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ON A THEOREM OF MURASUGI

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1. Let $l=k_1 \cup k_2$ be a 2-component link in S^3, with k_2 unknotted. The 2-fold cover of S^3 branched over k_2 is again S^3; let $k_1^{(2)}$ be the inverse image of k_1, and suppose that $k_1^{(2)}$ is connected. How are the signatures $\sigma(k_1)$, $\sigma(k_1^{(2)})$ of the knots k_1 and $k_1^{(2)}$ related? This question was considered (from a slightly different point of view) by Murasugi, who gave the following answer [Topology, 9 (1970), 283–298].

THEOREM 1 (*Murasugi*).

$$\sigma(k_1^{(2)})=\sigma(k_1)+\xi(l)\ .$$

Recall [4] that the invariant $\xi(l)$ is defined by first orienting l, giving, an oriented link $\bar{l}$, say, and then setting $\xi(l)=\sigma(\bar{l})+Lk(\bar{k}_1, \bar{k}_2)$, where σ denotes signature and Lk linking number.

In the present note we shall give an alternative, more conceptual, proof of Theorem 1, and in fact obtain it as a special case of a considerably more general result.

The idea of our proof is the following. If $l = l_1 \cup l_2$ is a link, partitioned into two sublinks l_1 and l_2, then the 2-fold branched covers over l_1, l_2, and the whole of l, are all quotients of a $\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$-cover branched over l. After possibly multiplying by 2, the diagram consisting of these branched covers bounds a corresponding diagram of 4-manifolds, and the signatures of the various links involved are expressible in terms of the signatures of these 4-manifolds (and the euler numbers of the branch sets); see e.g., [3]. The result is then a consequence of a relation among these 4-manifold signatures (Lemma 1).

This more general setting requires that we consider links in 3-manifolds other than homology spheres; in § 2 we discuss the signature in this context. (It becomes necessary to prescribe a particular 2-fold branched cover. However, we sacrifice some generality inasmuch as we restrict ourselves to oriented, null-homologous links: it would otherwise be necessary to prescribe a framing of the link as well.) In § 3 we set up the diagram of covering spaces, and in § 4 derive the relation between the signatures of the manifolds therein. Section 5 contains some consequences of this, including the appropriate generalization of Theorem 1.

All manifolds of dimensions 3 and 4 are to be oriented; manifolds of dimensions 1 and 2 are oriented only when this is explicitly stated, and those of dimension 2 need not even be orientable. We make no assumptions on the connectedness of our manifolds. If $\bar{l}$ is an oriented link, we denote the underlying nonoriented link by l.

2. Let $\bar{l} = \bar{k}_1 \cup \cdots \cup \bar{k}_m$ be an oriented link in a closed 3-manifold M, and suppose $\bar{l}$ is null-homologous. Let W be a 4-manifold and F a surface in W such that $\partial(W, F) = (M, l)$. Let F' be (the image of) a section of the normal S^1-bundle of F in W, with $\partial F' = l' = k'_1 \cup \cdots \cup k'_m$, say. Orient l' to obtain $\bar{l}' = \bar{k}'_1 \cup \cdots \cup \bar{k}'_m$ by requiring $\bar{k}'_i \sim \bar{k}_i$ in a tubular neighborhood of k_i, and define $\bar{e}(F) = -Lk(\bar{l}, \bar{l}')$. (Note that this is well-defined as $\bar{l}, \bar{l}'$ are both null-homologous in M.)

Now let $p: \tilde{M} \to M$ be some 2-fold covering of M branched along l, and suppose that p extends to a 2-fold covering $\tilde{W} \to W$ branched along F. Then

$$\sigma(\bar{l}, p) = \sigma(\tilde{W}) - 2\sigma(W) + \frac{1}{2}\bar{e}(F)$$

depends only on $\bar{l}$ and p. (If (W_1, F_1) and (W_2, F_2) are two pairs as above, apply the G-signature theorem [1] to the resulting involution on the closed 4-manifold $\tilde{W}_1 \cup_\partial -\tilde{W}_2$, together with Novikov additivity and the fact that the euler number of the normal bundle of $F_1 \cup F_2$ in $W_1 \cup -W_2$ is equal to $\bar{e}(F_1) - \bar{e}(F_2)$.)

We remark that if M is a homology sphere, p is unique, and $\sigma(\bar{l}, p)$ is just the signature of $\bar{l}$. Again, we may take l to be the empty link; $-\sigma(\phi, p)$ is the α-invariant [2] of the nontrivial covering translation of $\tilde{M}$.

3. Let l_1, l_2 be disjoint links in a 3-manifold M, and write $l = l_1 \cup l_2$. Let $\alpha: H_1(M - l) \to \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$ be a homomorphism which sends each meridian of l_1 (resp. l_2) to the nontrivial element of the first (resp. second) $\boldsymbol{Z}_2$. Let W be a 4-manifold and F_1, F_2 disjoint surfaces in W such that $\partial(W, F_1, F_2) = (M, l_1, l_2)$. Write $F = F_1 \cup F_2$, and suppose there exists a homomorphism $\beta: H_1(W - F) \to \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$ such that $\alpha = \beta i_*$, where $i: M - l \to W - F$ is inclusion. (We shall discuss this assumption later.)

Let $\tilde{W} \to W$ be the branched covering associated with β. The covering translations induce a $\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$-action on $\tilde{W}$. Let g_1 generate the second $\boldsymbol{Z}_2$ factor, g_2 the first, and let $g_3 = g_1 g_2$ be the remaining nontrivial element. Setting $W^{(i)} = \tilde{W}/(g_i)$, $i = 1, 2, 3$, we have the following commutative diagram of 2-fold branched coverings.

$$\begin{tikzcd}
& \widetilde{W} \arrow[dl, "q_2^{(1)}"'] \arrow[d, "\tilde{q}"] \arrow[dr, "q_1^{(2)}"] & \\
W^{(1)} \arrow[dr, "q_1"'] & W^{(3)} \arrow[d, "q"] & W^{(2)} \arrow[dl, "q_2"] \\
& W &
\end{tikzcd}$$

Here q_i is branched over F_i, $i = 1, 2$, and q is branched over F. If $F_1^{(2)}$ and $F_2^{(1)}$ are the inverse images of F_1 in $W^{(2)}$ and F_2 in $W^{(1)}$, then $q_1^{(2)}$, $q_2^{(1)}$ are branched over $F_1^{(2)}$, $F_2^{(1)}$ respectively. Finally, $\tilde{q}$ is unbranched.

Now suppose that l_1 and l_2 can be oriented to obtain null-homologous links $\bar{l}_1$ and $\bar{l}_2$ respectively. Let $\bar{l} = \bar{l}_1 \cup \bar{l}_2$. There are induced orientations of $l_1^{(2)} = \partial F_1^{(2)}$ and $l_2^{(1)} = \partial F_2^{(1)}$, giving null-homologous links $\bar{l}_1^{(2)}$ and $\bar{l}_2^{(1)}$ in $\partial W^{(2)}$ and $\partial W^{(1)}$ respectively.

Writing p's instead of q's to denote the restrictions of these coverings to the appropriate boundaries, we have the equations

$$\text{(i)} \qquad \sigma(\bar{l}_1, p_1) = \sigma(W^{(1)}) - 2\sigma(W) + \frac{1}{2}\,\bar{e}(F_1)$$

$$\text{(ii)} \qquad \sigma(\bar{l}_2, p_2) = \sigma(W^{(2)}) - 2\sigma(W) + \frac{1}{2}\,\bar{e}(F_2)$$

$$\text{(iii)} \qquad \sigma(\bar{l}, p) = \sigma(W^{(3)}) - 2\sigma(W) + \frac{1}{2}\,\bar{e}(F)$$

$$\text{(iv)} \qquad \sigma(\bar{l}_1^{(2)}, p_1^{(2)}) = \sigma(\widetilde{W}) - 2\sigma(W^{(2)}) + \frac{1}{2}\,\bar{e}(F_1^{(2)})$$

$$\text{(v)} \qquad \sigma(\bar{l}_2^{(1)}, p_2^{(1)}) = \sigma(\widetilde{W}) - 2\sigma(W^{(1)}) + \frac{1}{2}\,\bar{e}(F_2^{(1)})$$

$$\text{(vi)} \qquad \sigma(\phi, \tilde{p}) = \sigma(\widetilde{W}) - 2\sigma(W^{(3)})\ .$$

We now consider the question of the existence of a suitable homomorphism β. Suppose $H_1(W; \boldsymbol{Z}_2) = 0$. Then (see [3, § 1]) the cohomology exact sequence of the pair $(W, W-F)$, together with duality, gives an exact sequence

$$\begin{array}{c}
0 \longrightarrow H^1(W - F; \boldsymbol{Z}_2) \longrightarrow H_2(F, \partial F; \boldsymbol{Z}_2) \longrightarrow H_2(W, \partial W; \boldsymbol{Z}_2) \\
\wr\| \\
H^0(F; \boldsymbol{Z}_2)\ .
\end{array}$$

The existence of $\beta\colon H_1(W - F) \to \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$ taking a meridian of F_i to the nontrivial element of the ith $\boldsymbol{Z}_2$, $i = 1, 2$, is then seen to be equivalent to the condition that $[F_i, \partial F_i] = 0 \in H_2(W, \partial W; \boldsymbol{Z}_2)$, for $i = 1, 2$. (In particular, the assertion $H^1(W - F; \boldsymbol{Z}_2) \cong H^0(F; Z_2)$ in [3, p. 353] is incorrect.)

Now suppose, in addition, that $H_1(M; \boldsymbol{Z}_2) = 0$. Then β will automatically satisfy $\alpha = \beta i_*$. But β will not in general exist, for it is clear that if $[F_i, \partial F_i] = 0 \in H_2(W, \partial W; \boldsymbol{Z}_2)$, $i = 1, 2$, then $Lk_{Z_2}(l_1, l_2) = 0$. However, this condition is also sufficient; that is, given links $l_1, l_2 \subset M$ such that $H_1(M; \boldsymbol{Z}_2) = 0$ and $Lk_{Z_2}(l_1, l_2) = 0$, there exist W, F_1, F_2, β as above. To see this, let W be any 4-manifold with $\partial W = M$ and $H_1(W; \boldsymbol{Z}_2) = 0$, and let E_1, E_2 be connected surfaces in W with $\partial E_i = l_i$ and $[E_i, \partial E_i] = 0 \in H_2(W, \partial W; \boldsymbol{Z}_2)$, $i = 1, 2$. (For example, we could obtain E_i by starting with a connected surface in M bounded by l_i and pushing its interior slightly into W.) We may assume that E_1 and E_2 intersect transversally in points in int W. Since $Lk_{Z_2}(l_1, l_2) = 0$, there will be an even number of such intersection points, and these may be removed, a pair at a time, by adding a tube to (say) E_1 along an arc in E_2 connecting the two points in question.

REMARK. Section 5 contains equations, derived from (i)—(vi) above, involving link signatures and linking numbers. Since both are additive under disjoint union, these equations will still be valid if we only assume $\partial(W, F_1, F_2) = k(M, l_1, l_2)$, the disjoint union of k copies of (M, l_1, l_2), for some $k > 0$. Moreover, we have just seen that this weaker assumption is always satisfied (with $k = 2$) if M is a $\boldsymbol{Z}_2$-homology sphere. For notational simplicity, however, we shall continue to take $k = 1$, without further comment.

4. To deduce relations between the link signatures on the left of equations (i)—(vi), we must find relations between the quantities on the right. The main ingredient is the following.

LEMMA 1.

$$\sigma(\widetilde{W}) = \sum_{i=1}^{3} \sigma(W^{(i)}) - 2\sigma(W) .$$

Proof. If G is a finite group and N is a G-manifold, then a standard transfer argument shows that

$$(*) \qquad \sigma(N) = |G|\sigma(N/G) - \sum_{g \in G - \{1\}} \operatorname{sign}(g, N) .$$

Applying this to the $\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$-manifold $\widetilde{W}$, we have

$$(1) \qquad \sigma(\widetilde{W}) = 4\sigma(W) - \sum_{i=1}^{3} \operatorname{sign}(g_i, \widetilde{W}) .$$

For $i = 1, 2, 3$, $W^{(i)} = \widetilde{W}/(g_i)$ has an action of $(\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2)/(g_i) \cong \boldsymbol{Z}_2$, generated by h_i, say. Applying (*) again, we get

(2) $$\sigma(W^{(i)}) = 2\sigma(W) - \mathrm{sign}(h_i, W^{(i)}) , \qquad i = 1, 2, 3 .$$

By the proof of the proposition on page 415 of [2]

$$\mathrm{sign}(h_i, W^{(i)}) = \frac{1}{2}\sum_{j\neq i} \mathrm{sign}(g_j, \widetilde{W}) .$$

Hence

(3) $$\sum_{i=1}^{3} \mathrm{sign}(h_i, W^{(i)}) = \sum_{i=1}^{3} \mathrm{sign}(g_i, \widetilde{W}) .$$

The result now follows from equations (1), (2) and (3).

We also need

LEMMA 2.

$$\bar{e}(F_1^{(2)}) = 2\bar{e}(F_1) , \quad \bar{e}(F_2^{(1)}) = 2\bar{e}(F_2) ,$$
$$\bar{e}(F) = \bar{e}(F_1) + \bar{e}(F_2) - 2Lk(\bar{l}_1, \bar{l}_2) .$$

(Note that $Lk(\bar{l}_1, \bar{l}_2)$ is well-defined, since $\bar{l}_1$ and $\bar{l}_2$ are both null-homologous.)

Proof. To prove the first statement, let V_1 be an oriented surface in M with $\partial V_1 = -\bar{l}_1$. Let the inverse image of V_1 in $\partial W^{(2)}$ be $V_1^{(2)}$, a 2-fold branched cover (possibly disconnected) of V_1. Let $\bar{l}_1'$ be the (oriented) boundary of a section of the normal 1-sphere bundle of F_1; its inverse image $\bar{l}_1^{(2)\prime}$ in $\partial W^{(2)}$ is the boundary of a corresponding section for $F_1^{(2)}$. Then

$$\bar{e}(F_1^{(2)}) = \bar{l}_1^{(2)\prime}\cdot V_1^{(2)} = 2\bar{l}_1'\cdot V_1 = 2\bar{e}(F_1) .$$

Similarly, $\bar{e}(F_2^{(1)}) = 2\bar{e}(F_2)$. Finally, we may assume that $\bar{l}_1'$ does not meet l_2, and is homologous to $\bar{l}_1$ in $M - l_2$. Extending in the obvious way the notation already introduced, we then have

$$\begin{aligned}\bar{e}(F) &= (\bar{l}_1' \cup \bar{l}_2')\cdot(V_1 \cup V_2)\\ &= \bar{l}_1'\cdot V_1 + \bar{l}_2'\cdot V_2 + \bar{l}_1'\cdot V_2 + \bar{l}_2'\cdot V_1\\ &= \bar{e}(F_1) + \bar{e}(F_2) - 2Lk(\bar{l}_1, \bar{l}_2) .\end{aligned}$$

5. From equations (i)-(iv), together with Lemmas 1 and 2, one easily obtains

$$\sigma(\bar{l}_1, p_1) + \sigma(\bar{l}, p) + Lk(\bar{l}_1, \bar{l}_2) = \sigma(\bar{l}_2, p_2) + \sigma(\bar{l}_1^{(2)}, p_1^{(2)}) .$$

Now suppose $M = S^3$ and l_2 is the unknot. Then $\partial W^{(2)}$ is also S^3, and $\sigma(\bar{l}_2, p_2) = 0$, so the above equation becomes

$$\sigma(\bar{l}_1^{(2)}) = \sigma(\bar{l}_1) + \sigma(\bar{l}) + Lk(\bar{l}_1, \bar{l}_2) .$$

If, further, l_1 has only one component, then

$$\sigma(\bar{l}) + Lk(\bar{l}_1, \bar{l}_2) = \xi(l)$$

so we obtain

$$\sigma(\bar{l}_1^{(2)}) = \sigma(l_1) + \xi(l) .$$

Theorem 1 is the special case in which $l_1^{(2)}$ has only one component.

REMARK. Using equations (i), (ii), (iii) and (vi) we obtain instead the relation

$$\sigma(\phi, \tilde{p}) + \sigma(\bar{l}, p) + Lk(\bar{l}_1, \bar{l}_2) = \sigma(\bar{l}_1, p_1) + \sigma(\bar{l}_2, p_2) .$$

If $M = S^3$, this can be written as

$$\sigma(\phi, \tilde{p}) + \xi(l) = \xi(l_1) + \xi(l_2) .$$

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