ has no point spectrum we conclude that $T$ is 0.

COROLLARY 5. *Let $T \in \mathscr{L}(\mathscr{H})$ be a hyponormal weighted shift (unilaterial or bilaterial) with no point spectrum. Then $T$ is D-symmetric.*

*Proof.* Since $T$ is hyponormal the weights must be increasing in modulus. Since $T$ has no point spectrum they must all be nonzero. Since the modulus of the weights must converge as $n \to \pm\infty$; $T$ must be essentially normal. It is well known that $\{T\}'$ contains no trace class operators (see [19], page 62) which completes the proof.

REMARK. Before going further we would like to show that both of the hypotheses in Corollary 4 are necessary. To demonstrate the relevance of the condition $\sigma_p(A) = \varnothing$ is easy. Let $S$ be the unilateral shift operator. Set $T = S \oplus 0$ on $\mathscr{H}_1 \oplus \mathscr{H}_2$ where $\mathscr{H}_2$ can be finite or infinite dimensional. Clearly $T$ is subnormal and it follows immediately from Lemma 1 that $T$ is not $D$-symmetric.

The foregoing example raises an interesting question. If $A$ and $B$ are $D$-symmetric how about $A \oplus B$? The example demonstrates that some care must be exercised. There is one rather easy and obvious positive result which we state without proof. The reader may find Rosenblum's theorem [17], useful here.

PROPOSITION 4. *Let $A$ and $B$ be D-symmetric. If $\sigma(A) \cap \sigma(B) = \varnothing$; then $A \oplus B$ is D-symmetric.*

EXAMPLE 2. We now present a second example which is interesting on several counts. Let $\mathscr{H}_1 = L^2(\Delta, dA/\pi)$ where $\Delta$ denotes the unit disc and $dA$ denotes area measure. Define an operator $M$ on $\mathscr{H}_1$ as follows. For $f \in \mathscr{H}_1$; $M$: $f(z) \to zf(z)$. Let $S$ be the unilateral

shift operator on $\mathscr{H}_2$ where $\{e_n\}_{n=1}^{\infty}$ is the canonical basis for $S$. Now set $T = M \oplus S$ on $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$. Then $T$ is a subnormal operator and $\sigma_p(T) = \varnothing$. Moreover $T$ is the direct sum of two $D$-symmetric operators. However $T$ itself is *not* $D$-symmetric as we shall show. James Deddens has independently shown that the operator $M \oplus S \oplus S \oplus \cdots \notin \mathscr{S}$. It suffices to exhibit a trace class operator $L$ which commutes with $T$ but not with $T^*$. Our operator $L$ will have the form $\begin{vmatrix} 0 & J \\ 0 & 0 \end{vmatrix}$. Since $T = \begin{vmatrix} M & 0 \\ 0 & S \end{vmatrix}$ the condition $TL = LT$ is equivalent to $MJ = JS$. Let $\varphi = \chi_D$ when $D$ is the disc of radius $\alpha$ centered at 0 and $\alpha < 1$.

Define $Je_n = z^n\varphi$ for $n = 1, 2, \cdots$ ($J$ maps $\mathscr{H}_2$ into $\mathscr{H}_1$). Then $JSe_n = Je_{n+1} = z^{n+1}\varphi$ while

$$MJe_n = Mz^n\varphi = z^{n+1}\varphi .$$

By continuity and linearity $MJ = JS$. Observe that $\|z^n\varphi\|^2 = 1/2\pi \iint_D r^{2n+1} dr d\theta = \alpha^{2n+2}/(2n+2)$. Hence

$$\|J\|_{tr} \leqq \sum_{n=1}^{\infty} \|Je_n\| \leqq \sum_{n=1}^{\infty} \alpha^{n+1} \Big/ \sqrt{2n+2} < \infty .$$

Thus $T$ and hence $L$ is of trace class. It remains to show that $L$ does not commute with $T^*$. But $T^*L = LT^*$ is equivalent to $M^*J = JS^*$ which is equivalent to $SJ^* = J^*M$. However it follows from [22] Theorem 3 that no *nonzero* operator (trace class or not) can intertwine the shift and a normal in this way. Thus $L$ does not commute with $T^*$ and we are finished.

Although this example was included primarily to illustrate that the subnormal operator in Theorem 3 must have a cyclic vector; it also gives some indication of the subtleties involved in characterizing just when the direct sum of $D$-symmetric operators is $D$-symmetric since neither $M$ nor $S$ is a particularly pathological operator.

2. Compact symmetric operators. We will now give a classification of compact symmetric operators modulo one difficulty. The method does give rise to a situation where the direct sum of $D$-symmetric operators is $D$-symmetric.

The proof of the next lemma was suggested by B. B. Morrel and is more concise than the original. A more general result will appear in the Appendix. We also note that the first half of Lemma 5 was obtained independently by L. Fialkow [7].

LEMMA 5. *Let* $A = \int \lambda dE(\lambda)$ *be a normal operator and* $T$ *an arbitrary operator in* $\mathscr{L}(\mathscr{H})$. *If* $E[\sigma(A) \cap \sigma(T)] = 0$, *then the*

*equations $AX = XT$ and $YA = TY$ have only the trivial solution $X = Y = 0$. If $\sigma_p(T^*) \cap \sigma(A^*) = \varnothing$ and $A|E[\sigma(A) \cap \sigma(T)]$ has a complete set of eigenvectors, then the equation $AX = XT$ has only the solution $X = 0$.*

*Proof.* Let $u \in \mathscr{H}$ and assume that $AX = XT$. For all $\lambda \in \rho(T)$, the resolvent set of $T$, $(A - \lambda)X(T - \lambda)^{-1}u = Xu$. Since $Xu \in$ range $(A - \lambda)$ for all $\lambda \in f(T)$, it follows from [16] that

$$Xu \in E(\sigma(T))\mathscr{H} = E[\sigma(T) \cap \sigma(A)]\mathscr{H} = 0 .$$

Since $u$ was arbitrary; $X = 0$. To handle the case $YA = TY$, take adjoints and use the fact that range $A =$ range $A^*$ for a normal operator $A$.

In the second part, let $\{\varphi_n\}$ be a complete set of eigenvectors for $A|E[\sigma(A) \cap \sigma(T)]\mathscr{H}$. If $u \in \mathscr{H}$, then arguing as before $Xu \in E[\sigma(T)]\mathscr{H} = E[\sigma(T) \cap \sigma(A)]\mathscr{H}$, whence $Xu = \Sigma a_n\varphi_n$ (where $A\varphi_n = \lambda_n\varphi_n$). Note that $X^*(A - \lambda_n)^*\varphi_n = 0 = (T - \lambda_n)^*X^*\varphi_n$ which implies that $X^*\varphi_n = 0$ since $(T - \lambda_n)^*$ is injective. Thus

$$\begin{aligned} ||Xu||^2 &= (Xu, \Sigma a_n\varphi_n) \\ &= (u, \Sigma a_n X^*\varphi_n) = 0 \end{aligned}$$

whence $X = 0$.

THEOREM 6. *Let $A \in \mathscr{L}(\mathscr{H})$ be compact. Then $A$ is D-symmetric if and only if $A = N \oplus Q$ on $\mathscr{H}_1 \oplus \mathscr{H}_2 = \mathscr{H}$ where $N$ a is compact normal operator with no kernel and $Q$ is a quasinilpotent D-symmetric. operator If $A$ is D-symmetric and $Q$ is trace class then $Q = 0$. The decomposition is unique.*

*Proof.* Assume $A$ is $D$-symmetric. Let $(A - \lambda)f = 0$ where $\lambda \neq 0$, $f \in \mathscr{H}$.

*Claim.* $f$ reduces $A$. Since $A^*$ is compact there exists a vector $g \in \mathscr{H}$ such that $(A-\lambda)^*g=0$. It followsfrom Lemma 1 that $(A-\lambda)^*f=0$, thus $f$ reduces $A$. Now order the nonzero eigenvalues of $A$ in decreasing modulus $|\lambda_1| \geqq |\lambda_2| \geqq |\lambda_3| \geqq \cdots$. Use the fact just proved to show that $E_{\lambda_1} = \{f \in \mathscr{H}: Af = \lambda_1 f\}$ reduces $A$ and $A|E_{\lambda_1}$ is normal. Repeating the argument for $\lambda_2, \lambda_3 \cdots$, we find that

$$A = N \oplus Q \quad \text{on} \quad \mathscr{H}_1 \oplus \mathscr{H}_2$$

where $N$ is normal with no kernel, and $Q$ is quasinilpotent. Since $A$ is $D$-symmetric so are both $N$ and $Q$. This proves the first half of the theorem.

Assume now that $A = N \oplus Q$ where $N$ is normal with trivial kernel and $Q$ is quasinilpotent and $D$-symmetric. Since $Q$ is compact, the operator $A$ is essentially normal. It follows from Theorem 3; that $A$ is $D$-symmetric if and only if $A \sim T \Rightarrow A^* \sim T$ for $T$ of trace class. Write $T = \begin{vmatrix} T_1 T_2 \\ T_3 T_4 \end{vmatrix}$. If $AT = TA$ then

$$\begin{vmatrix} NT_1 - T_1N & NT_2 - T_2Q \\ QT_3 - T_3N & QT_4 - T_4Q \end{vmatrix} = 0 .$$

It follows from Lemma 5 that

$$T_2 = T_3 = 0. \quad \text{Thus} \quad T = \begin{vmatrix} T_1 & 0 \\ 0 & T_4 \end{vmatrix}.$$

Moreover $T_1 \sim N$ implies $T_1 \sim N^*$ and $T_4 \sim Q$ implies $T_4 \sim Q^*$ by Theorem 3. Thus $T \sim A^*$ and hence $A$ is $D$-symmetric, again by Theorem 3. Note that no use was made of the compactness of $N$ in the second half of the theorem.

Finally, let us consider the case when $Q$ is trace class. By Theorem 3, if $Q$ is $D$-symmetric and $Q \sim T$, $T$ of trace class then $Q \sim T^*$. But $Q$ itself is trace class. Thus $Q \sim Q^*$ whence $Q$ is normal.

The characterization just given is not altogether satisfactory since we do not know which quasinilpotent compact operators are $D$-symmetric. We note that the class is not vacuous. Indeed in Example 1 following Lemma 2, the weighted shift $S$ with weights $\{(\log n)^{-1}\}$ is $D$-symmetric and it is obviously compact and quasinilpotent. Of course the equally compact and quasinilpotent operator $S \oplus 0$ is not $D$-symmetric by Lemma 1.

Before going further we observe that the $D$-symmetric operators are not closed. To see this consider the following operators defined on the orthonormal basis $\{f_k\}_1^\infty$

$$S_n f_k = \begin{cases} n^{-1} f_2 & \text{for} \quad k = 1 \\ f_{k+1} & \text{for} \quad k > 1 \end{cases} \qquad \text{for} \quad n = 1, 2, \cdots .$$

In other words $S_n$ is just the unilateral shift with the first weight diminished. It is well known that $S_n$ is subnormal or hypernormal and since the other hypothesis are satisfied. $S_n$ is $D$-symmetric for all $n$ by Corollary 4 or 5. However $S_n \to S_0$ where the first weight of $S_0$ is zero and thus $S_0$ is not $D$-symmetric by Lemma 1. The fact that $\mathscr{S}$ is not closed was also noted in [1].

Next we wish to characterize the compact operators in $\bar{\mathscr{S}}$, the closure of the $D$-symmetric operators, We need thefollowing lemma which appears in [8] page 916 and is attributed to R. G. Douglas.

LEMMA 7. *Let $T \in \mathscr{L}(\mathscr{H})$ be a compact quasinilpotent operator. Then $T$ is the norm limit of, finite rank nilpotent operators.*

Before attacking $\bar{\mathscr{S}}$ we need the following sharpened version of Lemma 2.

LEMMA 8. *Let $T \in \mathscr{L}(\mathscr{H})$. Assume that $\mu$ is an isolated point of $\sigma(T)$ and furthermore that $(T - \mu)$ is Fredholm. If $T \in \mathscr{S}^-$ then $E_\mu = \{f \in \mathscr{H}: T_f = \mu f\}$ reduces $T$ and $T|E_\mu$ is normal.*

*Proof.* Let $\gamma = \operatorname{dist}[\mu, \sigma(T)\backslash\mu]$. Since the spectrum is an upper semi-continuous function of the operator, there exists a $\delta_1 > 0$ such that $||T - S|| < \delta_1$ implies that $\sigma(S) \subset \{\sigma(T) + \gamma/4\}$. Define idempotents as follows

$$P_T = \frac{1}{2\pi_i} \int_\Gamma (\lambda - T)^{-1} d\lambda$$

and

$$P_S = \frac{1}{2\pi_i} \int_\Gamma (\lambda - S)^{-1} d\lambda$$

where $\Gamma(t) = \mu + \gamma/2e^{it}$ for $0 \leqq t \leqq 2\pi$. (We are also assuming that $||S - T|| < \delta_1$.) It is well known that $||P_T - P_S|| \to 0$ as $||T - S|| \to 0$. Since $(T - \mu)$ is Fredholm; the subspace $P_T\mathscr{H}$ is finite dimensional and is invariant under $T$. Thus by continuity $P_S\mathscr{H}$ is finite dimensional for $S$ close to $T$ and $P_S\mathscr{H}$ is invariant for $S$. Assume $S \in \mathscr{S}$. The first part of the proof of Theorem 6 may be repeated to show that $P_S\mathscr{H}$ reduces $S$ and $S|P_S\mathscr{H}$ is normal. Choose $S_n \in \mathscr{S}$ where $S_n \to T$. Since $P_{S_n} \to P_T$ in norm, it follows that $P_T\mathscr{H}$ reduces $T$ and $T|P_T$ is normal which completes the proof.

THEOREM 9. *Let $T \in \mathscr{L}(\mathscr{H})$ be compact. Then $T \in \mathscr{S}^-$ if and only if $T = N \oplus Q$ where $N$ is normal with $\ker N = \{0\}$ and $Q$ is quasinilpotent.*

*Proof.* Order the nonzero points in the spectrum of $T$ by decreasing modulus as say $|\lambda_1| \geqq |\lambda_2| \geqq \cdots$. Then $\lambda_1$ is isolated in $\sigma(T)$ and $(T - \lambda_1)$ is Fredholm. It follows from the previous lemma that $E_{\lambda_1} = \{f \in \mathscr{H}: Tf = \lambda_1 f\}$ reduces $T$ and $T|E_{\lambda_1}$ is normal. Repeating the argument, just as in Theorem 6, we see that $T = N \oplus Q$ when $N$ is normal with trivial kernel and $Q$ is quasinilpotent.

To prove the sufficiency let $T = N \oplus Q$. Since $N$ is normal, in light of Theorem 6 it suffices to show that $Q \in \mathscr{S}^-$ for every compact

quasinilpotent operator $Q$. In view of Lemma 7 we need only consider finite rank nilpotent operator $Q$. Assume for the moment that $\mathscr{H}$ is separable. Choose an invertible operator $S$ such that

(1) $SQS^{-1}$ is in Jordan canonical form with respect to the basis $\{f_k\}_1^\infty$.

(2) The 1's in the matrix appear *below* the main diagonal.

(3) $SQS^{-1}f_{n-1} = f_n$ but $SQS^{-1}f_k = 0$ for $k \geqq n$.

Set $M = ||S|| \cdot ||S^{-1}||$. Let $\varepsilon > 0$ be given. We define a operator $V$ as follows:

(1) If $SQS^{-1}f_k = f_{k+1}$ then $Vf_k = f_{k+1}$

(2) If $k < n$ and $SQS^{-1}f_k = 0$ then $Vf_k = \varepsilon M^{-1}f_{k+1}$

(3) $Vf_k = a_k f_{k+1}$ for $k \geqq n$ where $a_k = \varepsilon M^{-1}(\log k)^{-1}$.

By construction, $V$ is a shift operator on $\mathscr{H}$ with no nonzero weights and $||SQS^{-1} - V|| < \varepsilon M^{-1}$. Moreover $V$ is compact and we claim that $V$ does not commute with any trace class operator. Note that $V$ differs from the operator in Example 1 at only a finite number of weights. It is easily seen that the commutant of a weighted shift is little influenced by modification of the first few weights, provided one does not make any nonzero weights zero. (See [19], page 62.) In particular the operator in Example 1 does not commute with any trace class operator (in fact any $C_p$ operator). Thus neither $V$ nor $S^{-1}VS$ commutes with a trace class operator. Hence $S^{-1}VS$ is $D$-symmetric. But $||Q - S^{-1}VS|| = ||S^{-1}(SQS^{-1} - V)S|| < \varepsilon$ which completes the proof.

3. Degree of approximation. Let $A$ be a normal operator. Let $AX - XA = W$ for some $X, W \in \mathscr{L}(\mathscr{H})$. Since $A$ is $D$-symmetric, there must exist a sequence of operators $\{Y_n\}$ such that

$$A^*Y_n - Y_nA^* \longrightarrow W .$$

In general the operators $\{Y_n\}$ are not uniformly bounded. For if they were then a subsequence $Y_{n_k}$ would converge weakly to $Y$ and hence

$$A^*Y - YA^* = W .$$

However it is known that $\mathscr{R}(\Delta_A)$ and $\mathscr{R}(\Delta_{A^*})$ are not equal for an arbitrary normal operator (see [10]). The following question thus arises: How is the norm of $Y_n$ related to the norm of $[A^*Y_n - Y_nA^* - W]$? Before attacking this question we need the following lemma, whose proof was suggested by Grahame Bennett.

LEMMA 10. *Let $M = [M_{ij}]$ be an $n \times n$ operator valued matrix on $\mathscr{H} = \mathscr{H}_1 \oplus \cdots \oplus \mathscr{H}_n$. Set $\mathring{M} = [\theta_{ij}M_{ij}]$ where $|\theta_{ij}| = 1$ for $i, j = 1, \cdots, n$. Then $||\mathring{M}|| \leqq n^{1/2}||M||$.*

*Proof.* Let $f = (f_1, \cdots, f_n)$ where $\|f\| = 1$ and $f_i \in \mathscr{H}_i$. Thus

$$\|\mathring{M}f\|^2 = \sum_i \left\| \sum_j \theta_{ij} M_{ij} f_j \right\|^2 .$$

Set $g^{(i)} = (\theta_{i1} f_1, \cdots, \theta_{in} f_n)$, whence $|g|^{(i)}\| = 1$. Let $P_i$ denote the projection of $\mathscr{H}$ in $\mathscr{H}_i$. Then

$$\sum_j \theta_{ij} M_{ij} f_j = P_i M g^{(i)} .$$

Thus

$$\left\| \sum_j \theta_{ij} M f_j \right\|^2 = \|P_i M g^{(i)}\|^2 \leqq \|M g^{(i)}\|^2 \leqq \|M\|^2 .$$

Hence

$$\|\mathring{M}f\|^2 \leqq \sum_i \|M\|^2 \leqq n\|M\|^2 .$$

Since $f$ was arbitrary we conclude that

$$\|\mathring{M}\| \leqq n^{1/2}\|M\| .$$

THEOREM 11. *Let $A$ be a normal operator in $\mathscr{L}(\mathscr{H})$ with $\sigma(A) \subset \Gamma$ where $\Gamma$ is a rectifiable curve and length $\Gamma = \ell$. Let $AX - XA = W$. Then there exists a $Y \in \mathscr{L}(\mathscr{H})$ such that*

$$\|Y\| \leqq n\|X\|$$

*and*

$$\|(A^*Y - YA^*) - W\| \leqq 3n^{-1}\ell\|X\| .$$

*Proof.* Choose $n^2$ distinct points $\lambda_1, \cdots, \lambda_{n^2}$ on $\Gamma$ such that the discs $D(\lambda_i, \ell n^{-2})$ cover $\Gamma$. Disjointify the discs to obtain sets $K_j$ (not necessarily open or closed) such that $\bigcup_1^{n^2} K_j \supset \Gamma$, $K_i \cap K_j = \varnothing$ for $i \neq j$, and $K_j \subset D(\lambda_j, \ell n^{-2})$. Let $A = \int \lambda dE(\lambda)$. Set $\mathscr{H}_j = E(K_j)\mathscr{H}$ and set $A_n = \Sigma \lambda_j E(K_j)$. Clearly $A_n$ is normal and $\|A - A_n\| \leqq n^{-2}\ell$, ($A_n$ is a matrix on $\mathscr{H}_1 \oplus \cdots \oplus \mathscr{H}_{n^2}$, $A = \operatorname{diag}(\lambda_1, \cdots, \lambda_{n^2})$). Let $X = [X_{ij}]$ on $\mathscr{H}_1 \oplus \cdots \oplus \mathscr{H}_{n^2}$. Clearly $\|(A_nX - XA_n) - W\| \leqq 2\|X\| \cdot n^{-2}\ell$. Moreover $(A_nX - XA_n) = [(\lambda_i - \lambda_j)X_{ij}]$. Set $Y = \theta_{ij}X_{ij}$ where $\Theta = (\lambda_i - \lambda_j)/\overline{(\lambda_i - \lambda_j)}$. Since $|\theta_{ij}| = 1$ for all $i$, it follows from Lemma 11, that $\|Y\| \leqq n\|X\|$. By definition of $\theta_{ij}$, $A_n^*Y - YA_n^* = A_nX - XA_n$. Moreover

$$\|[A_n^*, Y] - [A^*, Y]\| \leqq 2n^{-1}\ell\|X\| .$$

Combining the equations above we see that

$$\begin{aligned} ||(A^*Y - YA^*) - W|| &\leqq 2n^{-1}\swarrow||X|| + 2n^{-2}\swarrow||X|| \\ &\leqq 3n^{-1}\swarrow||X|| \quad \text{for} \quad n > 1 \end{aligned}$$

which completes the proof.

REMARK. R. Moore in [14] proved the following

PROPOSITION A. *Let $A$ be normal and let $||X_n|| \leqq 1$ for $n = 1, 2, \cdots$. If $||AX_n - X_nA|| \to 0$ then $||A^*X_n - X_nA^*|| \to 0$. The technique in the theorem above can be modified to show the following.*

PROPOSITION B. *Let $A$ be a normal operator with $\sigma(A) \subset \Gamma$ when $\Gamma$ is a rectifiable curve. Let $||X|| \leqq 1$. If $||AX - XA|| < \delta$ then $||A^*X - XA^*|| < C\delta^{1/2}$ when $C$ is a universal constant.*

Since the proof is similar to the above we omit the details.

4. **Ampliation.** Let $S_0 \in \mathscr{L}(\mathscr{H})$ and denote its ampliation $S_0 \oplus S_0 \oplus \cdots$ ($\infty$ many copies of $S$). In general $S_0$ $D$-symmetric does not imply $S$ is $D$-symmetric. Joel Anderson has been kind enough to point out to us that it follows immediately from Lemma 2 that the ampliation of the ($D$-symmetric) operator in Example 1 is not $D$-symmetric. We next show that operators close to the unilateral shift have $D$-symmetric ampliations. (It follows immediately from Theorem 3, that the unilateral shift is $D$-symmetric, a fact also observed in [1].) Note that condition 1 of the theorem is essential. If the condition is dropped the operator (let alone its ampliation) need not be $D$-symmetric as an example following the proof shows. The example also shows that $D$-symmetry is not preserved under similarity but much simpler examples will do that.

THEOREM 12. *Let $S_0$ be a weighted shift operator with weights $a_1, a_2, \cdots$. Assume that*

(1) $1 - S_0^*S_0$ *is compact*

(2) (a) $S$ *is similar to the unilateral shift or*

(b) $0 < M^{-1} \leqq ||S^n f|| \leqq M$ *for all* $f \in H$ *with* $||f|| = 1$ *and* $n = 1, 2, \cdots$, *or*

(c) $0 < \inf_{n,k} |a_n \cdot a_{n+1} \cdot \cdots a_{n+k}|$ and $\sup_{n,k} |a_n a_{n+1} \cdots a_{n+k}| < \infty$.

*Then $S$, the ampliation of $S_0$, is $D$-symmetric.*

*Proof.* We first observe that conditions (a), (b), (c) under 2 are equivalent. Indeed, the equivalence of (a) and (b) is implicitly contained in [15]; and the equivalence of (b) and (c) is easy and may be found

in [19]. Note that $S$ is unitarily equivalent to the operator which sends $(f_1, f_2, \cdots)$ to $(0, a_1 f_1, a_2 f_2, \cdots)$ on $\mathscr{H}_1 \oplus \mathscr{H}_2 \oplus \cdots$ where each $\mathscr{H}_i$ is a copy of $\mathscr{H}$. Let $SX - XS = W$. We must show that $\|(S^*Y - YS^*) - W\| < \varepsilon$ for any preassigned $\varepsilon > 0$. As a first approximation to $Y$ we try $-SXS$. Then $S^*(-SXS) - (-SXS)S^* = W + K_1XS - SXK_2$ where $K_1 = (1 - S^*S)$ and $K_2 = (1 - SS^*)$. (The operator $K_1$ and $K_2$ are not compact but they like to think of themselves that way.) We next show how to approximate the term $SXK_2$ by elements of the form $[S^*, V]$. Let $\varepsilon > 0$ be given. Note that

$$K_2 = \bigoplus_1^\infty (1 - |a_j|^2) I_j$$

and thus $\|K_2 - F_2\| < \varepsilon$ for

$$F_2 = \bigoplus_1^m (1 - |a_j|^2) I_j \oplus 0 \oplus 0 \cdots$$

and $m$ sufficiently large. Thus $\|SXK_2 - SXF_2\| < \varepsilon \|X\| \|S\|$ and $SXF_2$ is an operator valued matrix which has at most $m$ nonzero columns. We next show how to approximate a single nonzero column matrix. Assume that $T\mathscr{H}_j = 0$ for $j \neq 1$. Fix $n$ for moment and set $Y_1 = \sum_{j=0}^{n-1} (n - j)/n [S(S^*S)^{-1}]^{j+1} TS^{*j}$. Then $S^*Y_1 - Y_1S^*$

$$\begin{aligned} &= \sum_{j=0}^{n-1} \frac{n-j}{n} [S(S^*S)^{-1}]^j TS^{*j} \\ &\quad - \sum_{j=0}^{n-1} \frac{n-j}{n} [S(S^*S)^{-1}]^{j+1} TS^{*j+1} \\ &= T + n^{-1} \sum_{j=1}^{n} [S(S^*S)^{-1}]^j TS^{*j} . \end{aligned}$$

We wish to estimate the term on the right and we have yet to choose $n$. It is easy to see that $\|[S(S^*S)^{-1}]^j\| \leqq M$ for all $j$ and of course $\|S^{*j}\| \leqq M$. Let $f = \sum a_j f_j$ where $f_j \in \mathscr{H}_j$ and $\|f\| = 1$. Observe that $TS^{*j} f_k = 0$ for $k \neq j - 1$. Thus

$$\begin{aligned} &\left\| \sum_{j=1}^{n} [S(S^*S)^{-1}]^j TS^{*j} f \right\| \\ &\quad \leqq \left\| \sum_{j=1}^{n} [S(S^*S)^{-1}]^j T a_{j+1} S^{*j} f_{j+1} \right\| \\ &\quad \leqq \sum_{j=1}^{n} M \|T\| \, |a_{j+1}| \, \|S^{*j} f_{j+1}\| . \end{aligned}$$

Thus $\|[S^*, Y_1] - T\| \leqq M^2 \|T\| n^{-1/2}$. Note that $\|T\| \leqq \|SXF_2\|$ if $T$ is the first column of $SXF_2$. For $n$ sufficiently large (namely $n > \varepsilon^{-1} m N^2 \|SXF_2\|$) it is clear that $\|[S^*, Y] - T\| < \varepsilon/m$. But the

remaining columns of $F_2$ may be treated in exactly the same way. Combining these estimates we can choose $Y$ so that $\|[S^*, Y] - SXF_2\| < \varepsilon$. There still remains the term $K_1XS$ to be dealt with. Choose $F_1$ such that $\|K_1 - F_1\| < \varepsilon$ as before. Then the operator matrix $K_1XS$ has only finitely many nonzero *rows*. Rather than approximating $F_1XS$ by term $[S^*, Y]$ it is easier and clearly equivalent to approximate $S^*X^*F_1$ by terms of the form $[S, Y]$. Since $S^*X^*F_1$ has only finitely many nonzero columns, we again consider the case when $T$ has a single nonzero column, the first. To approximate $T$, set

$$Z_1 = \sum_{j=0}^{n-1} \frac{n-j}{n} S^j T[(S^*S)^{-1}S^*]^{j+1} .$$

Then repeating the previous argument shows that $\|[S, Z_1] - T\| \leqq n^{-1/2} \cdot \|T\| M^2$. Thus there exists an operator $Z$ such that $\|[S^*, Z] - F_1XS\| < \varepsilon$ for $n$ sufficiently large. Combining all the estimates we see that there exists an operator $L$ such that $\|[S^*, L] - W\| < \varepsilon(4 + \|X\|)$ whence $\mathscr{R}(\Delta_S) \subset \mathscr{R}(\Delta_{S^*})^-$. The argument to verify the reverse inclusion is identical to the above and thus the proof is complete.

EXAMPLE. Let $\{f_n\}_{n=1}^{\infty}$ be orthonormal basis for $\mathscr{H}$. Set $Tf_n = a_n f_{n+1}$ for $n = 1, 2, \cdots$, where $a_n = \begin{cases} 2 & n \text{ odd} \\ 2^{-1} & n \text{ even.} \end{cases}$ Then $T$ satisfies condition 2 of Theorem 12 but $T$ is easily seen to be not $D$-symmetric by Lemma 2.

REMARK. In §3 we saw it was possible to estimate the size $Y$ required to approximate $W = AX - XA$ by terms of the form $S^*Y - YS^*$, the estimate being given in terms of $\|X\|$ and $\|[S^*Y] - W\|$. If $S_0$ is a shift operator with $a_j = 1$ for all but $m$ of the $a_j$'s, then it is again possible to make such estimates. Indeed the explicit nature of the operators $Y_1$, $Y$, $Z_1$, $Z$ in Theorem 12 reduces the estimation process to routine bookkeeping which we will pursue no further.

COROLLARY. *The ampliation of the Bergman shift is D-symmetric.*

5. **Normal derivations.** Before proceeding to the next theorem we will need two lemmas of a nonoperator theoretic nature.

LEMMA 13. *Let $E$ be an uncountable compact set in the plane and let $S^1$ be the unit circle. Then there exists a continuous map $f$ of $E$ onto $S^1$.*

*Proof.* By first projecting $E$ onto one of the coordinate axes, we may assume $E$ is an uncountable compact subset of the reals. If $E$

contains an interval then the lemma is trivial. If not, then $E$ is the union of a countable set and a perfect set. Since the perfect set contains no intervals it is a Cantor set and the result now follows from Theorem 3-28 (page 126) of [9].

LEMMA 14. *Let $M = \{z_1, \cdots, z_k\}$ be a set of distinct points in the unit disc $D$. Let multiplicities $n_1, \cdots, n_k$ be preassigned where $1 \leqq n_j \leqq \aleph_0$ for $j = 1, \cdots, k$. Assume that $n_j = \aleph_0$ for $j = 1, \cdots, p$ where $p \geqq 3$ and further that the convex hull of $\{z_1, \cdots, z_p\}$ contains a neighborhood of 0. Then there exists a sequence $\{\zeta_j\}$ such that*

(1) *Each $\zeta_j \in M$.*

(2) *There are precisely $n_j$ of the $\zeta_i$'s equal to $z_j$.*

(3) $|\sum_{i=1}^{N} \zeta_i| \leqq 2$ *for* $N = 1, 2, \cdots$.

*Proof.* Since the lemma is intuitively obvious we present only a sketch of the proof.

1. We can choose either three or four points from the set $\{z_1, \cdots, z_p\}$ so that the convex hull of these three or four points contains a neighborhood of 0. Call these points guide points.

2. Note that, beginning with any point $\mu$ in $D$ we can choose a finite sequence of guide points $\lambda_1, \cdots, \lambda_m$ such that $|\mu + \sum_{i=1}^{q} \lambda_i| \leqq 2$ for $q = 1, \cdots, m$ and $(\mu + \sum_{i=1}^{m} \lambda_i)$ is in any preassigned quadrant. (If there are only 3 guide points this requires a certain amount of pulling and hauling, but it is all elementary.)

3. Let $\{\mu_i\}$ be any sequence of points which satisfies conditions (1) and (2) in the conclusion of the lemma.

4. Now to define the $\zeta$'s. Start with $\zeta_1 = \mu_1$. Next select a sequence of guide points $\lambda_1, \cdots, \lambda_m$ as in 2, so that $(\mu_1 + \sum_{i=1}^{m_1} \lambda_i)$ is in the quadrant opposite $\mu_2$. Set $\zeta_2, \cdots, \zeta_{m_1+1}$ equal to $\lambda_1, \cdots, \lambda_{m_1}$. Set $\zeta_{m_1+2} = \mu_2$. Next select a set of guide points which move the point

$$\sum_{1}^{m_1+2} \zeta_i = \left(\mu_1 + \sum_{1}^{m_1} \lambda_i + \mu_2\right)$$

into the quadrant opposite $\mu_3$ and continue on in this manner, thus achieving the desired goal.

THEOREM 15. *Let $A$ be a normal operator with no point spectrum, and let $N$ be an arbitrary normal operator. Then $W^*NW \in \mathscr{R}(\Delta_A)^-$ for some unitary $W$ if and only if $0 \in W_e(N) (= \bigcap_{K \in \mathscr{K}} W(N + K)^-)$.*

*Proof.* Assume that $N \in \mathscr{R}(\Delta_A)^-$. Choose a $\lambda_0 \in \boldsymbol{C}$ and an orthonormal sequence $\{f_n\}$ such that $\|(A - \lambda_0)f_n\| \to 0$. By slightly modifying the argument given in [21], we can show that $(Nf_{n_j}, f_{n_j}) \to 0$ as $n_j \to 0$ for some subsequence $\{f_{n_j}\}$. This proves that $0 \in X_e(N)$.

Now assume that $0 \in W_e(N)$. Since $A$ has no point spectrum, $\sigma(A)$ must be uncountable. Using Lemma 13, choose a continuous function $h: \sigma(A) \to \boldsymbol{C}$ such that $h(\sigma(A)) = S^1$ (the unit circle). Then $h(A) = V$ is unitary. Hence $V = U + K$ where $U$ is the bilateral shift and $K$ is compact by the Berg-von Neumann theorem. Note that $\mathscr{R}(\Delta_A)^- \supset \mathscr{R}(\Delta_V)^-$ by [10] page 118 or [1]. Since $\mathscr{R}(\Delta_A)^-$ also contains $\mathscr{K}$ by [23] corollary to Theorem 3, it follows that $\mathscr{R}(\Delta_A)^- \supset \mathscr{R}(\Delta_U)^-$. Thus it suffices to show that $\mathscr{R}(\Delta_U)^-$ contains a unitary copy of $N$. Let $\varepsilon > 0$ be given. Since $0 \in W_e(N)$, we can choose normal operator $N_1$ such that

(1) $\|N - N_1\| < \varepsilon$

(2) $N_1 = \sum_{j=-\infty}^{\infty} \lambda_j(\circ, \varphi_j)\varphi_j$

where the $\lambda_j$'s take on only finitely many values.

(3) $W_e(N_1)$ contains a neighborhood of 0.

Since we are only looking for a unitary copy of $N_1$, we may assume $U$ is defined by $U\varphi_k = \varphi_{k+1}$ for $k = 0, \pm 1, \cdots$. We now set

$$X\varphi_k = a_k\varphi_{k-1}$$

where

$$a_k = \sum_{j=0}^{k-1} \lambda_j \quad \text{for} \quad k \geqq 1$$

$$a_0 = 0$$

and

$$a_k = -\sum_{j=-1}^{k} \lambda_j \quad \text{for} \quad k < 0 .$$

Then $(XU - UX)\varphi_k = (a_{k+1} - a_k)\varphi_k = \lambda_k\varphi_k$ for $k = 0, \pm 1, \cdots$. The "operator" $X$ just defined does precisely what we want but it is not clear that $X$ is bounded. However, if we first rearrange the $\lambda_j$'s (we are only interested in a unitary copy of $N_1$) along the lines of Lemma 14 (treating the forward half and the backward half separately and adjusting multiplicities) then in fact $\|X\| \leqq 2\|N_1\|$. The proof is complete.

COROLLARY 1. *Let $A$ be a normal operator with $\sigma_p(A) = \varnothing$. Let $T$ be a pure isometry. Then $\mathscr{R}(\Delta_A)^-$ contains an operator unitarily equivalent to $T$.*

*Proof.* We know that $\mathscr{R}(\Delta_A)^- \supset \mathscr{R}(\Delta_U)^-$ where $U$ is the bilateral shift. We will show that $\mathscr{R}(\Delta_U)^-$ contains a unitary copy of $V$, where $V$ is the unilateral shift. The general case is then obvious. First we set

$$U = \begin{vmatrix} V^* & 0 \\ F & V \end{vmatrix}$$

where $F$ is a one dimensional operator. By modifying the techniques of the theorem it is easy to see that $\mathscr{R}(\Delta_V)^-$ contains a unitary copy of $V$ (let $B = \operatorname{diag}(1/2, -1/2, 1/2, \cdots)$ and consider $VB - BV$) and $\mathscr{R}(\Delta_{V^*})^-$ contains a unitary operator $W$. Since $\mathscr{R}(\Delta_U)^- \supset \mathscr{K}$ by [23], and $V \oplus W$ is unitarily equivalent $V \bmod \mathscr{K}$ (see [5]) it follows that $\mathscr{R}(\Delta_U)^-$ contains $V$, from whence the general case follows.

COROLLARY 2. *Let $W$ be a pure isometry. Let $N$ be a normal operator. Then $\mathscr{R}(\Delta_W)^-$ contains a unitary copy of $N$ if and only if $0 \in W_e(N)$. Moreover $\mathscr{R}(\Delta_W)^-$ contains a unitary copy of every pure isometry.*

*Proof.* Observe that $\{W\}'$ contains no trace class operators and hence $\mathscr{R}(\Delta_W)^- \supset \mathscr{K}$ by [23]. The rest of the proof follows as above and we omit it.

REMARK. Let $A$ be normal. The first part of the proof of the theorem can be modified to show that if $T \in \mathscr{L}(\mathscr{H})$ and $W_e(T) \neq \{0\}$, then $\mathscr{R}(\Delta_A)^-$ does *not* contain the set

$$\{W^*TW : W \quad \text{unitary}\}.$$

In particular, if $N$ is a noncompact normal operator, then $\mathscr{R}(\Delta_A)^-$ does not contain all unitarily equivalent copies of $N$ for any $A$ normal.

In Theorem 15, the condition $\sigma_p(A) = \emptyset$, implies that the spectrum of $A$ is uncountable. This hypothesis is not completely gratuitous as the next example shows.

EXAMPLE. Let $E = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}$ on $\mathscr{H} \oplus \mathscr{H}$ (i.e., $E$ is a projection). Let $N = \begin{vmatrix} 2 & 0 \\ 0 & -1 \end{vmatrix}$. Then $0 \in W_e(N)$ but we claim $W^*NW \notin \mathscr{R}(\Delta_E)^-$ for any unitary $W$. Indeed $\mathscr{R}(\Delta_E)$ is closed and if $A \in \mathscr{R}(\Delta_E)$ then $A = \begin{vmatrix} 0 & C \\ B & 0 \end{vmatrix}$. If we assume $A$ is self adjoint then $A = \begin{vmatrix} 0 & B^* \\ B & 0 \end{vmatrix}$. Let $U = 2^{-1/2}\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$. Then $U^*AU = \begin{vmatrix} D & 0 \\ 0 & -D \end{vmatrix}$, whence $\sigma(A) = -\sigma(A)$. Since $N$ does not enjoy this last property we conclude that $W^*NW \notin \mathscr{R}(\Delta_E)$.

**6. Irreducible $D$-symmetric operators.** An operator $T \in \mathscr{L}(\mathscr{H})$ is essentially normal if $T^*T - TT^*$ is compact. Joel Anderson has shown that the unilateral shift of infinite multiplicity is $D$-symmetric and this seems to be the only known example of a $D$-symmetric operator

which is not essentially normal. Of course it is far from reducible. (The shifts in Theorem 12 are also examples but they are mildly disguised versions of Anderson's example.) In this section we wish to present an example of an irreducible $D$-symmetric operator which is *not* essentially normal, the first such to the best of our knowledge.

Before getting to the example itself we will need one lemma.

LEMMA 16. *Let $T_j \in \mathscr{L}(\mathscr{H})$ for $j=1,\cdots,n$. Assume that $\|T_j\| \leqq M$ for each $j$. Assume also that there exist mutually orthogonal subspace $\mathscr{H}_1, \cdots, \mathscr{H}_n \subset \mathscr{H}$ such that $T_j|\mathscr{H}_j^{\perp} = 0$ for $j = 1, \cdots, n$. If $T = \sum_{j=1}^n T_j$ then $\|T\| \leqq n^{1/2}M$.*

*Proof.* Let $f \in \mathscr{H}$ where $\|f\| = 1$. Then $f = g \oplus \sum_{j=1}^n a_j f_j$ where $f_j \in \mathscr{H}_j$ and $\|f_j\| = 1$. Thus

$$\begin{aligned}\|Tf\| &= \left\|\sum_{j=1}^n a_j T_j f_j\right\| \\ &\leqq \left[\sum_{j=1}^n |a_j|^2\right]^{1/2}\left[\sum_{j=1}^n \|T_j f_j\|^2\right]^{1/2} \\ &\leqq n^{1/2}M\,.\end{aligned}$$

EXAMPLE 17. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for $\mathscr{H}$ and define $S$ as follows

$$Se_n = \begin{cases} 2e_{n+1} & \text{for} \quad n = 2^k \quad k = 1, 2, \cdots \\ 2^{-(1/2k-1)}e_{n+1} & \text{for} \quad 2^k < n < 2^{k+1}\,. \end{cases}$$

The operator $S$ is reminescent of the operators in §4; it is a shift operator and $2^{-1}\|f\| \leqq \|S^k f\| \leqq 2\|f\|$ for all $f \in \mathscr{H}$ and $k = 1, 2, \cdots$. Thus $S$ is similar to the unilateral shift, which ensures its irreducibility. It is easy to see $S$ is not essentially normal. It is clear that $S$ does not commute with a trace class operator and hence $R(\Delta_S)^- \supset \mathscr{K}$ by [23]. It remains to show $S$ is $D$-symmetric. Let $SX - XS = W$ and let $\varepsilon > 0$ be given. Without loss of generality assume $\|X\| = 1$. We wish to find a $Y$ such that $\|[S^*, Y] - W\| < \varepsilon$. We start with $-SXS$. Thus

$$[S^*, -SXS] = W + (1 - S^*S)XS - SX(1 - SS^*)\,.$$

We next show that $SX(1 - SS^*) \in R(\Delta_{S^*})^-$. Note that $(1 - SS^*)e_n = a_n e_n$ where

$$a_n = \begin{cases} 1 & \text{for} \quad n = 0 \\ -3 & \text{for} \quad n = 2^k + 0 \quad k = 1, 2, \cdots \\ 1 - 2^{-(2/2k-1)} & \text{for} \quad 2^k + 1 < n \leqq 2^{k+1}\,. \end{cases}$$

It is easy to see that $|1 - 2^{-(2/2k-1)}| \leqq 2 \cdot 2^{-k}$. Fix $m = 2^{k_0}$ (to be chosen later) and set

$$K = \text{projection on } \{e_1, \cdots, e_m\}$$
$$P = \text{projection on } \{e_n : n = 2^k + 1, k \geqq k_0\}$$

and

$$E = 1 - (K + P) .$$

Note that $E$, $K$, $P$ are orthogonal and $\|(1 - SS^*)E\| \leqq 2m^{-1}$. Set $R = (1 - SS^*)XS$. Then $R = RK + RE + RP$. The term $RK$ is compact, hence in $R(\Delta_{S^*})^-$; $\|RE\| \leqq 4m^{-1}$ so this term can be ignored. It remains to show $RP \in R(\Delta_{S^*})^-$. Choose $n$ so large that $8n^{1/2} < \varepsilon$ and fix $m > n$.

Set

$$Y_1 = \sum_{j=0}^{n-1} \frac{n - j}{n} [S(S^*S)^{-1}]^{j+1} RPS^{*j} .$$

Then

$$[S^*, Y_1] = RP + n^{-1} \sum_{j=1}^{n} [S(S^*S)^{-1}]^j RPS^{*j} .$$

Note that $\|[S(S^*S)^{-1}]^j\| \leqq 4$ for all $j$. We wish to estimate the term on the right. It can not be handled as in Theorem 12, since every summand has an infinite number of nonzero columns. Observe however that $PS^{*j}e_i = 0$ for $i \neq 2^k + 1 - j$. Thus we set

$$\mathscr{H}_j = sp\{e_i : i = 2^k + 1 - j \quad \text{for} \quad k \geqq k_0\}$$

and

$$T_j = [S(S^*S)^{-1}]^j RPS^{*j} \quad \text{for} \quad j = 1, 2, \cdots, n .$$

It is easy to see that $T_j | \mathscr{H}_j^{\perp} = 0$ and $\|T_j\| \leqq 8$. The $\mathscr{H}_j$'s are orthogonal since $|2^{k_1} - 2^{k_2}| > n$ for $k_1, k_2 > k_0$ by our choice of $k_0$. Thus it follows from Lemma 10 that

$$\left\| \sum_{j=1}^{n} [S(S^*S)^{-1}]^j RPS^{*j} \right\| \leqq 8n^{1/2}$$

whence $\|[S^*, Y_1] - RP\| \leqq 8n^{1/2} < \varepsilon$. This shows that $SX(1 - SS^*) \in R(\Delta_S)^-$. The term $(1 - S^*S)XS$ can be handled in the same way (see the proof of Theorem 12 for additional details). This completes the proof that $W \in R(\Delta_{S^*})^-$ whence $R(\Delta_S)^- \subset R(\Delta_{S^*})^-$. The reverse inequality is proved in exactly the same way.

REMARK. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for $\mathscr{H}$ and define $T$ as follows

$$Te_n = \begin{cases} 2e_{n+1} & \text{for } n = 2^k,\ k = 1, 2, \cdots \\ e_{n+1} & \text{for } 2^k < n < 2^{k+1}. \end{cases}$$

Then $T$ is a somewhat more presentable version of $S$. Since $T$ does not commute with a trace class operator, $R(\Delta_T)^- \supset \mathscr{K}$ by [23]. Clearly $T - S \in \mathscr{K}$. Thus $R(\Delta_T)^- = R(\Delta_S)^-$ and hence is $D$-symmetric. One might wonder why we did not start with $T$ in the first place since it is the more attractive candidate. Unfortunately $T$ is not power bounded and hence the proof above can not be applied directly.

APPENDIX. We wish to define decomposable operator and to do so we must first define spectral maximal subspace. However we never use the special property of spectral maximal subspaces as opposed to those of ordinary subspaces so the reader may skip this if he desires.

DEFINITION. A subspace $\mathscr{X}$ of $\mathscr{H}$ is a spectral maximal subspace for $T$ if

(1) $\mathscr{X}$ is invariant for $T$

and

(2) If $\mathscr{Y}$ is any other invariant subspace for $T$ with $\sigma(T|\mathscr{Y}) \subset \sigma(T|\mathscr{X})$. Then $\mathscr{Y} \subset \mathscr{X}$.

DEFINITION. An operator $T \in \mathscr{L}(\mathscr{H})$ is decomposable if for every open cover $\{G_i\}_1^n$ of $\sigma(T)$ there exist spectral maximal subspaces $\mathscr{X}_i$ such that

$$\sigma(T|\mathscr{X}_i) \subset G_i \quad \text{for} \quad i = 1, \cdots, n$$

and $\sum \mathscr{X}_i = \mathscr{H}$.

Further information on decomposable operators may be found in [6]. I am grateful to M. Radjabalipour for a suggestion concerning the next

THEOREM. *Let $A$ be a decomposable operator and $T$ an arbitrary operator in $\mathscr{L}(\mathscr{H})$. Assume there exists an open cover $\{G_\alpha\}$ of $\sigma(A) \cap \rho(T)$ where $G_\alpha \subset \sigma(A) \cap \rho(T)$ for each $\alpha$, and maximal spectral subspaces $\mathscr{X}_\alpha$ such that $\sigma(T|\mathscr{X}_\alpha) \subset G_\alpha$ and $V\mathscr{X}_\alpha = \mathscr{H}$. Then the equation $YA = TY$ has only the solution $Y = 0$.*

*Proof.* Assume $YA = TY$ and fix an $\mathscr{X}_\alpha$. Thus $\sigma(A|\mathscr{X}_\alpha) = F \subset G_\alpha \subset \rho(T)$. Let $u \in \mathscr{X}_\alpha$. Set $f(\lambda) = Y(A - \lambda)^{-1}u$ for $\lambda \in F'$ and observe that $f$ is analytic on $F'$. Set $g(\lambda) = (T - \lambda)^{-1}Yu$ and observe that $g$ is analytic for $\lambda \in \rho(T)$.

*Claim.* $f$ and $g$ coincide on $F' \cap \rho(T)$. For $\zeta \in F' \cap \rho(T)$ note that $(T-\zeta)[f(\zeta)-g(\zeta)] = [(T-\zeta)Y](A-\zeta)^{-1}u - Yu = Yu - Yu = 0$. Since $\zeta \in \rho(T)$ it follows that $f(\zeta) = g(\zeta)$. Thus $g$ has a bounded analytic extension to the entire plane. Since $g$ vanishes at $\infty$, $g$ must be identically zero whence $Yu = 0$. Since $\alpha$ was arbitrary, and $V\mathscr{K}_\alpha = \mathscr{H}$ it follows that $Y = 0$.

EXAMPLE. Note that in the context of the previous theorem it does not suffice to merely assume that $\sigma_p(T) \cap \sigma(A) = \varnothing$. Let $\{f_j\}_{-\infty}^{\infty}$ be an orthonormal basis for $\mathscr{H}$. Let $A$ be the bilateral shift; thus $Af_j = f_{j+1}$ for all $j$. Let

$$Tf_j = \begin{cases} 0 & j \leqq 0 \\ f_{j+1} & j \geqq 1 \end{cases}$$

and

$$Xf_j = \begin{cases} 0 & j \leqq 0 \\ f_j & j \geqq 1 \,. \end{cases}$$

Then $AX = XT$ although the point spectrum of $T$ (and $T^*$) does not overlap $\sigma(A)$.

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