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# WEAK FROBENIUS RECIPROCITY AND COMPACTNESS CONDITIONS IN TOPOLOGICAL GROUPS

ROLF WIM HENRICHS

**We study weak containment relations between unitary representations of a locally compact group $G$ and closed subgroups $H$. We prove that certain weak Frobenius properties and compactness conditions are equivalent. Moreover, for amenable $G$ having small invariant neighborhoods at $e$ weak Frobenius reciprocity (FP) defined by Fell holds for the pair $(G, H)$ if every element of $H$ has relatively compact conjugacy class in $G$.**

Introduction. In [4], Fell considers the following weak version of the Frobenius reciprocity property (FP): for every closed subgroup $H$ of a locally compact group $G$ and $\pi \in \hat{G}$, $\psi \in \hat{H}$ *$\pi$ is weakly contained in ${}_GU^{\psi}$, the unitary representation of $G$ induced by $\psi$, if and only if $\psi$ is weakly contained in the restriction $\pi|H$ of $\pi$ to $H$.*

Compact groups have property FP by the classical reciprocity theorem; Fell has shown that abelian groups satisfy FP.

In §2 we deal with a weaker property (RFP): reciprocity above holds for every $\psi \in \hat{H}$ and the trivial one dimensional representation $I_G$ of $G$ (not necessarily for arbitrary $\pi \in \hat{G}$). Property RFP is inherited by closed subgroups, we do not know whether this is true for FP. However, we have shown in [8] that for discrete groups $G$ properties FP and RFP are equivalent with $G$ to have only finite conjugacy classes. To get analogous results in the nondiscrete case we look at the normal subgroup $G_F$ of $G$, the union of all relatively compact conjugacy classes in $G$. $G_F$ is open if and only if there is a compact neighborhood of $e \in G$, invariant under the action of $G$ by inner automorphisms ($G \in$ [IN]; see [15], for a proof). It turns out for the class of IN-groups RFP to be a compactness condition.

THEOREM A. *For a locally compact group the following conditions are equivalent*

(1) $G \in [IN] \cap [RFP]$

(2) $G = G_F$.

Also for Lie groups $G \in$ [RFP] $G_F$ is open as it will be shown in [3]. Thus it follows from Theorem A, that for Lie groups or connected groups $G \in$ RFP is equivalent with $G$ to have only relatively compact conjugacy classes ($G \in [FC]^-$).

If $G$ is an IN-group there is a compact normal subgroup $K$ of

$G$ such that $G/K$ has small invariant neighborhoods at $e$ ($G \in$ [SIN]). The results in [8] for discrete groups can be generalized to SIN-groups. The following theorem shows that groups $G \in [FC]^- \cap$ [SIN] have property FP. Combining it with Theorem A one sees that for SIN-groups RFP and FP are equivalent.

THEOREM B. *Let $G$ be an amenable* SIN-*group. If $H$ is a closed subgroup of $G$ contained in $G_F$ and $\pi \in \hat{G}$, $\psi \in \hat{H}$, $\pi$ is weakly contained in ${}_GU^{\psi}$ if and only if $\pi \mid H$ weakly contains $\psi$.*

As a corollary we get that the direct product of an abelian group and a compact group has property FP. It remains an open problem whether arbitrary [FC]$^-$-groups have property FP. The methods used in §3 to prove the results for SIN-groups do not work in the general IN-group case.

In §2 we state some general weak containment relations for unitary representations of arbitrary locally compact groups and then prove that all conjugacy classes of an IN-group satisfying RFP have compact closure. Furthermore, we show that extensions of compact groups with groups satisfying RFP have property RFP. Therefore the proof of $2 \Rightarrow 1$ in Theorem A can be reduced to the SIN-group case.

**1. Preliminaries.** The following notations will be used throughout the paper:

| | |
|---|---|
| $C^*(G)$ | $= C^*$-algebra of the locally compact group $G$ |
| $\langle , \rangle$ | = canonical bilinear from on $L^\infty(G) \times L^1(G)$ |
| ${}_xf(y)$ | $= f(xy)$ and $f_x(y) = f(yx)$ for a function $f$ on $G$ |
| $f^\tau(y)$ | $= f(\tau^{-1}(y))$ for an automorphism $\tau$ of $G$ |
| $\operatorname{supp} f$ | = support of $f$ |
| $C_{00}(X)$ | = continuous functions on the locally compact space $X$ having compact support |
| $\operatorname{supp} \mu$ | = support of the measure $\mu$ |
| $\langle x \rangle$ | = subgroup generated by $x \in G$ |
| $C(x)$ | = centralizer of $x$ |
| $[G: H]$ | = index of the subgroup $H$ |
| $g \mid Y$ | = restriction of a mapping $g$ to $Y$ |
| $\operatorname{ex} C$ | = set of extreme points of the convex set $C$. |

Representation always means continuous unitary representation on a Hilbert space. $\hat{G}$ denotes the set of equivalence classes of irreducible representations of $G$. If $\pi$ is a representation of $G$, $\ker \pi$ denotes the kernel of $\pi$, considered as a representation of $C^*(G)$. If $S$, $T$ are sets of representations, we write $S \prec T$ if $S$ is weakly contained in $T$. By [2, § 18], $S \prec T$ if and only if $\bigcap_{\pi \in S} \ker \pi \supseteqq \bigcap_{\pi \in T} \ker \pi$.

Let $P(G)$ be the set of all continuous positive definite functions on $G$, $P(G) \subseteqq L^\infty(G)$ endowed with the weak *-topology. On $P^1(G) = \{\varphi \in P(G); \varphi(e) = 1\}$ this equals the topology of uniform convergence on compact sets in $G$, sometimes called Pontryagin topology. Every $\varphi \in P(G)$ defines a representation $\pi_\varphi$ of $G$ on a Hilbert space $\mathfrak{H}_\varphi$ with cyclic vector $\xi_\varphi$ such that

$$\varphi(x) = (\pi_\varphi(x)\xi_\varphi \mid \xi_\varphi) \qquad \text{for all } x \in G.$$

The positive functional on $C^*(G)$ corresponding to $\varphi \in P(G)$ is also denoted by $\varphi$, $M_\varphi = \{a \in C^*(G); \varphi(a^*a) = 0\}$ is a left ideal in $C^*(G)$.

Let $N$ be a closed normal subgroup of $G$; we set $f^x(n) = f(xnx^{-1})$ for a function $f$ on $N$ and $x \in G$. The extension to $C^*(N)$ of the mapping $f \to f^x$ of $C_{00}(N)$ will be written as $a \to a^x$. An ideal $M$ in $C^*(N)$ is called $G$-stable if $a \in M$ implies $a^x \in M$ for all $x \in G$. For a closed subgroup $H$ of $G$ we set $P(N, H) = \{\varphi \in P(N); \varphi^x = \varphi$ for all $x \in H\}$ and $P^1(N, H) = P(N, H) \cap P^1(N)$. $P_1(N, H) = \{\varphi \in P(N, H); \varphi(e) \leqq 1\}$ is convex and compact, $E(N, H)$ denotes the set of all nonzero extreme points of $P_1(N, H)$. We write $E(N)$ instead of $E(N, N)$.

Let $H$ be a closed subgroup of $G$; left Haar measures on $G$ and $H$, respectively, are denoted by $dx$ and $ds$ and let $\Delta_G$ and $\Delta_H$ be their modular functions. For $f \in C_{00}(G)$ let $T_H f \in C_{00}(G/H)$ be the function

$$T_H f(\dot{x}) = \int_H f(xs)ds\ , \qquad x \in G\ .$$

If $\psi$ is a representation of $H$ ${}_GU^\psi$ denotes the representation of $G$ obtained by inducing $\psi$ to $G$. For a function $f$ on $G$ we set $q(s) = (\Delta_G(s)/\Delta_H(s))^{1/2}$ and $R(f) = q(s)f(s)$, $s \in H$. For $\gamma \in P(H)$ let $\mu^\gamma$ be the Radon measure on $G$ defined by

$$\mu^\gamma(f) = \int_H \gamma(s)R(f)(s)ds\ , \qquad f \in C_{00}(G)\ .$$

By [1, Thm. 1], $\mu^\gamma$ is positive definite, i.e., $\mu^\gamma(f^* * f) \geqq 0$, let

$$N^\gamma = \{f \in C_{00}(G); \mu^\gamma(f^* * f) = 0\} \quad \text{and} \quad [f]^\gamma = f + N^\gamma.$$

The completion of $C_{00}(G)/N^\gamma$ with respect to the scalar product

$$([f]^\gamma \mid [g]^\gamma) = \mu^\gamma(g^* * f)\ , \qquad f,\ g \in C_{00}(G)$$

is denoted by $\mathfrak{H}^\gamma$. The representation ${}_GU^\gamma$ of $G$ on $\mathfrak{H}^\gamma$ such that

$$U^\gamma_x[f]^\gamma = [{}_{x^{-1}}f]^\gamma\ , \qquad f \in C_{00}(G)\ , \quad x \in G$$

is equivalent to ${}_GU^{\pi_\gamma}$ [1].

If $H$ is an open subgroup of $G$ we identify $\mathfrak{H}^{\gamma}$ with $\mathfrak{H}_{\varphi}$ by $[f]^{\gamma} \to \pi_{\varphi}(f)\xi_{\varphi}$, where $\varphi \in P(G)$ is the trivial extension of $\gamma$, $\varphi(x) = 0$ for $x \in G \backslash H$.

**2. Weak containment and the restricted Frobenius property RFP.** If a locally compact group $G$ satisfies FP it has the following (weaker) property RFP: for every closed subgroup $H$ of $G$ and $\psi \in \hat{H}$

$$I_G \prec {}_G U^{\psi} \quad \text{if and only if} \quad \psi = I_H .$$

Actually, if $\pi = I_G$, $\psi = I_H$ thus $\psi = \pi \mid H$, we have

$$I_G \prec {}_G U^{I_H} \qquad \text{for all closed subgroups } H \text{ of } G$$

(by [6], this property is satisfied if and only if $G$ is amenable and it is equivalent to the weak Frobenius property WF1 defined by Fell in [4]: for every closed subgroup $H$ of $G$ and $\pi \in \hat{G}$

$$\pi \prec {}_G U^{\pi \mid H}) .$$

Conversely, if $\psi \in \hat{H}$ and $I_G \prec {}_G U^{\psi}$, then FP implies

$$\psi \prec I_H \quad \text{therefore} \quad \psi = I_H .$$

We do not know whether FP is inherited by closed subgroups therefore we deal with the weaker property RFP.

LEMMA 2.1. *If $G$ has property RFP, closed subgroups $H$ and quotients $G/N$ have property RFP.*

*Proof.*

(a) Every closed subgroup of an amenable group is amenable and by [6] satisfies WF1. The same holds for every continuous homomorphic image of $G$.

(b) Let $K$ be a closed subgroup of $H$ and let $I_H \prec {}_H U^{\psi}$, $\psi \in \hat{K}$. By Theorem 4.3 in [4] and by the theorem on inducing in stages (see [18], for instance)

$${}_G U^{I_H} \prec {}_G U({}_H U^{\psi}) = {}_G U^{\psi} . \qquad \text{Since } G \text{ satisfies RFP}$$
$$I_G \prec {}_G U^{I_H} \quad \text{and} \quad I_G \prec {}_G U^{\psi} \quad \text{therefore} \quad \psi = I_K .$$

(c) Let $W$ be a closed subgroup of $G/N$, $N$ closed normal, and let $I_{G/N} \prec U^{\psi}$, $\psi = \pi_{\rho} \in \hat{W}$. Then $I_G \prec U^{\psi} \circ p$, $p: G \to G/N$ the canonical projection. If $H = p^{-1}(W)$ and $\gamma = \rho \circ p \in P^1(H)$, $\psi \circ p$ is the cyclic representation associated with $\gamma$. If left Haar measures of $G$ and $G/N$, $H$ and $W$, respectively, are normalized such that Weil's formula holds, ${}_G U^{\psi \circ p}$ and ${}_G U^{\psi} \circ p$ are easily seen to be equivalent: $[f]^{\gamma} \to [T_N f]^{\rho}$, $f \in C_{00}(G)$, defines the corresponding intertwining operator. Therefore

$I_G \prec {}_G U^{\psi \circ p}$ and $\psi = I_W$ follows from $\psi \circ p = I_H$.

Let $\mu$ be a positive definite Radon measure on $G$. If $\{f_i; i \in I\}$ is an approximate identity for $C_{00}(G)$ in the inductive limit topology we denote by $\pi_i$ the cyclic representation generated by $\pi_\mu$ and $[f_i]^\mu$.

LEMMA 2.2. *$\pi_\mu$ is weakly equivalent to the set of representations $\pi_i$, $i \in I$.*

*Proof.* Clearly $\{\pi_i; i \in I\} \prec \pi_\mu$. Let $a \in \bigcap_{i \in I} \ker \pi_i$ and $f \in C_{00}(G)$ be given. As

$$\| [f]^\mu - \pi_\mu(f)[f_i]^\mu \|^2 = \mu((f - f*f_i)^* * (f - f*f_i))$$

tends to zero and

$$\pi_\mu(a)\pi_\mu(f)[f_i]^\mu = \pi_i(a)\pi_i(f)[f_i]^\mu = 0$$

we get $\pi_\mu(a)[f]^\mu = 0$. $C_{00}(G)$ being dense in $\mathfrak{H}^\mu$ the assertion follows.

The left regular representation of $G$ is denoted by $\lambda_G$, or simply $\lambda$. The crucial step exploring which groups may have RFP is the following

PROPOSITION 2.3. *Let $N$ be an open normal subgroup of $G$ and let $x$ be an element of $G$, not in $G_F$. Then $\lambda \prec U^\gamma$ for every character $\gamma$ of $\langle x \rangle$ if one of the following conditions is satisfied*

(1) *$x$ has order $p$, $p$ prime number*

(2) *$xN \in (G/N)_F$ has infinite order and $\langle x \rangle \cap G_F = \{e\}$.*

*Proof.* In both cases $\langle x \rangle$ is discrete and $\langle x \rangle \cap G_F = \{e\}$. Let $\gamma$ be any character of $\langle x \rangle$ and let $\{f_i; i \in I\}$ be a usual approximative identity for $C_{00}(G)$ in the inductive limit topology. Since $N$ is open we may suppose $\operatorname{supp} f_i \subseteqq N$ for $i \in I$. By Lemma 2.2, since $\lambda$ is the representation corresponding to the positive definite measure $f \to f(e)$, $f \in C_{00}(G)$, $\lambda$ is weakly contained in the set of cyclic representations $\pi_i$ defined by $\lambda$ and $f_i$, $i \in I$. By [2, 18.1.4], it is sufficient to show that for every $i \in I$ the function defined by $\lambda$ and $f_i$ can be approximated uniformly on compact sets by positive definite functions associated with $U^\gamma$. Therefore let $f \in C_{00}(G)$ with $K = \operatorname{supp} f \subseteqq N$ be fixed and let $C$ be a compact set in $G$. For $c \in C$, $s \in \langle x \rangle$, $z \in G$ define

$$g(s, c, z) = \int_G f(c^{-1}y^{-1}z^{-1}sz)f^*(y)dy .$$

Then

$$\begin{aligned}(U_c^\gamma[f_z]^\gamma \mid [f_z]^\gamma) &= \sum_{s\in\langle x\rangle} \gamma(s)q(s)((f_z)^* * c^{-1}f_z)(s)\\ &= \sum_{s\in\langle x\rangle} \gamma(s)q(s)\int_G f(c^{-1}y^{-1}sz)\overline{f(y^{-1}z)}\Delta_G(y^{-1})dy\\ &= \sum_{s\in\langle x\rangle} \gamma(s)q(s)g(s, c, z)\Delta_G(z^{-1})\ .\end{aligned}$$

If $g(s, c, z)\neq 0$ $z^{-1}sz$ must be in the set $K^{-1}cK$.

*Case* (1). Let $|\langle x\rangle| = p$ and $k = (p-1)!$ then $x^k \notin G_F$ and there exists $z\in G$ such that $z^{-1}x^kz$ is not in the compact set $\bigcup_{i=1}^{p-1}(K^{-1}CK)^{k/i}$. It follows

$$z^{-1}x^iz \notin K^{-1}CK\ , \qquad 1 \leqq i \leqq p-1$$

therefore $g(s, c, z) = 0$ if $s\neq e$, $c\in C$. Thus for every $c\in C$

$$(\lambda(c)f \mid f) = (f^* *_{c^{-1}} f)(e) = g(e, c, z) = (U_c^\gamma[f_z]^\gamma \mid [f_z]^\gamma)\Delta_G(z)\ .$$

*Case* (2). We may assume that $xN$ is in the centre of $G/N$: as $G/N$ is discrete and $[G/N: C(xN)] < \infty$ $H = \{z\in G;\ zN\in C(xN)\}$ has finite index, therefore $\langle x\rangle \cap H_F = \{e\}$. Then if one can prove $\lambda_H \prec_H U^\gamma$ $\lambda \prec_G U^{\lambda_H} \prec {}_G U({}_H U^\gamma) = {}_G U^\gamma$ follows.

Now if $z^{-1}sz\in K^{-1}cK \subseteqq NcN$, it follows $c\in Nz^{-1}szN = sN$. Therefore $g(s, c, z) = 0$ for all $s\in\langle x\rangle$ and all $z\in G$ unless $c\in\langle x\rangle N$. If $c\in N$ and $g(s, c, z)\neq 0$ then $c\in sN$ forces $s = e$ as $\langle x\rangle\cap N = \{e\}$. Thus for all $z\in G$

$$\Delta_G(z)(U_c^\gamma[f_z]^\gamma \mid [f_z]^\gamma) = \begin{cases} 0 & c\notin\langle x\rangle N\\ (\lambda(c)f \mid f) & c\in N\ .\end{cases}$$

Finally, there is a finite set $\{k_i;\ 1\leqq i\leqq m\}$ of nonzero integers such that $C\cap(\langle x\rangle N\backslash N) \subseteqq \bigcup_{i=1}^m x^{k_i}N$. As $x^k\notin G_F$, $k = \prod_{i=1}^m k_i$, we may choose $z\in G$ such that

$$zx^kz^{-1} \notin \bigcup_{i=1}^m (K^{-1}CK)^{k/k_i}$$

therefore

$$zx^{k_i}z^{-1}\notin K^{-1}CK \qquad \text{for } 1\leqq i\leqq m\ .$$

Thus $g(x^{k_i}, c, z) = 0$, but if $s\neq x^{k_i}$ $g(s, c, z) = 0$ for $c\in C\cap(\langle x\rangle N\backslash N)$ as $c\notin sN$. Consequently

$$(U_c^\gamma[f_z]^\gamma \mid [f_z]^\gamma) = 0 \qquad \text{for } c\in C\cap(\langle x\rangle N\backslash N)\ .$$

As $(\lambda(c)f \mid f) = 0$ if $c\notin N$ we have proved: there is $z\in G$ such that $(\lambda(c)f \mid f) = \Delta_G(z)(U_c^\gamma[f_z]^\gamma \mid [f_z]^\gamma)$ for all $c\in C$.

COROLLARY 2.4. *Let $G$ be amenable and let $x\notin G_F$ satisfy one* *of the conditions in Proposition* 2.3. *Then for every* $\gamma \in \langle \hat{x} \rangle$ *the representation* $U^\gamma$ *of* $C^*(G)$ *is faithful* ($\ker U^\gamma = 0$).

COROLLARY 2.5. *If* $G$ *has property RFP every element of finite order belongs to* $G_F$.

*Proof.* If not, let $n$ be the smallest number $n \in N$ for which there exist a group $H \in$ [RFP] and $x \in H \backslash H_F$ of order $n$. Then $n$ cannot be a prime number. Otherwise there would exist a character $\gamma$ of $\langle x \rangle$, $\gamma \neq 1$, such that $I_H \prec U^\gamma$ in contrary to $H \in$ [RFP]. If $n = mr$, $n \neq m$, $r \in N$, $x^m \in H_F$ as $n$ is minimal. By [7, Thm. 3.11], there is a compact normal subgroup $K$ of $H$ with $x^m \in K$. As $H/K \in$ [RFP] and $|\langle xK \rangle| < n$
$xK \in (H/K)_F$ therefore $x \in H_F$, a contradiction.

For example, the euclidean group of the plane cannot have property RFP by Corollary 2.5.

LEMMA 2.6. *Let* $G$ *satisfy RFP and let* $\langle x \rangle$ *be isomorphic to* $\boldsymbol{Z}$. *Then* $x \in C(x^n)_F$ *for all* $n \in \boldsymbol{N}$.

*Proof.* By Lemma 2.1, the group $H = C(x^n)/\langle x^n \rangle$ has RFP and $x\langle x^n \rangle \in H_F$ follows from the last corollary. Let $K$ be compact such that

$$\{yxy^{-1};\, y \in C(x^n)\} \subseteqq K\langle x^n \rangle \subseteqq G\,.$$

If $yxy^{-1} = kx^{nm(y)}$, $k \in K$, $m(y) \in \boldsymbol{Z}$, it follows

$$x^n = k^n x^{n^2 m(y)} \quad \text{as} \quad y \in C(x^n)\,.$$

Thus $x^{n - n^2 m(y)}$ belongs to the finite set $K^n \cap \langle x^n \rangle$. Therefore there is a finite set $M \subseteqq \boldsymbol{Z}$ such that

$$\{yxy^{-1};\, y \in C(x^n)\} \subseteqq \{kx^{nm};\, k \in K,\ m \in M\}$$

which proves the lemma.

If $V$ is a normal vector group in $G$ and $x \in G_F$ $xvx^{-1}v^{-1}$ is a compact element of $V$ for every $v \in V$ so that $V \subseteqq C(x)$ [7, (3.4)]. Now we can prove

THEOREM 2.7. *If* $G \in [IN]$ *has property RFP then all conjugacy classes in* $G$ *have compact closure.*

*Proof.*
(a) First let $G$ be discrete and let $xG_F \in (G/G_F)_F$. By Proposition

2.3, (2) there exists $n \in N$ with $x^n \in G_F$ (take $N = G_F$). If $\langle x \rangle$ is not finite, $x \in C(x^n)_F$ by Lemma 2.6 thus $x \in G_F$ as $[G: C(x^n)] < \infty$, and if $\langle x \rangle$ is finite $x \in G_F$ by Corollary 2.5. Therefore $(G/G_F)_F$ consists of one element so that $G = G_F$ by Lemma 2 in [8].

(b) Let $G \in [\mathrm{IN}] \cap [\mathrm{RFP}]$, we may assume $G \in [\mathrm{SIN}]$. By [22], there exists a compact normal subgroup $K$ of $G$ and closed normal subgroups $V, D$ of $G/K$, $V$ a vector group and $D$ discrete, such that $(G/K)_F$ is the direct product of $V$ and $D$[1]. Again we may assume $K = \{e\}$. As $G_F$ is open $G/G_F = (G/G_F)_F$ by (1a), and Proposition 2.3 shows that for every element $x$ in $G$ there exists $n \in N$ with $x^n \in G_F$.

If the closed subgroup generated by $x$ is compact, $x^n$ is compact in $G_F$ and by [7, Thm. 3.11] $x^n$ generates a compact normal subgroup $K$ of $G$. As $xK$ has finite order $x \in G_F$, therefore $V \subseteq C(x)$. If $\langle x \rangle \cong \boldsymbol{Z}$, $x \in C(x^n)_F$ and again $V \subseteq C(x)$ as $V \subseteq C(x^n)$. Thus $V$ is contained in the centre of $G$.

If for $x \in G$ $x^n = vd$, $v \in V$, $d \in D$ we have $C(d) \subseteq C(x^n)$. As $d$ belongs to a finite conjugacy class $[G: C(x^n)] < \infty$ and as $x \in C(x^n)_F$ $x \in G_F$ follows.

It is an interesting question whether groups $G \notin [\mathrm{IN}]$ can have property RFP. It will be shown in [3] that every Lie group or connected group $G \in [\mathrm{RFP}]$ is an IN-group. Now let $H$ be a closed subgroup of an arbitrary locally compact group $G$, $\pi \in \hat{G}$, $\psi \in \hat{H}$. If $K$ is compact normal and $\psi(H \cap K) = \{I\}$

$$\dot{\psi}(\dot{s}) = \psi(s) , \qquad s \in H$$

defines a continuous irreducible representation $\dot{\psi}$ of the closed subgroup $HK/K$ in $G/K$.

PROPOSITION 2.8. *Let $\pi \in \hat{G}$ and let $K$ be a compact normal subgroup of $G$ such that $\pi(K) = \{I\}$. If $\pi \prec U^{\psi}$ for $\psi \in \hat{H}$ then $\psi(H \cap K) = \{I\}$ and $\dot{\pi} \prec U^{\dot{\psi}}$.*

*Proof.* Let $\pi = \pi_\varphi$ and $\psi = \pi_\gamma$, $\varphi \in P^1(G)$, $\gamma \in P^1(H)$. For $f \in C_{00}(G)$ define ${}^K\!f \in C_{00}(G)$ by

$${}^K\!f(x) = \int_K f(kx)dx, \ x \in G \text{ where } dk \text{ denotes the}$$

normalized Haar measure on $K$. As ${}^K\!f(xky) = {}^K\!f(xy)$ for all $k \in K$,

[1] I am indebted to the referee for pointing out that the proof in [22] contains an error (in the proof, on the fourth line of p. 328, that $L$ is $B$-invariant) and for giving a sketch of how to correct that error: it suffices to prove that when $W \times D$ is in $[FC]_B^-$ with $W \sim \boldsymbol{R}^n$ and $D$ discrete abelian, then $W$ has a $B$-invariant complement $D_1$. Observing first that $G = W \times D$ is also in $[\mathrm{SIN}]_B$ since $W$ is characteristic and open, one can then apply a splitting theorem of Hofmann and Mostert to $\hat{G} = \hat{W} \times \hat{D}$ to find a $B$-invariant complement $\hat{W}_1$ to $\hat{D}$. Then take $D_1 = \hat{W}_1^{\perp}$.

$x,\ y \in G$, an easy computation shows

$$(2.1)\qquad \int_K (U^{\gamma}_{xk^{-1}}[f]^{\gamma} \mid [f]^{\gamma})dk = (U^{\gamma}_{x}[{}^{K}f]^{\gamma} \mid [{}^{K}f]^{\gamma}) .$$

Now let a compact set $C \subseteqq G/K$ and $\varepsilon > 0$ be given. $C_{00}(G)$ being dense in $\mathfrak{H}^{\gamma}$ it follows from $\pi \prec U^{\psi}$ that there exist $f_i \in C_{00}(G)$, $1 \leqq i \leqq m$, such that

$$| \varphi(xk^{-1}) - \sum_{i=1}^{m} (U^{\gamma}_{xk^{-1}}[f_i]^{\gamma} \mid [f_i]^{\gamma}) | \leqq \varepsilon , \qquad \text{for } k \in K,\ x \in p^{-1}(C) ,$$

$p\colon G \to G/K$ the canonical projection. Since $\varphi(xk^{-1}) = \varphi(x)$, $k \in K$, and using (2.1) we get

$$(2.2)\qquad | \varphi(x) - \sum_{i=1}^{m} (U^{\gamma}_{x}[{}^{K}f_i]^{\gamma} \mid [{}^{K}f_i]^{\gamma}) | \leqq \varepsilon , \qquad x \in p^{-1}(c) .$$

At first, we conclude from (2.2) that there exists a function $f \in C_{00}(G)$ such that $[{}^{K}f]^{\gamma} \neq 0$, let $\| [{}^{K}f]^{\gamma} \| = 1$. By Blattner's theorem (see [18, Thm. 4.4]), $R(({}^{K}f)^* * {}^{K}f)$ is a positive element of $C^*(H)$, let $T = (R(({}^{K}f)^* * {}^{K}f))^{1/2}$. Then for $k \in H \cap K$

$$\begin{aligned}\psi(k)\psi(T^2) &= \int_H q(k^{-1}s)(({}^{K}f)^* * {}^{K}f)(k^{-1}s)\psi(s)ds \\ &= \int_H q(s)(({}^{K}f)^* * {}^{K}f)(s)\psi(s)ds = \psi(T^2) = \psi(T^2)\psi(k)\end{aligned}$$

therefore $\psi(T)$ commutes with $\psi(k)$ and for all $k \in H \cap K$

$$\begin{aligned}(\psi(k)\psi(T)\xi_{\gamma} \mid \psi(T)\xi_{\gamma}) &= (\psi(T^2)\xi_{\gamma} \mid \xi_{\gamma}) \\ &= \int_H R(({}^{K}f)^* * {}^{K}f)(s)\gamma(s)ds = \| [{}^{K}f]^{\gamma} \|^2 = 1 .\end{aligned}$$

But then $\| \psi(k)\psi(T)\xi_{\gamma} - \psi(T)\xi_{\gamma} \|^2 = 0$ thus

$$\psi(k)\psi(s)\psi(T)\xi_{\gamma} = \psi(s)\psi(s^{-1}ks)\psi(T)\xi_{\gamma} = \psi(s)\psi(T)\xi_{\gamma}$$

for all $s \in H$. Since $\psi$ is irreducible and $\psi(T)\xi_{\gamma} \neq 0$

$$\psi(k) = I \qquad \text{for all } k \in H \cap K .$$

If Haar measures on $G/K$ and $HK/K$, respectively, are suitable chosen and if $\rho \in P^1(HK/K)$ is defined by $\rho(p(s)) = \gamma(s)$, $s \in H$, it is easy to see that

$$(U^{\rho}_{p(x)}[T_K f_i]^{\rho} \mid [T_K f_i]^{\rho}) = (U^{\gamma}_{x}[{}^{K}f_i]^{\gamma} \mid [{}^{K}f_i]^{\gamma}) , \qquad x \in G$$

therefore (2.2) shows $\dot{\pi} \prec U^{\dot{\psi}}$.

COROLLARY 2.9. *If $G$ is an extension of a compact group $K$*

*with a group satisfying RFP, G has property RFP.*

*Proof.* $G$ is amenable, if $G/K$ is amenable, $K$ compact, normal. If $\psi \in \hat{H}$ is such that $I_G \prec {}_GU^{\psi}$, $I_{G/K} \prec U^{\dot{\psi}}$ holds by the proposition. $G/K \in [\text{RFP}]$ implies $\dot{\psi} = I_{HK/K}$ thus $\psi = I_H$.

If $\varphi \in P(G)$

$$\langle \varphi^x \mid H, h \rangle = \int_H h(s)(\pi_\varphi(s)\pi_\varphi(x)\xi_\varphi \mid \pi_\varphi(x)\xi_\varphi)ds$$

for $h \in L^1(H)$, $x \in G$. Thus for $a \in C^*(H)$, $x \in G$

$$(\varphi^x \mid H)(a) = ((\pi_\varphi \mid H)(a)\pi_\varphi(x)\xi_\varphi \mid \pi_\varphi(x)\xi_\varphi)$$

so that $a \in M_{\varphi^x|H}$ if and only if $(\pi_\varphi \mid H)(a)\pi_\varphi(x)\xi_\varphi = 0$. Since $\xi_\varphi$ is cyclic for $\pi_\varphi$ we get a characterization of $\ker \pi_\varphi \mid H$ by left ideals corresponding to positive definite functions on $H$

$$\ker \pi_\varphi \mid H = \bigcap_{x \in G} M_{\varphi^x|H} \,. \tag{2.3}$$

If $\varphi$ is a class function on $G$

$$\ker \pi_\varphi \mid H = M_{\varphi|H} = \ker \pi_{\varphi|H} \,. \tag{2.4}$$

We shall make frequent use of these formulas. We apply (2.3) to prove the following lemma which will be used in § 3.

LEMMA 2.10. *Let $H$ be a closed subgroup of a locally compact group $G$. Then $\pi_\varphi \prec {}_GU^{\varphi|H}$ for $\varphi \in P(G)$ if either*

*$G/H$ has finite volume or*
*$H$ is normal and $G/H$ is amenable.*

*Proof.* First let $H$ be a normal subgroup of $G$, $G/H$ amenable. By (2.3) we have

$$\ker \pi_\varphi \mid H = \bigcap_{x \in G} M_{(\varphi|H)^x} = \bigcap_{x \in G} \bigcap_{s \in H} M_{((\varphi|H)^x)^s}$$

$$= \bigcap_{x \in G} \ker \pi_{(\varphi|H)^x}$$

therefore $\pi_\varphi \mid H$ is weakly equivalent to the set of representations $(\pi_{\varphi|H})^x$, $x \in G$. Since the representations induced by $(\pi_{\varphi|H})^x$, $x \in G$, are equivalent to ${}_GU^{\varphi|H}$

$${}_GU^{\pi_\varphi|H} \prec {}_GU^{\varphi|H}\,, \quad \text{and} \quad \pi_\varphi \prec {}_GU^{\pi_\varphi|H} \text{ as } G/H \text{ is amenable [6].}$$

Now let $G/H$ have finite volume. We state

$$\|[f]^\varphi\|^2 \leqq \nu(G/H)\|[f]^\gamma\|^2\,, \qquad f \in C_{00}(G)$$

where $\nu$ is an invariant measure on $G/H$ and $\gamma = \varphi \mid H$: considering $\pi_\gamma$ as a subrepresentation of $\pi_\varphi \mid H$ and using the fact that $\Delta_G$ and $\Delta_H$ coincide on $H$ it is easy to check

$$\begin{aligned}\|[f]^\varphi\|^2 &= \int_G\int_G \varphi(y^{-1}x)f(x)\overline{f(y)}dydx \\ &= \int_G\int_G b(x)b(y)(\pi_\varphi(x)\pi_\gamma(R({}_xf))\xi_\gamma \mid \pi_\varphi(y)\pi_\gamma(R({}_yf))\xi_\gamma)dydx\end{aligned}$$

where $b$ denotes a Bruhat function for $H$. Therefore

$$\begin{aligned}\|[f]^\varphi\| &\leqq \int_G b(x)\,\|\pi_\varphi(x)\pi_\gamma(R({}_xf))\xi_\gamma\|\,dx \\ &= \int_{G/H}\int_H b(xs)\,\|\pi_\gamma(R({}_{xs}f))\xi_\gamma\|\,dsd\nu(\dot{x})\,.\end{aligned}$$

Since the function $x \to \|\pi_\gamma(R({}_xf))\xi_\gamma\|$ is constant on cosets (as $q(s) = 1$, $s \in H$) and $\int_H b(xs)ds = 1$, $x \in G$

$$\begin{aligned}\|[f]^\varphi\|^2 &\leqq \left(\int_{G/H}\|\pi_\gamma(R({}_{\dot{x}}f))\xi_\gamma\|\,d\nu(\dot{x})\right)^2 \\ &\leqq \nu(G/H)\int_{G/H}\|\pi_\gamma(R({}_{\dot{x}}f))\xi_\gamma\|^2\,d\nu(\dot{x}) \\ &= \nu(G/H)\int_G b(x)\,\|\pi_\gamma(R({}_xf))\xi_\gamma\|^2\,dx\end{aligned}$$

but

$$\int_G b(x)\,\|\pi_\gamma(R({}_xf))\xi_\gamma\|^2\,dx = \|[f]^\gamma\|^2$$

by Blattner's theorem (see [18, Thm. 4.4]). Now let $\{f_i,\ i \in I\}$ be an approximate identity for $C_{00}(G)$ in the inductive limit topology and for $i \in I$ let

$$\begin{aligned}\varphi_i(x) &= (\pi_\varphi(x)[f_i]^\varphi \mid [f_i]^\varphi)\,, \\ \rho_i(x) &= (U_x^\gamma[f_i]^\gamma, [f_i]^\gamma)\,, \qquad x \in G\,.\end{aligned}$$

Then for $f \in C_{00}(G)$

$$\varphi_i(f^* * f) = \|[f * f_i]^\varphi\|^2 \leqq \nu(G/H)\rho_i(f^* * f)$$

thus $\pi_{\varphi_i}$ is a subrepresentation of $\pi_{\rho_i}$ by [2, 2.5.1]. Since $\pi_{\rho_i}$ is contained in $U^\gamma$ and $\pi_\varphi \prec \{\pi_{\varphi_i},\ i \in I\}$ (by Lemma 2.2) $\pi_\varphi \prec U^\gamma$.

REMARK 2.11. If $G$ is first countable we can choose $r_i > 0$, $i \in N$, such that $f_0 = \sum_{i \in N} r_i f_i^* * f_i \in C_{00}(G)$. Then one shows as in [11] that $[f_0]^\varphi$ is a cyclic vector for $\pi_\varphi$ (the lemma used in [11] is correct if the measure is defined by a positive definite function). Therefore $\pi_\varphi$ is a subrepresentation of $U^\gamma$ in the case $G/H$ to have finite volume.

COROLLARY 2.12. *Let $G = G_{n+1}$ be amenable and let $G_i$, $1 \leqq i \leqq n$, be an ascending chain of closed subgroups of $G$. If $G_i$ is normal in $G_{i+1}$ or if $G_{i+1}/G_i$ has finite volume, $1 \leqq i \leqq n$, then $\pi_\varphi \prec {}_G U^{\varphi|G_1}$ for all $\varphi \in P(G)$.*

*Proof.* Let $\rho = \varphi \mid G_n$ and suppose

$$\pi_\rho \prec {}_{G_n} U^{\rho|G_1}$$

then

$${}_G U^\rho \prec {}_G U({}_{G_n} U^{\rho|G_1}) = {}_G U^{\varphi|G_1} .$$

Using Lemma 2.10 the assertion follows by induction.

By Corollary 2.9, in order to prove that groups $G \in [FC]^-$ have RFP we may suppose $G \in [\mathrm{SIN}]$.

## 3. Topological Frobenius properties for SIN-groups.

Let $H$ be a closed subgroup of a SIN-group $G$ and $\psi$ be a unitary representation of $H$. It has been shown in [9] that the restriction to $H$ of ${}_G U^\psi$ contains $\psi$ as a subrepresentation therefore

THEOREM 3.1. SIN-*groups have property* $WF2$ (*defined by Fell in* [4]: *for every closed subgroup $H$ and $\psi \in \hat{H}$ $\psi \prec {}_G U^\psi \mid H$*).

Representations corresponding to positive definite measures of metric groups are known to be cyclic. What we shall need is the following fact.

PROPOSITION 3.2. *Let $G \in [\mathrm{SIN}]$ be first countable. If $\gamma \in P^1(H)$ is indecomposable then there exists an extension $\varphi \in P(G)$ of $\gamma$ such that $\pi_\varphi$ is weakly equivalent to ${}_G U^\gamma$.*

*Proof.* As $G \in [\mathrm{SIN}]$ there is an approximate identity for $C_{00}(G)$ in the inductive limit topology consisting of class functions (see [7] or [9]). Moreover, we can choose $f_i \in C_{00}(G)$ and $r_i > 0$ such that supports $S_i$ of $f_i^* * f_i$ are contained in a compact set $K$ and $g_n = \sum_{i=1}^n r_i f_i^* * f_i$ converges uniformly on $K$ to a class function $f \in C_{00}(G)$. Since $f_i$ is a class function for $x \in G$

$$\begin{aligned}\rho_i(x) := (U_x^\gamma[f_i]^\gamma \mid [f_i]^\gamma) &= \mu^\gamma(f_i^* *{}_{x^{-1}} f_i) \\ &= \mu^\gamma((f_i^* * f_i)_{x^{-1}}) .\end{aligned}$$

We define

$$\varphi(x) = \mu^\gamma(f_{x^{-1}}) , \qquad x \in G$$

then $\varphi$ is continuous as $x \to f_{x^{-1}}$ is continuous and $\mu^\gamma$ is a Radon measure. Furthermore, $\varphi$ is positive definite as

$$\varphi(x) = \lim_{n\to\infty} \sum_{i=1}^{n} r_i \rho_i(x) \qquad \text{for } x \in G .$$

By Lemma 2.1 in [9] $\rho_i \mid H = \rho_i(e)\gamma$ and by the proof of that lemma we may assume $\mu^\gamma(f) = 1$ therefore

$$\varphi \mid H = \gamma \sum_{i=1}^{\infty} r_i \rho_i(e) = \gamma \sum_{i=1}^{\infty} r_i \mu^\gamma(f_i^* * f_i) = \gamma .$$

Now let $g \in C_{00}(G)$, $S = \operatorname{supp} g$ then

$$\begin{aligned} \left| \langle \varphi, g \rangle - \sum_{i=1}^{n} r_i \langle \rho_i, g \rangle \right| &\leqq \int_S | g(x) | \, | \mu^\gamma((f - g_n)_{x^{-1}}) | \, dx \\ &\leqq \int_S | g(x) | \int_H | \gamma(s) | \, | (f - g_n)(sx^{-1}) | \, ds dx \\ &\leqq \sup_{y \in K} | (f - g_n)(y) | \cdot \int_{H \cap KS} ds \cdot \| g \|_{L^1(G)} \end{aligned}$$

hence for all $a \in C^*(G)$

$$\varphi(a) = \sum_{i=1}^{\infty} r_i \rho_i(a) .$$

Since $\varphi^x(a) = \varphi(a^{x^{-1}})$, $x \in G$, by [17, 1.8],

$$\varphi^x(a) = \sum_{i=1}^{\infty} r_i \rho_i^x(a) \qquad \text{for } a \in C^*(G),\ x \in G.$$

As $r_i > 0$ $\varphi^x(a^* a) = 0$ if and only if $\rho_i^x(a^* a) = 0$ for $i \in N$ thus

$$\ker \pi_\varphi = \bigcap_{x \in G} M_{\varphi^x} = \bigcap_{i \in N} \ker \pi_{\rho_i} .$$

By Lemma 2.2, $U^\gamma$ is weakly equivalent to $\{\pi_{\rho_i}, i \in N\}$ hence $U^\gamma$ and $\pi_\varphi$ are weakly equivalent.

Let $N$ be a closed normal subgroup of $G \in [\mathrm{SIN}]$ contained in $G_F$ and let $\operatorname{Aut}(N)$ be the group of all topological automorphisms of $N$ with the Birkhoff topology [10, §26]. $I(N, H)$ denotes the subgroup of all $n \to xnx^{-1}$, for $x$ in a closed subgroup $H$ of $G$, then $B = \overline{I(N, H)}$ is compact in $\operatorname{Aut}(N)$ [7, Thm. (0.1)] and we define as in [17]:

$f^H(n) = \int_B f^\tau(n) d\tau$ where $d\tau$ is the normalized Haar measure on $B$. If $\rho \in P(N)$ $\rho^H \in P(N, H)$ and $\rho \to \rho^H$ is a continuous affine mapping from $P_1(N)$ onto $P_1(N, H)$ [17, 1.9].

Furthermore, for $a \in C^*(N)$

$$\rho^H(a) = \int_B \rho^\tau(a)d\tau .$$

Since $\tau \to \rho^\tau(a)$ is continuous on $B$

$$M_{\rho^H} = \bigcap_{\tau \in B} M_{\rho^\tau} = \bigcap_{x \in H} M_{\rho^x}$$

combining this with (2.3) we get for $\varphi \in P(G)$

$$\ker(\pi_\varphi \mid N) = M_{(\varphi \mid N)^G} = \ker \pi_{(\varphi \mid N)^G} . \tag{3.1}$$

If $\varphi \in P^1(G)$ is associated with $\pi \in \hat{G}$, $(\varphi \mid N)^G \in E(N, G)$ by Lemma 1 in [13]. Conversely, if $\alpha \in E(N, G)$ we can find an indecomposable function $\rho \in P^1(N)$ satisfying $\rho^G = \alpha$. By [9, Satz 2] there exists an extension $\varphi \in \operatorname{ex} P^1(G)$ of $\rho$, thus $(\varphi \mid N)^G = \alpha$. The mapping $\varphi \to (\varphi \mid N)^G$, $\varphi \in \operatorname{ex} P^1(G)$, is continuous and $\alpha \to M_\alpha$ defines a homeomorphism of $E(N, G)$ onto $G - \operatorname{Max} C^*(N)$ the set of all maximal modular $G$-stable ideals of $C^*(N)$ endowed with hull-kernel topology [17, Proposition 4.8]. Therefore

PROPOSITION 3.3. *$\pi \to \ker(\pi \mid N)$ defines a continuous map from $\hat{G}$ onto $G$-$\operatorname{Max} C^*(N)$.*

REMARK 3.4. If $N$ is open we can consider $C^*(N)$ as a subalgebra of $C^*(G)$ thus $\ker(\pi \mid N) = \ker \pi \cap C^*(N)$. In this case the map $\pi \to \ker(\pi \mid N)$ has been studied in [13] and has some more properties stated in [13, Thm. 1].

Let $H$ be a closed subgroup of $G$ and $\rho \in E(N, H)$. Since $P_1(N)$ is compact, convex there exists $\varphi \in \operatorname{ex} P_1(N)$ satisfying $\varphi^H = \rho$. By changing order of integration, for $n \in N$

$$\rho^G(n) = \int_{\overline{I(N,G)}} \varphi^H(\tau^{-1}(n))d\tau = \int_{\overline{I(N,H)}} \left(\int_{\overline{I(N,G)}} \varphi^{\tau\sigma}(n)d\tau\right)d\sigma$$
$$= \varphi^G(n) \qquad \text{thus } \rho^G = \varphi^G \in E(N, G) \text{ [17, 5.1].}$$

In the following lemma we summarize such functorial properties and further known facts concerning $E(N, H)$ used in this paper.

LEMMA 3.5. *Let $H$ be a closed subgroup of $G \in [\mathrm{SIN}]$ and let $N$ be a closed normal subgroup of $G$ contained in $G_F$.*

(1) *$\varphi \to \varphi \mid H$ maps $E(G, H)$ onto $E(H)$ [9, Lemma 1.3 and Satz 2][2].*

[2] Lemma 1.3 in [9] holds for arbitrary locally compact groups. The notation $I(H)$ in [9] does not refer to the inner automorphisms of $H$ but rather to the inner automorphisms of $G$ induced by elements of $H$.

(2) *$\varphi \to (\varphi \mid N)^G$ maps* $\operatorname{ex} P^1(G)$ *onto* $E(N, G)$.

(3) *If $\rho \in E(N, H)$, $\rho^G$ is in $E(N, G)$.*

(4) *The closure $F(N, H)$ of $E(N, H)$ with respect to the Pontryagin topology is locally compact and $F(N, H) \cup \{0\}$ is equal to the weak $*$-closure of* $\operatorname{ex} P_1(N, H) = E(N, H) \cup \{0\}$ [9, *Korollar* 2.8].

(5) *If $N$ is contained in $H$,* $\operatorname{ex} P_1(N, H)$ *is compact* [17, 4.2; 12, *Satz* 1; 21, *Satz* 1].

Let $N$ be contained in $H$. Then it is well known that for given $\beta \in P^1(N, H)$ there exists a unique normalized positive Radon measure $\mu$ on $P_1(N, H)$ such that $\mu$ has resultant $\beta$, i.e.,

$$\langle \beta, f \rangle = \int_{P_1(N,H)} \langle \gamma, f \rangle d\mu(\gamma) \qquad \text{for all } f \in L^1(N) ,$$

and $\operatorname{supp} \mu \subseteqq \operatorname{ex} P_1(N, H)$ holds [20, Satz 1; 17, 2.2]. If $N = H$ the unique measure $\mu$ is denoted by $\mu_\beta$. For arbitrary subgroups $H$ of $G$ maximal measures on $P_1(N, H)$ (with respect to Choquet ordering) having resultant $\beta$ don't need to be unique.

LEMMA 3.6. *Let $N$ be a closed normal subgroup of $G \in$ [SIN] contained in $G_F$ and for $\beta \in P^1(N, G)$ let $\mu$ be the unique maximal measure on $P_1(N, G)$ with resultant $r(\mu) = \beta$.*

(1) *If $H$ is a closed subgroup of $G$ and if $\nu$ is any maximal measure on $P_1(N, H)$ such that $r(\nu)^G = \beta$ then*

$$\operatorname{supp} \mu = (\operatorname{supp} \nu)^G = \{\rho^G ; \rho \in \operatorname{supp} \nu\} .$$

(2) *For $\alpha \in E(N, G)$*

$$\pi_\alpha \prec \pi_\beta \textit{ if and only if } \alpha \in \operatorname{supp} \mu .$$

*Proof.*

(1) The image $\nu^G$ of $\nu$ corresponding to the continuous affine mapping $\rho \to \rho^G$ from $P_1(N, H)$ onto $P_1(N, G)$ has resultant $r(\nu)^G = \beta$ and

$$\operatorname{supp} \nu^G = (\operatorname{supp} \nu)^G \subseteqq (\overline{\operatorname{ex} P_1(N, H)})^G \subseteqq E(N, G) \cup \{0\}$$

(this follows from Choquet theory and Lemma 3.5). By uniqueness $\mu = \nu^G$ and the assertion follows.

(2) Since $\mu$ has resultant $\beta$

$$\beta(a) = \int_{P_1(N,G)} \gamma(a) d\mu(\gamma) \qquad \text{holds for } a \in C^*(N)$$

thus

$$M_\beta = \bigcap_{\gamma \in \operatorname{supp} \mu} M_\gamma = \bigcap_{0 \neq \gamma \in \operatorname{supp} \mu} M_\gamma$$

as $\gamma \to \gamma(a)$ is continuous on $P_1(N, G)$ for every $a \in C^*(N)$. Since $\alpha, \beta$ are class functions $\ker \pi_\alpha = M_\alpha \supseteqq M_\beta = \ker \pi_\beta$ if $\alpha \in \operatorname{supp} \mu$. Conversely, if $\pi_\alpha \prec \pi_\beta$ $M_\alpha$ is in the closure of $\{M_\gamma, \gamma \in \operatorname{supp} \mu \backslash \{0\}\}$ in $G$-Max $C^*(N)$ with respect to hull-kernel topology, therefore $\alpha \in \operatorname{supp} \mu$.

THEOREM 3.7. *Suppose $G \in [\mathrm{SIN}]$ and let $H$ be a closed subgroup of $G$ contained in $G_F$. If $\psi \in \hat{H}$, and $\pi \in \hat{G}$ is weakly contained in ${}_GU^\psi$ then $\pi \mid H$ weakly contains $\psi$.*

*Proof.* By [7, Thm. 2.11; 16, Lemma 4.3] any SIN-group $G$ is a projective limit of Lie groups $G/K_j$, $j \in J$, $K_j$ compact normal. In particular, every $G/K_j$ is first countable. By Proposition 2.3 in [16], there exists $j \in J$ such that $\pi(K_j) = \{I\}$. Since $K_jH/K_j$ is contained in $(G/K_j)_F$, by Proposition 2.8 we may assume $G$ to be first countable.

Now let $\psi = \pi_\gamma$, $\gamma \in P^1(H)$, and let $\varphi \in P^1(G)$ be an extension of $\gamma$ such that $\pi_\varphi$ is weakly equivalent to $U^\psi$ (such a function $\varphi$ exists by Proposition 3.2). Then

$$\pi \prec U^\psi \quad \text{implies} \quad \pi \mid G_F \prec \pi_\varphi \mid G_F \,.$$

By (3.1) $\ker(\pi_\varphi | G_F) = \ker \pi_{(\varphi | G_F)^G}$ and there exists $\alpha \in E(G_F, G)$ such that $\ker \pi_\alpha = \ker \pi \mid G_F$ (see Remark 3.4). Next, take some maximal measure $\nu$ on $P_1(G_F)$ with resultant $\varphi \mid G_F$. By Lemma 3.6 there is $\rho \in \operatorname{supp} \nu$ with $\rho^G = \alpha$ ($H = \{e\}$, $\beta = (\varphi \mid G_F)^G$), therefore

$$\ker \pi_\rho = \bigcap_{x \in G_F} M_{\rho^x} \supseteqq \bigcap_{x \in G} M_{\rho^x} = M_{\rho^G} = \ker \pi \mid G_F$$

and then

$$\pi_\rho \mid H \prec \pi \mid H \,. \tag{3.2}$$

As in the proof of Lemma 4.4 in [15] one shows: there exists a net $\{\rho_i\} \subseteqq P_1(G_F)$ and $r_i \geqq 0$, $i \in I$, with

$$r_i(\varphi \mid G_F) - \rho_i \in P(G_F)$$

such that $\rho$ is the weak $*$-limit of $\{\rho_i\}$. Since

$$\| \rho_i \| = \rho_i(e) \leqq 1 \quad \text{and} \quad \liminf \| \rho_i \| \geqq \| \rho \| = \rho^G(e) = 1$$

we may assume $\rho_i(e) = 1$. Then $\rho = \lim \rho_i$ uniformly on compact sets in $G$ thus $\rho \mid H = \lim \rho_i \mid H$. Since $\gamma$ is indecomposable and $\varphi \mid H = \gamma$, $r_i\gamma - \rho_i \mid H \in P(H)$, $i \in I$, implies $\rho_i \mid H = \gamma$ therefore $\rho \mid H = \gamma$. Then $\psi = \pi_\gamma$ is a subrepresentation of $\pi_\rho \mid H$ and by (3.2) $\psi \prec \pi \mid H$ follows.

REMARK. Since groups $G \in [FC]^- \cap [\mathrm{SIN}]$ are amenable [14] it follows from Theorem 3.7 that they have property RFP. For arbitrary $G \in [FC]^-$ there exists a compact normal subgroup $K$ of $G$ such that $G/K \in [FC]^- \cap [\mathrm{SIN}]$ thus $G$ satisfies RFP by Corollary 2.9. This completes the proof of Theorem A.

LEMMA 3.8. *Let $H$ be a closed subgroup of $G \in [\mathrm{SIN}]$ such that $H = H_F$ and for $\beta \in P^1(G, H)$ let $\nu$ be a maximal measure on $P_1(G, H)$ representing $\beta$. If $0 \notin \operatorname{supp} \nu$ then*

$$\operatorname{supp} \mu_{\beta|H} = \{\sigma \in E(H); \sigma = \rho \mid H, \rho \in \operatorname{supp} \nu\}$$

*in particular, $0 \notin \operatorname{supp} \mu_{\beta|H}$.*

*Proof.* The restriction map from $P_1(G)$ into $P_1(H)$ is not weak $*$-continuous in general, but if $0 \notin \operatorname{supp} \nu$

$$\operatorname{supp} \nu \subseteqq F(G, H) \subseteqq P^1(G, H)$$

therefore the map $R: \rho \to \rho \mid H$ from $\operatorname{supp} \nu$ into $P_1(H, H)$ is continuous. Since $E(H)$ is closed in Pontryagin topology the image $\nu^R$ of $\nu$ has support

$$R(\operatorname{supp} \nu) \subseteqq R(F(G, H)) \subseteqq E(H)$$

by Lemma 3.5. By the proof of Lemma 2.9 in [9]

$$\beta(x) = \int_{\operatorname{supp} \nu} \rho(x) d\nu(\rho) \quad \text{for} \quad x \in G \quad \text{thus}$$

$$\beta(s) = \int_{E(H)} \gamma(s) d\nu^R(\gamma) \quad \text{for} \quad s \in H \quad \text{and then}$$

$$\langle \beta \mid H, h \rangle = \int_{P_1(H,H)} \langle \gamma, h \rangle d\nu^R(\gamma) \quad \text{for} \quad h \in L^1(H)$$

hence $\nu^R = \mu_{\beta|H}$.

COROLLARY 3.9. *Let $N$ be a closed normal subgroup of $G \in [\mathrm{SIN}]$ contained in $G_F$ and let $\alpha \in E(N, G)$. If $F, H$ are closed subgroups of $N$, $F \subseteqq H$, and if $\nu$ is a maximal measure on $P_1(H, F)$ with resultant $\alpha \mid H$ then $0 \notin \operatorname{supp} \nu$.*

*Proof.* Let $\nu_1$ be a maximal measure on $P_1(N, H)$ with $r(\nu_1) = \alpha$, then $\{\alpha\} = (\operatorname{supp} \nu_1)^G$ by Lemma 3.6, therefore $0 \notin \operatorname{supp} \nu_1$. By Lemma 3.8 $0 \notin \operatorname{supp} \mu_{\alpha|H}$ and again by Lemma 3.6 $0 \notin \operatorname{supp} \nu$.

REMARK. The same holds if $\alpha$ is the resultant of a probability measure $\mu$ on $P_1(N, G)$ with $\operatorname{supp} \mu \subseteqq E(N, G)$.

G. Schlichting has pointed out to me the following corollary.

COROLLARY 3.10. *Let $G, N, \alpha$ as in Corollary 3.9 and let $H$ be a compact subgroup of $N$. Then $\mu_{\alpha|H}$ has finite support.*

*Proof.* By [12, Satz 3], $E(H)$ is discrete and

$$\operatorname{supp} \mu_{\alpha|H} \subseteqq E(H) \qquad \text{(Corollary 3.9)} .$$

REMARK 3.11. Let $G \in [\mathrm{SIN}]$ and $N \subseteqq G_F$ be a discrete normal subgroup of $G$. Since every element in $N/Z(N)$ has finite order, $Z(N)$ the center of $N$, every finite set in $N/Z(N)$ generates a finite subgroup [19, Thm. 4.3.2 and Corollary 2, p. 45]. Thus every finite subset of $N$ is contained in a normal subgroup $M$ of $G$ such that

$$Z(N) \subseteqq M \subseteqq N \quad \text{and} \quad [M: Z(N)] < \infty .$$

THEOREM 3.12. *Let $G$ be an amenable SIN-group and $H \subseteqq G_F$ be a closed subgroup. If $\pi \in \hat{G}$, and if $\psi \in \hat{H}$ is weakly contained in $\pi \mid H$, then ${}_G U^{\psi}$ weakly contains $\pi$.*

*Proof.* Take $\alpha \in E(G_F, G)$, $\sigma \in E(H)$ such that $\pi \mid G_F$ is weakly equivalent to $\pi_\alpha$ and $\psi$ is weakly equivalent to $\pi_\sigma$ (see Remark 3.4 and the remarks preceding Proposition 3.3). By (2.4), $\psi \prec \pi \mid H$ implies $\pi_\sigma \prec \pi_\alpha \mid H \prec \pi_{\alpha|H}$ therefore

$$\sigma \in \operatorname{supp} \mu_{\alpha|H} \qquad \text{by Lemma 3.6.}$$

It is sufficient to prove

$$\pi_\alpha \prec \{({}_{G_F} U^{\sigma})^x,\ x \in G\} . \tag{3.3}$$

Actually, since the representations of $G$ induced by $({}_{G_F} U^{\sigma})^x$, $x \in G$ are equivalent to ${}_G U({}_{G_F} U^{\sigma}) = {}_G U^{\sigma}$ it follows from (3.3) and [6]

$$\pi \prec {}_G U^{\pi|G_F} \prec {}_G U^{\pi_\alpha} \prec {}_G U^{\sigma} \prec {}_G U^{\psi} .$$

Therefore let $Y$ be a compact subset of $G_F$. By [22] there exist normal subgroups $V, L$, and $K$ of $G$ such that $V$ is a vector group, $K$ is compact open in $L$, $L/K \subseteqq (G/K)_F$ and $G_F = VL$ is a direct product of $V$ and $L$[3]. Then by Remark 3.11 we can choose normal subgroups $M, Z$ of $G$, $K \subseteqq Z \subseteqq M \subseteqq L$, such that $[M: Z] < \infty$, $Z/K$ is the centre of $L/K$ and $Y$ is contained in $N = VM$. $VZ$ is an open subgroup as it contains $VK$. Now we consider the chain of subgroups

$$H \subseteqq HK \subseteqq HVZ \subseteqq HN .$$

[3] See the footnote to the proof of Theorem 2.7.

Since SIN-groups are unimodular $HK/H$ and $HN/HVZ$ have finite volume. $HK$ is normal in $HVZ$ as $Z/K$ is the centre of $L/K$ and $V$ is central in $G_F$. Therefore by Corollary 2.12

$$\pi_\rho \prec {}_{HN}U^{\rho|H} \quad \text{for} \quad \rho \in P(HN)\ . \tag{3.4}$$

Now let $\nu$ be a maximal measure on $P_1(HN, H)$ with resultant $\alpha|HN$. By Corollary 3.9 and Lemma 3.8, there exists $\rho \in \operatorname{supp} \nu$ such that

$$\rho \mid H = \sigma\ .$$

Since $\alpha \mid HN$ is a class function on $HN$ $\rho^{HN} \in \operatorname{supp} \mu_{\alpha|HN}$ by Lemma 3.6, thus $\pi_{\rho^{HN}} \prec \pi_{\alpha|HN}$. As $\ker \pi_\rho = \ker \pi_{\rho^{HN}}$ we get $\pi_\rho \prec \pi_{\alpha|HN}$, and $\pi_\rho \prec {}_{HN}U^\sigma$ follows from (3.4). Since $HN$ is open in $G_F$ we obtain by inducing up to $G_F$

$$\pi_\varphi \prec \pi_\beta \quad \text{and} \quad \pi_\varphi \prec {}_{G_F}U^\sigma$$

where $\varphi \in P(G_F)$ and $\beta \in P(G_F)$, respectively, denote the trivial extensions of $\rho$ and $\alpha \mid HN$, $\varphi(x) = 0 = \beta(x)$ if $x \notin HN$. Since $\pi_{\varphi^G}$ is weakly equivalent to $\{(\pi_\varphi)^x, x \in G\}$ therefore

$$\pi_{\varphi^G} \prec \pi_{\beta^G} \quad \text{and} \quad \pi_{\varphi^G} \prec \{({}_{G_F}U^\sigma)^x; x \in G\}\ .$$

Finally, take $\gamma \in E(G_F, G)$ such that $\pi_\gamma \prec \pi_{\varphi^G}$, then

$$\pi_{\gamma|N} \prec \pi_{\beta^G|N}\ .$$

But if $B = \overline{I(N, G)}$ and $n \in N$

$$\beta^G(n) = \int_B \beta(\tau^{-1}(n))d\tau = \int_B \alpha(\tau^{-1}(n))d\tau = \alpha(n)$$

therefore $M_{\gamma|N} \supseteqq M_{\alpha|N}$. Since $E(N, G)$ is homeomorphic to $G$-Max $C^*(N)$ and $\gamma \mid N$, $\alpha \mid N \in E(N, G)$

$$\gamma \mid N = \alpha \mid N$$

thus $\gamma$ and $\alpha$ agree on $Y$ and $\pi_\gamma \prec \{({}_{G_F}U^\sigma)^x; x \in G\}$ consequently

$$\pi_\alpha \prec \{({}_{G_F}U^\sigma)^x; x \in G\}\ .$$

REMARK. Theorem B follows from Theorem 3.7 and Theorem 3.12.

COROLLARY 3.13. *For* SIN-*groups* $G$ *the following conditions are equivalent*

1. $G \in [FP]$
2. $G \in [RFP]$
3. $G = G_F$.

*Proof.* Clearly, $1 \Rightarrow 2$, $2 \Rightarrow 3$ by Theorem 2.7 and $3 \Rightarrow 1$ follows from Theorem B.

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