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PERMUTATIONS OF THE POSITIVE INTEGERS WITH RESTRICTIONS ON THE SEQUENCE OF DIFFERENCES, II

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In this paper we discuss the following conjecture:

Conjecture: **Let** $D=\{D_1, \cdots, D_n\}$, $D\subset N$, N **the set of positive integers. Then there exists a permutation of** N, **call it** $(a_k: k\in N)$ **such that** $\{|a_{k+1}-a_k|: k\in N\}=D$ **iff** $(D_1, \cdots, D_n)=1$.

We also consider the following question:

Question: **For what sets** $D=\{D_1, \cdots, D_n\}$ **does there exist an integer** $M\in N$ **and a permutation** $\{b_k: k=1, \cdots, M\}$ **of** $\{1, \cdots, M\}$ **such that** $\{|b_{k+1}-b_k|: k=1, \cdots, M-1\}=D$.

We answer the conjecture and the following question in the affirmative if the set D **has the following property: For each** $D_r\in D$ **there is a** $D_s\in D$ **such that** $(D_r, D_s)=1$.

In the following, we shall say that $(a_k: k\in N)$, N the set of positive integers, is a permutation if every integer $n\in N$ appears once and only once in the sequence $(a_k: k\in N)$. Set $d_k=|a_{k+1}-a_k|$.

In a previous paper, [1], we proved the following theorem.

THEOREM 1. *Let $(m_j: j\in N)$ be any sequence of positive integers. Then there exists a permutation $(a_k: k\in N)$ such that $|\{i|d_i=j\}|=m_j$.*

In constructing such permutations we could use infinitely many differences. We now ask if permutations of N can be constructed where the set of differences comes from a finite set. We make the following conjecture.

CONJECTURE. Let $D=\{D_1, \cdots, D_n\}$, $D\subset N$. Then there exists a permutation $(a_k: k\in N)$ such that $\{d_k: k\in N\}=D$ iff $(D_1, \cdots, D_n)=1$, where $(D_1, \cdots, D_n)$ denotes the g.c.d. of the numbers, $D_1, \cdots, D_n$.

In this paper we show that the condition is necessary and that it is sufficient if corresponding to each $D_r\in D$, there is a D_s such that $(D_r, D_s)=1$.

For $n=1$, the condition that the g.c.d. be 1 gives that $D=\{1\}$. For the set, $D=\{1\}$, set $a_k=k$. Clearly, $\{d_k: k\in N\}=0$.

LEMMA 1. *Let $D=\{D_1, \cdots, D_n\}$, $D\subset N$. If there exists a permutation $(a_k: k\in N)$ such that $\{d_k: k\in N)=D$, then $(D_1, \cdots, D_n)=1$.*

Proof. Set $d = (D_1, \cdots, D_n)$. If $d > 1$, then by inducting on k one can show that $a_k \equiv a_1 (\operatorname{mod} d)$, which implies that $(a_k: k \in N)$ cannot be a permutation. Thus $d = 1$.

Lemma 1 shows that the conjecture is true for $n = 1$. That the conjecture is true for $n = 2$, will follow from Theorms 2 and 3.

THEOREM 2. *Let* $D = \{D_1, D_2\}$, $(D_1, D_2) = 1$ *and set* $M = D_1 + D_2 + 1$. *Then there exists a permutation* $(b_k: k = 1, \cdots, M)$ *of the set* $\{1, \cdots, M\}$ *with the following properties*:

(a) $\{|b_{k+1} - b_k|: k = 1, \cdots, M - 1\} = D$,

(b) $b_1 = 1, b_M = M$.

Proof. We may assume that $D_1 < D_2$. Since $(D_1, D_2) = 1$, we have that $(D_1, M - 1) = 1$. Thus, D_1 generates, under addition, the complete residue system modulo $M - 1$. Set $b'_1 = 0$, $b'_2 = D_1$. If b'_k has been defined, then, if $b'_k + D_1 \leqq M - 1$, set $b'_{k+1} = b'_k + D_1$. If $b'_k + D_1 > M - 1$, set $b'_{k+1} = b'_k + D_1 - (M - 1)$.

Since $b'_1 = 0$ and D_1 is a generator for the complete residue system modulo $M - 1$, it is clear that $b'_M = M - 1$. Furthermore, if $b'_k + D_1 \leqq M - 1$, then $|b'_{k+1} - b'_k| = D_1$, and if $b'_k + D_1 > M - 1$, then $|b'_{k+1} - b'_k| = |b'_k + D_1 - (M - 1) - b'_k| = |D_1 - (D_1 + D_2)| = D_2$. Thus $(b'_k: k = 1, \cdots, M)$ has the properties,

(a) $|b'_{k+1} - b'_k| \in D$ and for each i, there is a k such that $D_i = |b'_{k+1} - b'_k|$, and

(b′) $b'_1 = 0, b'_M = M - 1$.

Set $b_k = b'_k + 1$. Clearly $(b_k: k = 1, \cdots, M)$ has properties (a) and (b).

THEOREM 3. *Let* $D = \{D_1, \cdots, D_n\}$ *with* $(D_1, \cdots, D_n) = 1$. *If there exists an* $M \in N$ *and a permutation* $\{b_k: k = 1, \cdots, M\}$ *of* $\{1, \cdots, M\}$ *with the properties*

(i) $\{|b_{k+1} - b_k|: k = 1, \cdots, M - 1\} = D$ *and*

(ii) $b_1 = 1, b_M = M$,

then there exists a permutation $(a_k: k \in N)$ *such that* $\{d_k: k \in N\} = D$ *and each* D_i *appears infinitely often in the sequence* $(d_k: k \in N)$.

Proof. We shall give a recursive definition for a permutation $(a_k: k \in N)$. Since $b_1 = 1$ and $|b_2 - b_1| = |b_2 - 1| \in D$, we must have that $b_2 - 1 \in D$. Without loss of generality we may assume that $b_2 = 1 + D_1$. Set $a_k = b_k$, $k = 1, \cdots, M$. We now define a_{M+j}, $j = 1, \cdots, M - 1$. Set $a_{M+j} = M - 1 + a_{j+1}$, $j = 1, \cdots, M - 1$. Clearly, $(a_{M+j}: j = 1, \cdots, M - 1)$ is a permutation of the set $\{M + 1, \cdots, 2M - 1\}$. Thus $(a_k: k = 1, \cdots, 2M - 1)$ is a permutation of the set $\{1, \cdots, 2M - 1\}$ with the property that $a_1 = 1$, $a_{2M-1} = 2M - 1$.

Note that $d_M = |a_{M+1} - a_M| = |(M - 1 + D_1 + 1) - M| = D_1 = d_1$, and for $M - 1 \geqq j \geqq 1$, $d_{M+j} = |a_{M+j+1} - a_{M+j}| = |a_{j+2} - a_{j+1}| = d_{j+1}$. Thus $(d_j\colon j = 1, \cdots, M - 1) = (d_{M+j}\colon j = 0, \cdots, M - 2)$, as sequences. Thus, each D_i occurs twice as many times in the sequence $(d_k\colon k = 1, \cdots, 2M - 2)$ as in the sequence $(d_k\colon k = 1, \cdots, M - 1)$.

We now apply the procedure again and define $a_{2M-1+j} = 2M - 2 + a_{j+1}$, $j = 1, \cdots, M - 1$. Continuing this process one obtains a permutation $(a_k\colon k \in N)$ with the properties that $\{|a_{k+1} - a_k|\colon k \in N\} = D$ and each D_i occurs infinitely often in the sequence $(d_k\colon k \in N)$.

COROLLARY 1. *The conjecture is true for* $n = 2$.

Proof. Apply Theorems 2 and 3.

We can also verify the conjecture for a large class of sets $D = \{D_1, \cdots, D_n\}$, as the following result shows.

THEOREM 4. *Let* $D = (D_1, \cdots, D_n)$, *where* $(D_1, \cdots, D_n) = 1$ *and for each* r *there exists a* s *such that* $(D_r, D_s) = 1$. *Then there exists an* M *and a permutation* $(b_k\colon k = 1, \cdots, M)$ *of* $\{1, \cdots, M\}$ *with the properties,*

(i) $\{|b_{k+1} - b_k|\colon k = 1, \cdots, M - 1\} = D$ *and*

(ii) $b_1 = 1$, $b_M = M$.

Proof. For each r there is a s such that $(D_r, D_s) = 1$. Set $M_r = D_r + D_s + 1$. Then there exists for each $r = 1, \cdots, n$, by Theorem 2, a permutation $(b_k^{(r)}\colon k = 1, \cdots, M_r)$ of the set $\{1, \cdots, M_r\}$ such that $\{|b_{k+1}^{(r)} - b_k^{(r)}|\colon k = 1, \cdots, M_r - 1\} = \{D_r, D_s\}$ and $b_1^{(r)} = 1$, $b_{M_r}^{(r)} = M_r$.

Set $b_i = b_i^{(1)}$, $i = 1, \cdots, M_1$, $b_{M_1+j} = (M_1 - 1) + b_{j+1}^{(2)}$, for $j = 1, \cdots, M_2 - 1$. Thus $b_1 = 1$, $b_{M_1+M_2-1} = M_1 + M_2 - 1$ and $D \supset \{|b_{k+1} - b_k|\colon k = 1, \cdots, M_1 + M_2 - 2\} \supset \{D_1, D_2\}$.

Suppose that we have defined $(b_k\colon k = 1, \cdots, R_l - (l - 1))$, where $(b_k\colon k = 1, \cdots, R_l - (l - 1))$ is a permutation of $\{1, \cdots, R_l - (l - 1)\}$, $l < n$ and $R_l = M_1 + \cdots + M_l$, with the following properties:

(i) $D \supset \{|b_{k+1} - b_k|\colon k = 1, \cdots, R_l - (l - 1) - 1\} \supset \{D_1, \cdots, D_l\}$, and

(ii) $b_1 = 1$, $b_{R_l-(l-1)} = R_l - (l - 1)$.

Let $R_{l+1} = M_{l+1} + R_l$ and $b_{R_l-(l-1)+j} = R_l - (l - 1) - 1 + b_{j+1}^{(l+1)}$, $j = 1, \cdots, N_{l+1} - 1$.

Thus, we have that if $M = M_1 + \cdots + M_n + (n - 1)$, then $(b_k\colon k = 1, \cdots, M)$ has properties (i) and (ii).

COROLLARY 2. Let $D = \{D_1, \cdots, D_n\}$, where $(D_1, \cdots, D_n) = 1$ and for each r there exists a s such that $(D_r, D_s) = 1$. Then there exists a permutation $(a_k: k \in N)$ such that $\{d_k: k \in N\} = D$ and each element in D occurs infinitely often in $(d_k: k \in N)$.

Theorem 2 gives rise to the following questions.

Question 1. Given $(D_1, \cdots, D_n) = 1$, does there exist an $M \in N$ and a permutation $(b_k: k = 1, \cdots, M)$ of $\{1, \cdots, M\}$ such that

$$\{|b_{k+1} - b_k|: k = 1, \cdots, M-1\} = \{D_1, \cdots, D_n\}?$$

Question 2. Same as Question 1 but we require that $b_1 = 1$, $b_M = M$?

Theorems 2 and 4 yield some information concerning these two questions. Of course, an affirmative answer to Question 2 would yield an affirmative answer to our conjecture, as Theorem 3 shows.

Consider the set $\{6, 10, 15\}$. Even though $(6, 10, 15) = 1$, we cannot apply the procedure of Theorem 4 to this triple. However, Question 2 can be answered in the affirmative for the triple $\{6, 10, 15\}$, as the following construction shows.

Note that $(6, 10) = 2(3, 5)$. For $D = \{3, 5\}$, construct the sequence $(b'_k: k = 1, \cdots, 9)$ of Theorem 2. We obtain, $(0, 3, 6, 1, 4, 7, 2, 5, 8)$. Multiply every element by $2 = (6, 10)$, obtaining $(0, 6, 12, 2, 8, 14, 4, 10, 16)$. Now, 16 just happens to be $1 + 15$. Thus, add 1 to the sequence $(0, 6, 12, \cdots)$ and juxtapose it with $(0, 6, 12, \cdots)$ obtaining $(0, 6, 12, 2, 8, 14, 4, 10, 16, 1, 7, 13, 3, 9, 15, 5, 11, 17)$. Now add one to the sequence, obtaining $(1, 7, 13, 3, 9, 15, 5, 11, 17, 2, 8, 14, 4, 10, 16, 6, 12, 18)$, and call this new sequence $(b_k: k = 1, \cdots, 18)$. Note that $\{|b_{k+1} - b_k|: k = 1, \cdots, 17\} = \{6, 10, 15\}$ and $b_1 = 1$, $b_{18} = 18$.

From the above construction, one can glean a proof for the following lemma.

LEMMA 2. *Let* $D = \{D_1, D_2, D_3\}$, *where* $(D_1, D_2, D_3) = 1$, $(D_1, D_2) = d \neq 1$ *and* $D_3 + 1 = k(D_1 + D_2)$, *for some positive integer* k. *Then Question* 2 *and our conjecture can be answered in the affirmative for the set* $\{D_1, D_2, D_3\}$.

REFERENCE

1. P. J. Slater and W. Y. Vélez, *Permutations of the positive integers with restrictions on the sequence of differences*, Pacific J. Math., **71** (1977), 193-196.

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