KO-EQUIVALENCES AND EXISTENCE OF NONSINGULAR BILINEAR MAPS

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We show how to use homotopy theoretic methods to construct maps between various truncated projective spaces that induce isomorphisms in KO cohomology theory. We then use these maps to establish the existence of new families of nonsingular bilinear maps.

1. **Introduction.** Bilinear maps $f: R^a \times R^b \to R^c$ with the *nonsingular property* that $f(x, y) = 0$ implies $x = 0$ or $y = 0$ have been of interest for several reasons: (1) they generalize the multiplication map of the classical division algebras over R; (2) they provide estimates for the geometric dimension of vector bundles over real projective spaces [10], and are hence instrumental in the study of immersions of such spaces into R^n; (3) those maps with the additional property that $|f(x, y)| = |x| \cdot |y|$ can sometimes be used to produce essential harmonic maps between Euclidean spheres, in the sense of Eells and Sampson [5]. See [14] and [9, Theorem 4.2].

Furthermore, it has been realized for some time that there is an interesting relationship between nonsingular bilinear maps and stable homotopy theory [8], [9]. The purpose of this article is to further explore that relationship. Whereas [9] deals with *known* bilinear maps and their implications in homotopy theory, the present paper describes, in § 5, how homotopy theory could in turn be used to establish the existence of *new* families of nonsingular bilinear maps. These new families are distinct from the classical examples of Hurwitz and Radon [7], and yet exhibit the same "Clifford periodicity" phenomenon which is characteristic of the Hurwitz-Radon family.

Since the Hurwitz-Radon family can be used to produce essential harmonic maps between Euclidean spheres, it would be interesting to ask whether there exist families of nonsingular bilinear maps, occurring in the same dimension ranges as the ones established in this paper, that will yield further examples of essential harmonic maps.

The main homotopy tools employed in the paper are certain maps between truncated projective spaces called KO-equivalences. Roughly, a KO-equivalence is a map inducing an isomorphism in KO-cohomology. The methods in §§ 3, 4 for constructing KO-equivalences follow closely the techniques of [6, § 4]. Such constructions might have some independent interest, and could perhaps be read on their own right.

2. **Some homotopy lemmas.** A finite connected CW complex X in which all cells apart from the base point occur in dimensions between m and n is written $X = X_m^n$. A typical example is the real truncated projective space $P_m^n = P^n/P^{m-1}$. We call X_m^n *stable* if $n \leq 2m - 1$, in which case X_m^n is a suspension. We say that X is *2-torsion*, if $\tilde{H}^q(X; Z)$ is a 2-torsion group for all q. We shall also need the following standard notations:

$\mathscr{A}$ = the mod 2 Steenrod algebra;

$\mathscr{A}_0$ = the subalgebra of $\mathscr{A}$ generated by Sq^1;

$\phi(n)$ = the number of integers in the interval $[1, n]$ congruent to $0, 1, 2$ or 4 modulo 8.

LEMMA 2.1. *Suppose $X = X_m^n$ is a stable 2-torsion complex and that $\tilde{H}^*(X; Z_2)$ is free over $\mathscr{A}_0$. If $f: X \to X$ can be factored into a composite $f_p \cdots f_2 f_1$, where $p > \phi(n - m)$ and each f_i is a map $X \to X$ with induced homomorphism $f_i^* = 0$ on $\tilde{H}^*(X; Z_2)$, then f is homotopic to zero.*

Proof. Since $\tilde{H}(X; Z_2)$ is $\mathscr{A}_0$-free, one can apply Adams' vanishing theorem [1, Thm. 2.1] to conclude that

$$\mathrm{Ext}_{\mathscr{A}}^{s,t}(\tilde{H}^*(X; Z_2), Z_2) = 0 \text{ for } s > \phi(t - s - m) \ .$$

This means, in particular, that if

$$X = E_0 \longleftarrow E_1 \longleftarrow E_2 \longleftarrow \cdots \longleftarrow E_s \longleftarrow \cdots$$

is a modified Postnikov tower over X modelled on a minimal $\mathscr{A}$-free resolution of $\tilde{H}^*(X; Z_2)$, as explained in [13, Chapter 18], then each k-invariant of dimension $\leq n$ is located in $H^*(E_s; Z_2)$ for some $s \leq \phi(n - m)$. On the other hand, the hypothesis on f_i^* allows one to prove inductively on p that $f_p \cdots f_2 f_1$ lifts to E_p. If $p > \phi(n-m)$, then all subsequent k-invariants will have dimension exceeding $n(=\dim X)$, and hence constitute no further obstruction to lifting $f_p \cdots f_2 f_1$ arbitrarily high into the tower. Since the connectivity of E_s eventually increases beyond any bound as $s \to \infty$, such lifting is enough to imply $f \simeq 0$.

COROLLARY 2.2. *Let ι be the identity map of X where X is as in Lemma 2.1. Then the order of ι, considered as an element in the track group $[X, X]$, is a divisor of $2^{\phi(n-m)+1}$.*

COROLLARY 2.3. *Let $k\iota$ denote k times the identity map ι in the track group $[X, X]$. If X is as in Lemma 2.1, and $n - m \leq 7$, then $16\iota \simeq 0$.*

It goes without saying that by taking a closer look into the Postnikov tower of an individual X, it is sometimes possible to improve upon Corollary 2.2. A typical example is given by

PROPOSITION 2.4. *If X is the truncated projective space P_m^{m+5}, where $m \geq 9$ is congruent to 1 modulo 4, then $8\iota \cong 0$.*

The proof is practically the same as Lemma 2.1, except that in the present case, $\mathrm{Ext}_{\mathscr{A}}^{s,t}(H^*(P_m^{m+5}; Z_2), Z_2)$ has somewhat better vanishing properties than enunciated by Adams' vanishing theorem. Alternatively, it is a routine exercise to write down a Postnikov tower for P_m^{m+5} explicitly [13], and to check that $8\iota = (2\iota)(2\iota)(2\iota)$ lifts arbitrarily high into that tower.

Many of Toda's results in [15] on the "order of the identity class" of a suspension space are straightforward consequences of the above results.

3. Construction of KO-equivalences.

DEFINITION 3.1. A map $g: Y \to Z$ between two connected complexes is called a *KO-equivalence* if $g^*: KO(Z) \to KO(Y)$ is an isomorphism.

NOTATION 3.2. If L is a skeleton of a truncated projective space P_m^n, the collapsing map from P_m^n to P_m^n/L will always be denoted by c. The inclusion map $L \to P_m^n$ will be denoted by i, i', i'', j, etc., or by an unmarked arrow.

THEOREM 3.3. *Let m be odd and n be even. If $m + 9 \leq n \leq 2m - 10$, then there exists a KO-equivalence $g: P_{m+8}^{n+8} \to P_m^n$ such that the composition*

$$(*) \qquad P_m^{n+8} \xrightarrow{\ c\ } P_{m+8}^{n+8} \xrightarrow{\ g\ } P_m^n \xrightarrow{\ i\ } P_m^{n+8}$$

is equal to 16ι.

Proof. Note that the hypotheses on m and n guarantee that all complexes appearing in $(*)$ are 2-torsion and stable, and that their mod 2 reduced cohomology rings are $\mathscr{A}_0$-free. If we make $16\iota: P_m^{n+8} \to P_m^{n+8}$ cellular, and then collapse the n-skeleton of both its domain and range to a point, we get a map $P_{n+1}^{n+8} \to P_{n+1}^{n+8}$ which, being 16 times the identity, is trivial on account of Corollary 2.3. It follows that one can homotopically deform 16ι into a map $g_0: P_m^{n+8} \to P_m^n$. Since $m + 7 < n$ by hypothesis, we can further

assume that throughout the course of the deformation, P_m^{m+7} stays within P_m^n. It follows that the composite

$$P_m^{m+7} \xrightarrow{\ i\ } P_m^{n+8} \xrightarrow{\ g_0\ } P_m^n$$

is just a deformation of $P_m^{m+7} \xrightarrow{16\iota} P_m^{m+7} \xrightarrow{i} P_m^n$, which is trivial by Corollary 2.3. Hence g_0 factors through the cofiber P_{m+8}^{n+8} of i, and the resulting map is taken as g in (*).

The $\widetilde{KO}$ groups of the complexes occurring in (*) are finite cyclic and 2-torsion. Since i^* is onto and cokernel c^* has order 16, and $(igc)^*$ is multiplication by 16, g^* must be an isomorphism in KO theory. This concludes the proof.

REMARK 3.4. If m and n satisfy the hypothesis of the theorem, so would $m + 8$ and $n + 8$. Hence there is an infinite sequence of KO-equivalences

$$\cdots \xrightarrow{\ g\ } P_{m+8k}^{n+8k} \xrightarrow{\ g\ } \cdots \xrightarrow{\ g\ } P_{m+16}^{n+16} \xrightarrow{\ g\ } P_{m+8}^{n+8} \xrightarrow{\ g\ } P_m^n .$$

REMARK 3.5. It is possible to use $2^r\iota$, $(r > 4)$, rather than 16ι, to construct KO-equivalences between truncated projective spaces. The approach is similar.

4. Further construction of KO-equivalences. The method of § 3 can sometimes be adapted in order to construct a KO-equivalence $P_{m+8}^{n+8} \to P_m^n$, even if m and n do not satisfy the technical condition $m + 9 \leqq n$ imposed in Theorem 3.3. However, a case by case presentation of such adaptations will be inelegant and unnecessary. For if $n - m = 1, 3, 5$ or 7, P_{m+8}^{n+8} is homeomorphic to $\Sigma^8 P_m^n$ [4]. Now Adams and Luz have a unified procedure for constructing KO-equivalences $\Sigma^N X \to X$ when X is a suitable 2-torsion finite complex and N is a suitable power of 2 [2], [11]. Such a procedure applies nicely when $X = P_m^n$ with $n - m \leqq 7$, m odd, n even, provided that m is not too small.

In this section we therefore confine our efforts to show how the ideas developed in § 3 can be used to prove Theorem 4.1 below. This theorem has previously been obtained by Adams in [2, § 12], but we believe that it is of some interest to compare the two different proofs.

THEOREM 4.1. *For any odd integer* $m \geqq 9$ *there is a* KO-*equivalence* $g \colon P_{m+8}^{m+9} \to P_m^{m+1}$.

Proof. For $m \equiv 1 \pmod 8$ and $m > 9$, we construct g by imita-

ting the proof of Theorem 3.3. Thus we "compress" $16\iota\colon P_m^{m+9} \to P_m^{m+9}$ into a map $g_0\colon P_m^{m+9} \to P_m^{m+1}$, using a cellular homotopy. The following two lemmas imply that g_0 can be choosen such that its restriction to P_m^{m+7} is trivial.

LEMMA 4.2. *For $m \equiv 1 \pmod 8$ and $m > 9$, the compression map g_0 can be chosen such that*

$$P_m^{m+7} \xrightarrow{\ i'\ } P_m^{m+9} \xrightarrow{\ g_0\ } P_m^{m+1} \xrightarrow{\ i\ } P_m^{m+7}$$

is a trivial composite.

LEMMA 4.3. *Assume $m \equiv 1 \pmod 8$ and $m > 9$. Then a map $f\colon P_m^{m+7} \to P_m^{m+1}$ is trivial if and only if its composition with the inclusion map $P_m^{m+1} \xrightarrow{i} P_m^{m+7}$ is trivial.*

Postponing the proofs of these lemmas to the end of the section, we now form the cofiber map $g\colon P_{m+8}^{m+9} \to P_m^{m+1}$ induced by g_0. As in Theorem 3.3, g is a KO-equivalence. In fact, it is not difficult to see that g is at the same time a KU-equivalence and a KS_P-equivalence.

The case $m = 9$ was excluded from the above, because P_m^{m+9} fails to be stable when $m = 9$. To handle this case, begin with a KO-equivalence $g\colon P_{25}^{26} \to P_{17}^{18}$, and consider its restriction $g'\colon S^{25} \to P_{17}^{18}$. The Freudenthal theorem allows one to desuspend g' into a map $g''\colon S^{17} \to P_9^{10}$. But it is not hard to show that $\pi_{17}(P_9^{10})$ is a direct sum of four copies of Z_2, so that g'' extends to a map $\hat{g}\colon P_{17}^{18} \to P_9^{10}$. The eight-fold suspension $\Sigma^8 \hat{g}\colon P_{25}^{26} \to P_{17}^{18}$ agrees with g on the bottom sphere S^{25}, and must therefore be a KO-equivalence because g is such. By Bott periodicity, $\hat{g}$ is also a KO-equivalence.

We conclude by remarking that, in case the odd number k is not congruent to $1 \pmod 8$, a KO-equivalence $P_{k+8}^{k+9} \to P_k^{k+1}$ can still be constructed by suitably suspending one of the previously constructed KO-equivalences $g\colon P_{m+8}^{m+9} \to P_m^{m+1}$, with $m \equiv 1 \pmod 8$.

REMARK 4.4. For a comparison between these KO-equivalences and those constructed by Adams in [2, § 12], the following observation is pertinent. While our construction involves a number of technical homotopy lemmas, it avoids Toda brackets and e-invariant considerations. Moreover, it falls into the general construction pattern of § 3, and yields KO-equivalences $g\colon P_{m+8}^{m+9} \to P_m^{m+1}$ with the *special property* that

$$P_m^{m+9} \xrightarrow{\ c\ } P_{m+8}^{m+9} \xrightarrow{\ g\ } P_m^{m+1} \xrightarrow{\ i\ } P_m^{m+9}$$

is exactly equal to 16ι.

It is now time to prove Lemmas 4.2 and 4.3. The proof of 4.2 is based on the factorization $16\iota = (2\iota)(8\iota)$. Consider the following homotopy commutative diagram

$$
\begin{array}{ccccc}
 & & P_m^{m+1} & & \\
 & \nearrow^{g_2} & \downarrow & \searrow^{i} & \\
 & P_m^{m+3} \xrightarrow{2\iota} & P_m^{m+3} & \longrightarrow & P_m^{m+7} \\
 \nearrow^{g_1} & \downarrow & & \nearrow & \\
P_m^{m+7} \xrightarrow{i'} P_m^{m+9} \xrightarrow{8\iota} & P_m^{m+9} \xrightarrow{2\iota} & P_m^{m+9} & &
\end{array}
$$

Here, g_1 exists because 8ι followed by the collapsing map $c: P_m^{m+9} \to P_{m+4}^{m+9}$ is trivial (cf. Proposition 2.4). There are two obstructions to compressing 2ι into the map g_2. The first is $(2\iota)^*x$, where x generates $H^{m+2}(P_m^{m+3}; Z_2)$. This obstruction obviously vanishes. The second obstruction is the functionalization of Sq^2 on x with respect to 2ι. This vanishes because of indeterminacy.

We now pick $g_0 = g_2 g_1$ to be our compression of 16ι, and proceed to show that $ig_2 g_1 i'$ is trivial. To this end, notice that there is an alternative way to factorize $ig_2 g_1 i'$, as given in the following commutative diagram:

$$
\begin{array}{ccccccc}
P_m^{m+7} \xrightarrow{i'} & P_m^{m+9} \xrightarrow{g_1} & P_m^{m+3} \xrightarrow{g_2} & P_m^{m+1} & \longrightarrow & P_m^{m+3} \\
\downarrow^{8\iota=(2\iota)^3} & & \downarrow & & \searrow^{i} & \downarrow \\
 & & P_m^{m+7} \xrightarrow{\quad 2\iota \quad} & & P_m^{m+7} & \\
 & & \downarrow^{i_0} & & \nearrow^{h} & \\
P_m^{m+7} \xrightarrow{\quad i_0 \quad} & & P_m^{m+8} \xrightarrow{\quad i'' \quad} & & P_m^{m+9} &
\end{array}
$$

Since the two ways of going around the big square become homotopic when composed with i'', the big square itself is homotopy commutative for dimensional reasons. The dotted map h exists because P_m^{m+8} is obtained from P_m^{m+7} via an attaching map of order 2 (cf. [6, p. 58, bottom diagram]). Commutativity now gives $ig_2 g_1 i' = hi_0(8\iota) = (2\iota)(8\iota) = 16\iota$, which is therefore trivial by Theorem 2.1.

The proof of Lemma 4.2 can be accomplished by showing that the corresponding dual statement is true. The Spanier-Whitehead duals of P_m^{m+7} and P_m^{m+1} can be taken as P_t^{t+7} and P_{t+6}^{t+7} respectively,

where $t + m + 8$ is a large power of 2 (see [4]), so that $t \equiv 7 \pmod{8}$ in the present context. From the hypothesized triviality of the composite

$$P_t^{t+7} \xrightarrow{c} P_{t+6}^{t+7} \xrightarrow{Df} P_t^{t+7} ,$$

we must now argue for the triviality of Df, the dual of f. It suffices to show that Df lifts arbitrarily highly into the modified Postnikov tower

$$P_t^{t+7} \longleftarrow E_1 \longleftarrow E_2 \longleftarrow E_3 \longleftarrow \cdots$$

over P_t^{t+7}. The following lists all the k-invariants of dimension $\leq t + 7$ arising in this tower, together with their defining relations, compare [12, p. 60]. (We use a subscript to indicate the dimension of a k-invariant.)

(0) primary k-invariant:

the generator $x_t \in H^t(P_t^{t+7}; Z_2)$;

(1) k-invariants in $H^*(E_1; Z_2)$:

$$k_{t+2}: Sq^2 Sq^1 x_t = 0 ,$$

$$k_{t+4}: Sq^4 Sq^1 x_t = 0 ,$$

$$k_{t+5}: Sq^4 Sq^2 x_t = 0 ,$$

$$k_{t+7}: Sq^8 x_t = 0 ;$$

(2) k-invariants in $H^*(E_2; Z_2)$:

$$\kappa_{t+3}: Sq^2 k_{t+2} = 0$$

$$\kappa_{t+4}: Sq^1 k_{t+4} + Sq^2 Sq^1 k_{t+2} = 0 ,$$

$$\kappa_{t+7}: Sq^4 k_{t+4} + Sq^6 k_{t+2} = 0 ;$$

(3) k-invariant in $H^*(E_3; Z_2)$:

$$\theta_{t+4}: Sq^1 \kappa_{t+4} + Sq^2 \kappa_{t+3} = 0 .$$

With this list, it is no longer difficult to check that if any of these k-invariants constitute an obstruction to trivializing Df, then it is actually an obstruction to trivializing $(Df)c$. In this way Lemma 4.3 is obtained.

5. Existence of nonsingular bilinear maps. It has been remarked by M. Mahowald that KO-equivalences can be used to estimate the geometric dimension of vector bundles over projective

spaces, via a procedure which is also known to J. F. Adams. Specifically, consider the following diagram:

$$(\#)$$

$$
\begin{array}{ccccccc}
P^{n+8k} & & & & P^{n+8} & P^n & BO(d) \\
\downarrow c & & & & \downarrow c & \downarrow c \; \zeta \nearrow & \downarrow \\
\xrightarrow{\;g\;} P^{n+8k}_{m+8k} \xrightarrow{\;g\;} & \cdots & \xrightarrow{\;g\;} & P^{n+8}_{m+8} \xrightarrow{\;g\;} & P^n_m & \xrightarrow{\;\zeta\;} & BO
\end{array}
$$

where m, n are fixed integers, the g's are KO-equivalences, ζ represents the generator of $\widetilde{KO}(P^n_m)$, and d is the geometric dimension of ζ, that is, the smallest integer for which a lifting $\hat{\zeta}$ of ζ into $BO(d)$ exists. According to the computation of [3], $\widetilde{KO}(P^n_m)$ is cyclic of order $2^{\phi(m,n)}$, where $\phi(m, n) = \phi(n) - \phi(m - 1)$. It follows that $\zeta g^k c = \zeta(g \cdots g)c$ can be interpreted as a stable vector bundle over P^{n+8k} represented by $(2^{\phi(n+8k)}/2^{\phi(m,n)})\xi = 2^{4k+\phi(m-1)}\xi$, where ξ denotes the Hopf line bundle. This stable bundle has geometric dimension $\leq d$, due to the existence of $\hat{\zeta}$. As long as $2^{\phi(m-1)} > n$, the fact can be re-stated as:

($\dagger$) $2^{4k+\phi(m-1)}\xi$ over P^{n+8k} has at least $2^{4k+\phi(m-1)} - d$ everywhere independent sections.

Notice that d depends on m and n alone, and is independent of k.

To convert existence of sections into existence of nonsingular bilinear maps, we invoke the result of [8], which asserts that given ($\dagger$), there is a (large) power of 2, say N_0, such that for any positive multiple qN_0, a nonsingular bilinear map $R^a \times R^{c-d} \to R^c$ exists with $a = 8k + n + 1$ and $c = 2^{4k+\phi(m-1)} + qN_0$ respectively. Letting k run through $0, 1, 2, \cdots, l$ and picking N to be the maximum of the N_0's so incurred, we obtain

THEOREM 5.1. *Let m, n be fixed integers satisfying $2^{\phi(m-1)} > n$, such that there exists a sequence of KO-equivalences*

$$
\cdots \xrightarrow{\;g\;} P^{n+8k}_{m+8k} \xrightarrow{\;g\;} \cdots \xrightarrow{\;g\;} P^{n+8}_{m+8} \xrightarrow{\;g\;} P^n_m \, .
$$

Then for every $l > 0$ there is a power of 2, denoted by N, corresponding to which there exists a family of nonsingular bilinear maps

$$
\phi_k \colon R^{8k+n+1} \times R^{2^{4k+\phi(m-1)}+N-d} \longrightarrow R^{2^{4k+\phi(m-1)}+N} \, ,
$$

with $k = 0, 1, \cdots, l$. Here d depends only on m and n, and is given by $(\#)$ above.

Let $a = 8k + n + 1$, $c = 2^{4k+\phi(m-1)} + N$ and $b = c - d$. The numerical features of the family $\phi_k \colon R^a \times R^b \to R^c (0 \le k \le l)$ becomes striking if we recall that the classical Hurwitz-Radon maps constitute a nonsingular family

$$\Psi_k \colon R^\alpha \times R^\beta \longrightarrow R^\gamma$$

where $\alpha = 8k + 2^\varepsilon$, $\beta = \gamma = 2^{4k+\varepsilon}$, with $\varepsilon = 0, 1, 2$ or 3 and k arbitrary. This family has the "Clifford periodicity" in that as β and γ are multiplied by 16, α increases by 8. If, by ignoring the constants N and d, we introduce $a' = 8k + n + 1$, $b' = 2^{4k+\phi(m-1)}$, $c' = 2^{4k+\phi(m-1)}$ and call (a', b', c') the *essential ranges* of the family $\{\phi_k\}_{0 \le k \le l}$, then we can summarize Theorem 5.1 above by saying that the essential ranges of $\{\phi_k\}$ exhibit the same Clifford periodicity as the dimension ranges (α, β, γ) of Hurwitz-Radon maps.

As an illustration let us specify diagram (♯) to the case $m = 9$ and $n = 10$, using the KO-equivalences g constructed in § 4. Using $\pi_9(BO(7)) = Z_2 \oplus Z_2$, it is not difficult to show that the generator $\zeta \colon P_9^{10} \to BO$ lifts (optimally) to $BO(7)$. Consequently we take $d = 7$ in (♯) to obtain

Special case 5.2. For every $l > 0$ there is a power of 2, denoted by N, corresponding to which there exists a family of nonsingular bilinear maps

$$\phi_{h-1} \colon R^{8h+3} \times R^{2^4h+N-7} \longrightarrow R^{2^4h+N}$$

for $h = 1, 2, \cdots, l$.

Here for simplicity we have introduced h to stand for $k + 1$ in Theorem (5.1).

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