PACIFIC JOURNAL OF MATHEMATICS
Vol. 82, No. 2, 1979

A CONVERSE TO (MILNOR-KERVAIRE THEOREM) $\times R$ ETC...

MICHAEL H. FREEDMAN

One of the most puzzling questions in low dimensional topology is which elements $\alpha \in \pi_2(M)$, where M is a smooth compact 4-manifold, may be represented by a smoothly imbedded 2-sphere. This paper treats a stable version of the problem: When is there a smooth proper imbedding, h: $S^2 \times \boldsymbol{R} \hookrightarrow M \times \boldsymbol{R}$ by which the ends of $S^2 \times R$ are mapped to the ends of $M \times R$, and for which the composition

$$S^2 \xrightarrow{x \to (x,0)} S^2 \times \boldsymbol{R} \xrightarrow{h} M \times \boldsymbol{R} \xrightarrow{\pi} M$$

represents α?

If there is an h as above, we say that α is stably represented. We are able to determine precisely which α are stably represented when M is simply connected. In general, the vanishing of a finiteness obstruction ($\in K_0(Z[\pi_1(M)])$) yields Poincare' imbeddings of S^2 in M. In the nonsimply connected case, sufficient information is obtained to carry out surgery $\times \boldsymbol{R}$, yielding an alternative construction of manifold structures on (4-dimensional Poincare' spaces) $\times \boldsymbol{R}$. All terminology will be smooth.

We say a class $\alpha \in \pi_2(M)$ is characteristic if the composition,

$$\pi_2(M) \xrightarrow{\text{Hur}} H_2(M; Z) \xrightarrow{(2)} H_2(M; Z_2) \xrightarrow{\partial} H_2(M, \partial; Z_2) \xrightarrow{P.D.^{-1}} H^2(M; Z_2) ;$$

carries α to $w_2(\tau(M))$. Otherwise, we say α is ordinary.

If α is characteristic and $N(\alpha) = 0$ there is a well defined number, $Arf(q(\alpha)) = 0$ or 1, which is the Arf invariant of a certain Z_2-quadratic form. When M is closed this Arf invariant is related to more familiar invariants by the formula $\text{Arf}(q(\alpha)) = \text{Hur}\,\alpha \cdot \text{Hur}\,\alpha$—signature $(M^4)/8 \pmod 2$. See [2] for details.

We say α has a spherical dual if there is a $\beta \in \pi_2(M)$ with $\lambda(\alpha, \beta) = 1$, where λ is the Wall-intersection form taking values in $Z[\pi_1(M)]$, see [5].

MAIN THEOREM (case: $\pi_1(M) = 0$). *α is stably represented if and only if α is ordinary or α is characteristic and* $\text{Arf}(q(\alpha)) = 0$.

MAIN THEOREM (case: $\pi_1(M) \neq 0$). *If α is stably represented, the Wall self intersection form $\mu(\alpha')$ is 0 for some immersion α' homotopic to α and if α is characteristic* $\text{Arf}(q(\alpha)) = 0$. *Conversely* *if either* (1) α *is ordinary*, $\mu(\alpha') = 0$, *has a spherical dual* β *and* $\pi_2(M) \xrightarrow{\text{Hur.}} H_2(M; Z_2)$ *is epi or* (2) α *is characteristic*, $\mu(\alpha') = 0$, α *has a spherical dual* β, *and* $\text{Arf}(q(\alpha)) = 0$ *then* α *is stably represented.*

ADDENDUM. Any h constructed by the main theorem may be chosen so that ker: $\pi_1(M \times R - h(S^2 \times R)) \xrightarrow{\text{inc}} \pi_1(M \times R)$ is central and cyclic, and the two ends of ($M \times R$ – open tube $(h(S^2 \times R))$ each determine isomorphisms from the usual inverse limit to $\pi_1(M \times R - h(S^2 \times R))$.

Other stable imbeddings are descirbed at the end in Remarks A and B.

We may drop the Hurewicz map from our notation, writing intersection numbers as $\alpha \cdot \alpha$, for example.

A theorem of Milnor and Kervaire [3] states: If M is closed and if α is characteristic and is represented by a smoothly imbedded 2-sphere, then $\alpha \cdot \alpha - \sigma(M)/8 \equiv 0 \pmod 2$.

The converse of this theorem is false (for example, if γ genrates $H_2(CP^2, Z)$, then $7\gamma - \sigma(CP^2)/8 = 49\text{–}1/8 \equiv 0 \pmod 2$; but by a theorem of Tristram's [4] no surface of genus < 10 may be smoothly imbedded to represent 7γ), so we are required to cross with R to obtain our main theorem.

Proof, case $\pi_1(M) = 0$, *"only if" direction.* In [2], we defined the quadratic form, q, on $H_1(\text{Surface}; Z_2)$, for any imbedded surface K presenting a characteristic class in $H_2(M; Z)$, essentially as follow; if A is a simple closed curve on K let B be an immersed oriented surface with $\partial B = A$ and B meeting K transversely, except at A where the meeting is "normal". Define $q([A]) =$ (# intersections of interior (B) with K) + (the Euler obstruction to extending $\nu_{A \hookrightarrow K}$ as a section of $\nu_{B \hookrightarrow M}$ $[B, \partial]$) (mod 2). The two terms in this definition will be denoted $\pitchfork_B$ and x_B respectively; subscripts will be omitted when only one disk, B, is being discussed.

Assume there is an imbedding, $h: S^2 \times R \to M \times R$. Make h transverse to $M \times 0$. Since h maps ends to ends, a connectivity argument shows that $h^{-1}(M \times 0)$ is homologous (in $S^2 \times R$) to $S^2 \times 0$. So with suitably chosen orientation, $h(S^2 \times R) \cap M \times 0$ represents Hur(α). Put $K = h(S^2 \times R) \cap M \times 0$. $K = \partial \bar{K}$, where $\bar{K} = h(S^2 \times R) \cap M \times [0, \infty)$. A homological argument shows that $\ker(\text{inc}_* H_1(K; Z) \to H_1(\bar{K}; Z))$ is a summand of $H_1(K; Z)$ generated by the "first half" of a symplectic basis, $a_1, \cdots, a_k$. To show $\text{Arf}(q(\alpha)) = 0$ it is sufficient to show that $q(a_i) = 0$, $1 \leqq i \leqq k$. We do this.

Let A_i be a simple closed curve representing a_i and let A_i bound an immersed oriented surface, B, in $M \times 0$ (as above) and a imbedded surface $B' \subset \bar{K}$. Let $\bar{B} = B \cup_{A_i} B'$. Since $[h(S^2 \times \boldsymbol{R})] \in H^{\mathrm{int}}{}_2(M \times R; Z)$ is dual to $w_2(\tau(M \times R)) \in H^2(M \times R; Z)$, the intersection number $\bar{B} \cdot h(S^2 \times R) = w_2(\tau(M \times R)/\bar{B})[\bar{B}] = w_2(\nu_{\bar{B} \hookrightarrow M \times R})[\bar{B}] = x + x' (\bmod 2)$ where x' is the obstruction to extending v, the orthogonal complement to the inward vector field on ∂B in $\nu_{K \hookrightarrow M \times 0}$, to a section of $\nu_{\bar{K} \hookrightarrow M \times [0,\infty)}/B'$ evaluated on $[B', \partial]$. Let $\pitchfork'$ be the intersection number of B' and $\bar{K}$ when B' is displaced from $\bar{K}$ by pushing along v. Then,

$$\bar{B} \cdot h(S^2 \times R) = \pitchfork + \pitchfork'$$

so

$$\pitchfork + \pitchfork' \equiv x + x' \pmod 2$$

but clearly $\pitchfork' = x'$ so

$$\pitchfork + x \equiv 0 \pmod 2$$

so $q([a_i]) = 0$.

Note. When $\pi_1(M) \neq 0$ $q(\alpha)$ may still be defined (see [2]) and essentially the same argument shows $\mathrm{Arf}(q(\alpha))$ must be zero if h exists.

If direction: α characteristic and $\pi_1 M \cong 0$.

Let $h_0: T_0 \to M \times 0$ be an imbedding of an oriented surface representing $\alpha \otimes [0] \in H_2(M \times 0; Z)$. Since $\mathrm{Arf}(q(\alpha)) = 0$ there is a symplectic basis, $\{a_i, a_i'\}$, for $H_1(h_0(T_0); Z)$ with $q[a_i]_2 = q[a_i']_2 = 0$ $\forall_i$. Let A_i and A_i' imbedded circles representing a and a_i' respectively. Assume $A_1 \cap A_j = \phi$ for $i \neq j$, $A_i \cap A_j' = \phi$ if $i \neq j$, A_i and A_i' meet transversley in one point. Such circles will be called a standard family for the symplectic basis. Surgery on $\{A_i\}$ or $\{A_i'\}$ would convert T_0 to a 2-sphere. The usual low dimensional problems make it impossible to carry out these surgeries ambiently. We will see, however, that after suitably enlarging the genus of T_0 (to improve the complement) the surgeries based on $\{A_i\}$ and $\{A_i'\}$ may be carried out ambiently, and the trace of these surgeries is imbedded in $M \times [-1, 1]$. Once more, we add 1-handles to improve the complement and 2-handles to reduce the first homology of the trace. In the limit the trace finally becomes $S^2 \times R$. The resulting imbedding is analogous to an imbedding of R in $I \times R$ built from {3 points}$\subset$ $I \times 0$ by attaching 0 and 1-handles (see Diagram 1).

Let $\{A_i, A_i'\}$ be the boundaries of immersed disks in $M \times 0$, $\{B_i, B_i'\}$. We use the same letter (B_i) to denote the image and map. We require that the inward normals to A_i in B_i and A_i' in B_i' lie in $\nu_{h_0(T_0) \hookrightarrow M \times 0}$ and that interior (B_i) and interior (B_i') be transverse to $h_0(T_0)$. Since $\pi_1(M) = 0$, $\nu_{A_i \hookrightarrow B_i}$ (or $\nu_{A_i' \hookrightarrow B_i'}$) may be arbitrarity specified as a section of $\nu_{h_0(T) \hookrightarrow M \times 0/A_i (\text{or } A_i')}$. This allows us to pick $\{B_i, B_i'\}$ so

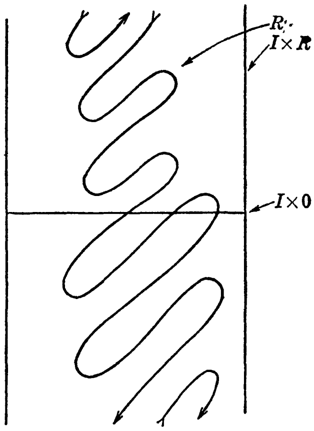

DIAGRAM 1

that the relative Euler classes, x_{B_i} and $x_{B_i'}$, are zeor for all i.

Let B denote the union of immersions $B = \cup_i (B_i \cup B_i')$.

LEMMA 1. *The family of immersions B is regularly homotopic* (rel ∂, *the homotopy taking place in* M) *to a family of disjoint imbeddings* $\bar{B} = \bigcup_i(\bar{B}_i \cup \bar{B}_i')$ (*we will drop the bar after the proof of this lemma*) *in transverse postion with* $\pitchfork_{\bar{B}_1} = \pitchfork_{\bar{B}_i'} = 0$ $\forall_i$ *and* $x_{\bar{B}_i} = x_{\bar{B}_i'} = 0$ $\forall_i$.

Notation. We arrange the intersection points of $\bar{B}_i$ and $h_0(T_0)$ in oppositely signed pairs (p_{ij}, p_{ij}') and the intersection points of $\bar{B}_i'$ and $h_0(T_0)$ in oppositely signed pairs $(p_{ij'}, p_{ij'}')$.

Proof. All immersions are assumed to be in transverse position. The formula $0 \equiv q(a) = \pitchfork_B + x_B$ ($\equiv$ is mod 2), shows that x_B is even as we have chosen B with $\pitchfork_B = 0$. Adding a $\pm$ double point to B in a chart changes x_B by ± 2. So we add sufficiently many double points to make $x_{B_i} = x_{B_i'} = 0$ $\forall_i$. Now we need to push the double points of B off ∂B.

The necessary construction is well known and may be summarized as follows. The double point set of B, $D^+ \subset M \times 0$, will consist of finitely many points. Let $D \subset D^+$ be the double point lying in $M \times 0 - h_0(T_0)$. For each double point $q \in D$ choose an arc, c, imbedded in B_i or B_i' to ∂B_i or $\partial B_i'$. Make sure the arc meets D^+ only at q. Now push one sheet of $\bigcup_i (B_i \cup B_i')$ at q (the one in which the arc c does not lie) along the arc and past its end point. This replaces q with an oppositely signed pair of intersection points (p, p') with $h_0(T_0)$. To obtain $\bar{B}$, do this for every q. Since pushing off double points in this manner does not change x or $\pitchfork$, the lemma is proved.

Let $\{\gamma\}$ stand for a disjointly imbedded family of n arcs in B. Each arc γ_{ij} or $\gamma_{i'j'}$, should have a pair (P_{ij}, P'_{ij}) or $(p_{ij}, p'_{i'j})$ as its boundary, and every such pair should occur as the boundary of some γ. Furthemore each arc, γ, should meet $h_0(T_0)$ only at its endpoints.

Now consider $h_{[-1,1]}: T_0\times[-1, 1]\to M\times[-1, 1]$ defined by $h_{[-1,1]}(t, r)=(h_0(t\times 0)) + r(+$is the obvious additive action of R on $M\times R)$. Let $\{\gamma_{\pm 1}\}$ be the family of arcs $\{\gamma\}\pm 1$. Let $T_{[-1,1]} = T_0\times[-1, 1]\bigcup_{T_0\times 1} n$(1-handles) $\bigcup_{T_0\times -1} n$(1-handles). Define a map

$$h^+_{[-1,1]}: T_{[-1,1]}\longrightarrow M\times[-1, 1]$$

so that

$$h^+_{[-1,1]}|T_0\times[-1, 1] = h_{[-1,1]} ,$$

and

$h^+_{[-1,1]}$(1-handles) form small relative tubular neighborhoods

of the arcs $\{\gamma_1\}$ and $\{\gamma_{-1}\}$. Actually, we would like $(h^+_{[-1,1]})^{-1}(M\times\{-1, 1\})$ to consist only of $(T_{[-1,1]})$.

The definition of $h^+_{[-1,1]}$ may be modified by deforming the interiors of the 1-handles toward the $M\times 0$-level to achieve this. Let $\partial(T_{[-1,1]}) = \partial_1 T_{[-1,1]}\cup\partial_{-1}T_{[-1,1]}$. Then $\partial_{\pm 1}T_{[-1,1]}$ is the result of ambient 0-surgery on $h_0(T_0)\pm 1$ along $\{\gamma_{\pm 1}\}$.

Now $B_i + 1$ and $B'_i - 1$ are imbedded in $M\times(\pm 1)$ with their interiors disjoint from $\partial_\pm T_{[-1,1]}$. $B_i + 1$ and $B'_i - 1$ (still) have zero Euler obstructions so we may form the trace $\bar{T}_{[-1,1]}$ of ambient 1-surgeries along $\bigcup_i(B_i + 1)\cup(B'_i - 1)$ to obtain:

$$\bar{h}_{[-1,1]}: (\bar{T}_{[-1,1]}, \partial)\longrightarrow(M\times[-1, 1], \partial) .$$

(In other words, we attach thickenings $(B_i + 1)\times I$ and $(B'_i - 1)\times I$ to $h^+_{[-1,1]}(\partial_\pm T_{[-1,1]})$ and push $(B_i + 1)\times\dot{I}$ and $(B'_i - 1)\times\dot{I}$ into $M\times(-1, 1)$.) So as a handle body, $\bar{T}_{[-1,1]} = T_{[-1,1]}\bigcup_{\partial+}(\text{2-handles})_i\bigcup_{\partial-}(\text{2-handles})_{i'}$.

Observation. $\ker(\text{inc}_*: H_1(\partial_\pm\bar{T}_{[-1,1]}; Z)\to H_1(\bar{T}_{[-1,1]}; Z))$ is generated by the first half of a symplectic basis on which the quadratic form q vanishes (setting $q(x) = ([x]_2)$).

Proof. Since $\bar{h}_{[-1,1]}(\bar{T}_{[-1,1]})$ is dual to $w_2(\tau(M\times[-1, 1]))$. The (only if)-argument shows that q vanishes on the kernel. It is easy to see that the kernel is generated by the first half of a symplectic basis for $H_1(\partial_\pm\bar{T}_{[-1,1]}; Z)$.

We denote $\partial_\pm(\bar{T}_{[-1,1]})$ by $T_{\pm 1}$. Since $\text{Arf}(q) = 0$, there is a summand $X_{\pm 1}\subset H_1(\bar{h}_{[-1,1]}(T_{\pm 1}); Z)$ which is complementary to the kernel on which the integral intersection form and q both vanish. Let

$\{a_j^1\}$ and $\{a_j^{-1}\}$ be basis for X_1 and X_{-1} and let $\{A_j^1\}$ and $\{A_j^{-1}\}$ be disjointly imbedded circles representing $\{a_j^1\}$ and $\{a_j^{-1}\}$. We now construct a map $\bar{h}_{[1,2]}$ to $M\times[1, 2]$ in a manner analogous to the construction of $\bar{h}_{[-1,1]}$. The construction may be outlined in three stages:

1. Construct $h_{[1,2]}$: $T_1\times[1, 2]\rightarrow M\times[1, 2]$ by:

$$h_{[1,2]}(t_1, r) = \bar{h}_{[-1,1]}(t_1) + (r - 1) \text{ (i.e., as a product) .}$$

2. Find immersed 2-disks B_j^2 with $\partial B_j^2 = A_j + 1$, and zero Euler obstruction. Use Lemma 1 to make $\{B_j^2\}$ disjointly imbedded. Form the trace of ambient 0-surgeries at the 2-level to make interior B_i^2 and $h_{[1,2]}(T_1\times 2)$ disjoint. Call the result:

$$h^+_{[1,2]}\colon T_{[1,2]} \longrightarrow M\times[1, 2] \text{ .}$$

3. Now form the trace of ambient 1-surgeries along $\{B_i^2\}$. Call the result:

$$\bar{h}_{[1,2]}\colon \bar{T}_{[1,2]} \longrightarrow M\times]1, 2] \text{ .}$$

In an analogous way we obtain a mapping $\bar{h}_{[n,n+1]}$: $\bar{T}_{[n,n+1]}\rightarrow M\times[n, n + 1]$ for every integer $n > 0$ and mappings $\bar{h}_{[n,n-1]}$: $\bar{T}_{[n,n-1]}\rightarrow M\times[n, n - 1]$ for every integer $n < 0$.

$$h \overset{\text{def}}{=\!=} \bar{h}_{(-\infty,+\infty)} \overset{\text{def}}{=\!=} \bar{h}_{[-1,1]} \bigcup_{n>0} \bar{h}_{[n,n+1]} \bigcup_{n<0} \bar{h}_{[n,n-1]} \text{ .}$$

The domain of h is $T_{(-\infty,+\infty)} \overset{\text{def}}{\text{---}} \bar{T}_{[-1,1]} \bigcup_{n>0} \bar{T}_{[n,n+1]} \bigcup_{n<0} \bar{T}_{[n,n-1]}$.

LEMMA 2. $T_{(-\infty,+\infty)} = S^2\times \boldsymbol{R}$.

Proof. Let F be an oriented surface and $\{a_i, a_i'\}$ be a sumplectic basis for $H_1(F; Z)$, and $\{A_i, A_i'\}$ be a standard family of imbedded circles representing the basis. One may check that the result of attaching complementary handles to $F\times[-1, 1]$ is $S^2\times I$, i.e.: $(F\times[-1, 1]) \bigcup_{1-\text{level}}$ (2-handles attached to $A_i\times 1) \bigcup_{-1-\text{level}}$ (2-handles attached to $A_i'\times -1) = S^2\times I$.

If one considers $h(T_{(-\infty,+\infty)})$ in a neighborhood of the levels $M\times -1$ and $M\times -1$, two things happen; complementary 2-handles are attached and new 1-handles are attached. But ambiently, these occur in the opposite order, i.e., the index 1 critical values have slightly smaller absolute value than the index 2 critical values. Abstractly, however, the order may be reversed since the detcending 1-spheres of the 2-handles are disjoint from the transverse 1-spheres of the 1-handles. In that case, $\bar{T}_{[-1,1]}$ is $S^2\times[-1, 1] \bigcup_{1-\text{level}}$ 1-handles $\bigcup_{-1-\text{level}}$ 1-handles.

The same considerations apply at each pair of levels $(n, -n)$. In this way we finally straighten out all of $T_{(-\infty,+\infty)}$ by mapping nested compact sections diffeomorphically to $S^2\times[-n, n]$.

It is clear from the construction that h is a proper map. Also $\text{inc}_*[T_0] = \text{generator} \in H_2(T_{(-\infty,+\infty)}; Z)$ and $h_{0*}[T_0] = \alpha \otimes 0$, so h stably represents the desired class.

Proof of "if" direction when $\pi_1(M) = 0$ and α is ordinary. The proof hinges on the fact that the quadratic form associated to a surface, $K \hookrightarrow M$, $q: H_1(K; Z_2) \to Z_2$ is *not* well defined when $\alpha = i_*[K]$ is ordinary. If α is ordinary, there is a (spherical) class $\gamma \in H_2(M;Z)$ with $\gamma\cdot\gamma \not\equiv \gamma\cdot\alpha$ (mod 2). So if $a \in H_1(K; Z_2)$ and (B, A) are as before and (B', A) is obtained by forming an immersed connected sum of B and a representative of γ, then $q(a)$ defined in terms of (B, A) and $q(a)$ defined in terms of (B', A) are different. Therefore, by picking B or B' we may arrange $q(a) = 0$. Furthermoɹe we may arrange that the terms x and $\pitchfork$ are both zero.

This allows us to repeat the proof of Theorem 1, for whenever we need q to vanish on a particular subspace we can make it do so by choosing the disks (B, A) suitably. This completes the proof of the main theorem in the simply connected case.

LEMMA 3. *If $\alpha \in \pi_2(M)$ is stably represented, then $\mu(\alpha') = 0$ for some immersion α' homotopic to α.*

Proof. As before, make h transverse to $M\times 0$ and put $T_0 = h^{-1}(M\times 0)$. Consider $h_0: T_0 \to M\times 0$. Let $A_1, \cdots, A_n$ be disjointly imbedded circles in T_0 representing the first half of a symplectic basis of $H_1(T_0; Z)$. For all i, A_i are null-homotopic in $S^2\times R$. By composing with h, $h(A_i)$ bounds an immersed, normal disk, $b_i: D_i^2 \to M\times R$, denoted simply as B_i. The composition:

$$S^2 \xrightarrow{\text{collapse}} T_0 \bigcup_{A_i} D_i^2 \xrightarrow{h_0 \cup b_i} M \times R$$

represents α. One sees that the composition may be approximated by an immersion α' with a pair (cancelling in the group ring $Z[\pi_1(M)]$) of double points for each intersection of $\bigcup_i$ (interior (B_i) with T_0), and two pairs (also cancelling in the group ring) of double points for each self-intersection of $\bigcup_i B_i$. Therefore, $\mu(\alpha') = 0$. (To check cancellation, observe that the dual curves to $\{A_i\}$ are also null-homotopic in $S^2\times R$.)

THEOREM A. *If $\pi_1(M) \neq 0$ and $\alpha \in \pi_2(M)$ then α is stably re-*

presented if either of the following sets of hypotheses hold: (1) α *is ordinary,* $\mu(\alpha') = 0$ *some* $\alpha' \simeq \alpha, \alpha$ *has a spherical dual* β, *and* $\pi_2(M) \to H_2(M; Z_2)$ *is onto, or* (2) α *is a characteristic,* $\mu(\alpha') = 0$ *for some* $\alpha' \simeq \alpha, \alpha$ *has a spherical dual* β, *and* $\mathrm{Arf}(q(\alpha)) = 0$.

Proof. There is a geometric trick (due to Larry Taylor) whereby ambient 1-surgery is performed on a spherical immersion α with $\mu(\alpha) = 0$ to produce an imbedded oriented surface, which we call $h_0\colon T_0 \hookrightarrow M \times 0$ with $h_{0\#}\colon \pi_1(T_0) \to \pi_1(M \times 0)$ the zero map. Roughly, one tubes together the pairs of algebraically cancelling double points. Let $x_1, \cdots, x_n$ be the transverse circles of the resulting tubes.

We start with Case 2. Observe that $q([x_i]) = x + \pitchfork_{B_i} = 0 + 1 = 1$. By the following algebraic lemma we may extend $\{x_1, \cdots, x_n\}$. To a standard family of circles $\{x_1, \cdots, x_n, A_1, \cdots, A_n\}$ representing a symplectic basis with $q(A_i) = 0$ for all $1 \leqq i \leqq n$.

LEMMA 4. *Assume* L *is a free integer latus of dimension* $2n$ *equipped with a unimodular, shew-symmetric bilinear form,* $\cdot$, *and an associated quadratic form* $q\colon L \to Z_2$ *(satisfying* $q(\delta + \gamma) = q(\delta) + q(\gamma) + (\delta \cdot \gamma) \pmod 2$*). Assume* $\mathrm{Arf}(q) = 0$. *If* $\alpha_1, \cdots, \alpha_n$ *is a basis for a summand (as a lattice) of* L *satisfying:* $\alpha_i \cdot \alpha_j = 0$ *and* $q(\alpha_i) = 1$, *then there is an extension to a symplectic basis,* $\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n$ *satisfying* $q(\beta_i) = 0$.

Proof. Extend $\alpha_1, \cdots, \alpha_n$ to a symplectic basis, $\alpha_1, \cdots, \alpha_n, \beta'_1, \cdots, \beta_n$. By definition $\mathrm{Arf}(q) = \sum q(\alpha_i)q(\beta'_i)$ so $q(\beta'_i) = 1$ for an even number, $2k$, of i. Reorder the basis to put these first. We use the notation:

$$\alpha_1, \cdots, \alpha_k, \bar{\alpha}_1, \cdots, \bar{\alpha}_k, \alpha_{2k+1}, \cdots, \alpha_n ,$$
$$\beta'_1, \cdots, \beta'_k, \bar{\beta}'_1, \cdots, \bar{\beta}'_k, \beta'_{2k+1}, \cdots, \beta'_n$$

for the reordered basis. Set:

$$\beta_j = \beta'_j + \bar{\alpha}_j$$

and

$$\bar{\beta}_j = \bar{\beta}'_j + \alpha_j \quad \text{when} \quad 1 \leqq j \leqq k .$$

It is easily checked that the basis:

$$\alpha_1, \cdots, \alpha_k, \bar{\alpha}_1, \cdots, \bar{\alpha}_k, \alpha_{2k+1}, \cdots, \alpha_n ,$$
$$\beta_1, \cdots, \beta_k, \bar{\beta}_1, \cdots, \bar{\beta}_k, \beta_{2k+1}, \cdots, \beta_n$$

has the desired properties.

Let $\{A_1, \cdots, A_n, A'_i, \cdots, A_n\}$ be a standard family of circles extending $\{A_1, \cdots, A_n\}$ and representing a symplectic basis with $q(A_i) = q(A'_i) = 0$ for all $1 \leqq i \leqq n$. We now consider how this situation may be achieved in the ordinary case.

In Case 1 the hypothesis that $\pi_2(M) \rightarrow H_2(M; Z_2)$ is onto enables us to find a spherical class $\gamma \in H_2(M; Z_2)$ with $\gamma \cdot \gamma \not\equiv \gamma \cdot \alpha \pmod 2$. As in the simply connected case we use γ to redefine q. Therefore in both Cases 1 and 2 we may find immersed 2-disks $\{B_1, \cdots, B_n, B'_1, \cdots, B'_n\}$ bounding $\{h_0(A_1), \cdots, h_0(A_n), h_0(A'_1), \cdots, h_0(A'_n)\}$ with $q(A_i^{(\prime)}) = x +$ interior $B_i^{(\prime)}$. $T_0 = 0 + \text{even} = 0 \pmod 2$. The prime in parentheses means the statement holds omitting or rataining the prime. However, since $\pi_1(M) \neq 0$, the intersections of int $B_i^{(\prime)}$ and T_0 are actually defined over the group ring $Z[\pi_1(M)]$. At this point the hypothesis on the existence of spherical duals β euables us to change interior $(B_i^{(\prime)})$. T_0 so that this intersection element is zero, and keep $x = 0$. Now 1-handles may be added to $h(T_0) \pm 1$ to kill algebraically cancelling intersection points of $(\text{int} \cup B_i^{(\prime)}) \pm 1$ and $(T_0 \pm 1)$. Notice that this perpetuates the condition that $\pi_1(h^{-1}(M \times n) \xrightarrow{h/} \pi_1(M \times n))$ is the zero map. Now the proof proceeds as when $\pi_1(M) = 0$.

To verify that the resulting infinite construction stably represents α and not just a homologous class, we must see that the composition,

$$g: S^2 \xrightarrow{\text{collapse}} T_0 \bigcup_i B_i \xrightarrow{h_0 \cup b_i} M \times 0$$

represents α. But g and α' cobound a map on $S^2 \cup$1-handles$\cup$2-handles where $x_1, \cdots, x_n$ are the transverse 1-spheres of the 1-handles and $A_1, \cdots, A_n$ are the descending 1-spheres of the 2-handles. Since $\{x_1, \cdots, x_n, A_1, \cdots, A_n\}$ represent a symplectic basis of $H_1(T_0; Z)$, the cobordism is a product and the map a homotopy.

Proof of Addendum to Main Theorem.

LEMMA 5. *$K \hookrightarrow M$ be any oriented surface imbedded in compact 4-manifold. There is a surface $K' \hookrightarrow M$ obtained from $K \hookrightarrow M$ by finitely many ambient 0-surgeries satisfying* ker: $\pi^1(M - K') \xrightarrow{\text{inc}} \pi_1(M)$ *is a central cyclic subgroup generated by a small circle linking K'.*

Proof. Let $\cdot$ denote integral intersection number with K.

The two sequences in the following diagram are exact.

∂ (kernel$(\cdot)$) is a normal subgroup, which we will call G, of $\pi_1(M - K)$. Since $\pi_2(M, M - K)$ is finitely generated, kernel$(\cdot)$ is normally generated by finitely many elements and so G is also in

$$\begin{array}{ccccccc} 0 & & & & & & \\ \downarrow & & & & & & \\ \text{kernel}(\cdot) & & & & & & \\ \downarrow & & & & & & \\ \pi_2(M, M - K) & \xrightarrow{\partial} & \pi_1(M - K) & \longrightarrow & \pi_1(M) & \longrightarrow & 0 \\ \downarrow \cdot & & & & & & \\ Z & & & & & & \\ \downarrow & & & & & & \\ 0 & & & & & & \end{array}$$

DIAGRAM 2

the normal closure of finitely many elements $g_1, \cdots, g_t$. Let $d_1, \cdots, d_t$ be singular 2-disks, $(D^2, \partial) \to (M, M - K)$, with d_k representing an element of $\partial^{-1}(g_k)$ for $1 \leqq k \leqq t$. We many assume that each d_k is transverse to K, intersecting it in equally many points of positive and negative sign. On each d_k draw disjointly imbedded arcs pairing points of opposite sign. K' is obtained from K by ambient framed 0-surgeries along the aggregate of these arcs. K' and d_k are disjoint for all K. It follows from general position that $\pi_1(M - K')$ is a quotient of $\pi_1(M - K)$ and that G belongs to the kernel. This together with Diagram 2 yields the cxact sequence:

$$Z \longrightarrow \pi_1(M - K') \longrightarrow \pi_1(M) \longrightarrow 0 .$$

Since the action of $\pi_1(M - K')$ on $\pi_2(M, M - K')$ preserves intersection numbers with K, conjugation leave the kernel fixed; centrality follows.

To complete the proof of the addendum and also for Remark C at the end it will be necessary to describe the complement, $X = M \times R$ − open tube $(n(S^2 \times R))$ as a handle body on $M \times 0$ − open tube $(h_0(T_0))$. K will be used to denote an arbitrary level surface of $h(S^2 \times R)$. The construction of h is as a product imbedding except for isolated levels where 1-handles or 2-handles are attached (core disks are assumed to descend towards the zero level). As a consequence the complement is a product except for isolated levels where 2-handles or 3-handles (again core disk ore assumed to descend towards the zero level) are attached. (The general rule: a k-handle on a submanifold gives rise locally to a $(k + c - 1)$-handle in the complement where c is the codimension of the imbedding.) Any 2 (or 3)-handle of the complement is attached to a circle (or 2-sphere) obtained from a normal S^1-bundle, $S^0 \times S^1_{\text{normal}}$ (or $S^1 \times S^1_{\text{normal}}$), by ambient surgery. The normal S^1-bundle $S^0\times S^1_{\text{normal}}$ (or $S^1\times S^1_{\text{normal}}$) is the set of normal vectors to K in its level of length ε based on the descending S^0(or S^1) of a 1(or 2)-handle of $h(S^2\times R)$. The ambient surgery is along that 1(or 2)-handle.

It follows that if ker: $\pi_1(M-K)\to\pi_1(M)$ is central and cyclicly generated a linking circle γ and the 2-handles are attached to $\gamma^{-1}(\beta\gamma\beta^{-1})\sim 0$ in $(M-K)$ $(\beta\in\pi_1(M-K))$.

The construction of h involves the attaching of finitely many 1-handles to the level surfaces $K_{n-\varepsilon}$ and $K_{-n+\varepsilon}$ and finitely many 2-handles to the level surfaces $K_{n-\varepsilon/2}$ and $K_{n+\varepsilon/2}$, where the subscript denotes the height of the level, n is a positive integer and ε a small positive number. To establish the addendum we will add additional 1-handles to K_0, $K_{n-\varepsilon}$ and $K_{-n+\varepsilon}$ (and the additional 2-handles forced by the construction of h). First use Lemma 5 to begin the construction with a $K_0(=h_0(T))$. Satisfying: Kernel: $(\pi_1(M-K_0)\to\pi_1(M))$ is is central and cyclic. By the preceding paragraph, the complement $M\times[0,1]$-$h(S^2\times R)$ is $(M-K_0)\cup$ 2-handles $\cup$ 3-handles with the 2-handles trivially attached. So $\pi_1(M-K_0)\xrightarrow{\text{inc}}\pi_1(M\times[0,1]$-$h(S^2\times R))$ is an isomorphism $\pi_1(M\times 1-h(S^2\times R))$ is an epimorphism and the kernel is normally generated by a finite family of closed curves $\{c_i\}$ each of which bounds a 2-disk having intersection number 0 with K_1. This means that the proof of Lemma 5 may be applied (and more 1-handles added to the 1-level) to make both maps above isomorphisms.

Analogously the complement, $M\times R-h(S^2\times R)$ may be fixed up in "boxes" so that on π_1 each integral level maps isomorphically to the region bounded by it and the next larger (or smaller) integral level.

The Addendum follows immediately from Van Kampen's theorem.

Concluding remarks.

(A) There are other notions of "stable" imbeddings of 2-spheres in 4-manifolds for which similar theorems hold. For example, if a stable imbedding of $\alpha\in\pi_2(M^4)$ is taken to mean enlarging M by connected sum with arbitrarily many $(S^2\times S^2)$'s and then representing the class, α, by an imbedding; or if a stable imbedding of α is an imbedding $S^2\times CP^2\hookrightarrow M^4\times CP^2$ homotopic to $\alpha\times\mathrm{id}_{CP^2}$, then (at least in the case $\pi_1(M)=0$, α is characteristic) the theorem would be the same, i.e., α is "stably" represented if and only if $\mathrm{Arf}\,q(\alpha)A=0$. The first example is proved in [2]; the second follows from the regular neighborhood theory of [1].

(B) Our technique for imbedding spheres cross R may be extended to find a proper imbedding (with good control of fundamental groups) of $(\natural_{k\text{-copies}}(S^2\times S^2-\mathrm{int}(D^4)))\times \boldsymbol{R}$ in $M\times\boldsymbol{R}$ which "stably represents" the entire kernel of a 2-connected, degree one, normal map with vanishing surgery obstruction, $f\colon (M, \partial) \to (P, \partial)$ (where M is a smooth 4-manifold, and P is a Poincare' space). This allows us to recover many results obtained by crossing 4-dimensional surgery problems with S^1, completing surgery, and taking infinite cyclic covers.

(C) Let $\alpha \in \pi_2(M)$ be a class for which the main theorem (and addendum) constructs a stable imbedding $h\colon S^2 \times R \to M \times R$ with control of the fundamental group. Let $F = M \times 0 - h(S^2 \times R)$. Give F the structure of a finite 3-complex. By the proof of the addendum $X = M \times R - h(S^2 \times R) \cong F \vee_{i \in I} S_i^2 \bigcup_{j \in J} D_j^3$ where I and J are (presumably infinite) index sets. Let F^2 be the 2-skeleton of F. Let $\Lambda = Z[\pi_1(M)]$.

LEMMA 6. *$H_*(X, F^2; \Lambda) = 0$ for $* = 0, 1,$ or 2. $H_3(X, F^2; \Lambda)$ is a finitely generated projective module. Furthermore $\partial(H_3(X, F^2; \Lambda)) \subset H_2(F^2; \Lambda)$ is spherical.*

Proof. Let the tilde be used to denote coverings pulled back from the universal covering of M. From our construction, the inclusions $(\widetilde{h^{-1}(M \times 0)}) \to (\widetilde{h^{-1}(M \times [0, \infty)})$ and $(h^{-1}(M \times 0)) \to \widetilde{h^{-1}(M \times (-\infty, 0])}$ induce maps on integral homology which are epimorphisms with finitely generated kernels as modules over Λ. It follows that the map on complements $H_*(F; \Lambda) \to H_*(X; \Lambda)$ is also an epimorphism with finitely generated kernel. By construction, $\pi_1(F) \to \pi_1(X)$ and $\pi_0(F) \to \pi_0(X)$ are isomorphisms. This establishes the first assertion. That $H_3(X, F^2; \Lambda)$ is projective follows from Lemma 2.3 of [5].

Consider the diagram:

$$\begin{array}{ccc}
H_3(X, F^2; \Lambda) & \longrightarrow & H_3(X, F^2 \vee_{i \in I} S_i^2; \Lambda) \\
\| \wr & & \| \wr \\
H_3(\tilde{X}, \tilde{F}^2; Z) & & H_3(\tilde{X}, \widetilde{F \vee_{i \in I} S_i^2}; Z) \\
\big\downarrow \partial & & \big\downarrow \partial' \\
H_2(\tilde{F}^2; Z) & \xrightarrow{\text{injection}} & H_2(\widetilde{F \vee_{i \in I} S_i^2}, Z)
\end{array}$$

Since $X = (F^2 \vee_{i \in I} S_i^2) \cup$ 3-cells, the image of ∂' is spherical; It follows that image ∂ is also spherical.

Let $k \in \tilde{K}_0(\Lambda)$ be the class of $H_3(X, F^3, \Lambda)$. Since $\partial(H_3(X, F^2; \Lambda))$ is spherical, if k is trivial finitely many 2 and 3-cells may be attached to F^2 to yield a Λ-homology equivalence ($F^2 \cup 2$ and 3-cells) $\to$

X. X is not always dominated by a finite complex and Lemma 3.1 of [6] does not seem to admit an appropriate generalization. Consequently, we do not claim that k, the obstruction to being Λ-homology equivalent to a finite complex, is independent of our choice of X and F^2. However, if $\tilde{K}_0(\Lambda) = 0$, we will always be able to find a finite complex X' and a map $X' \to X$ inducing isomorphisms on π_1 and $H_*(;\Lambda)$. Whenever such an X' exist, there is a homotopy equivalence $g: E \bigcup_{\partial E} X' \to M$, where E is the total space of a 2-disk bundle over S^2 with zero section $= S$, and $g_\sharp[S] = \alpha$. Since $\pi_1(X') \to \pi_1(E \bigcup_{\partial E} X')$ is an epimorphism it is possible to attach additional 2 and 3-cells to X' to realize the negative of the torsion of g, $[g] \in \mathrm{wh}(\pi_1(M))$. So whenever we have produced a complement X with finiteness obstruction $k = 0 \in \tilde{K}_0(\Lambda)$ we also have a simple homotopy equivalence, $g': E_{\partial E} \cup X'' \to M$ with $g'[S] = \alpha$, i.e., α is represented by a Poincare' imbedding.

REFERENCES

1. S. Cappell and J. Shaneson, *Singularities and immersions*, Annals of Math., **105** (1977), 539-552.
2. M. Freedman and R. Kirby, *A geometric proof of Rochlin's theorem*, Proc. of Sumposium in Pure Math., Vol. XXXII, AMS; Part 2, (1978), 85-98.
3. J. Milnor and M. Kevaire, *On 2-spheres in 4-manifolds*, Proc. Nat. Acad. Sci. U.S., **47** (1961), 1651-1657.
4. A. Tristram, *Some cobordism invariants for links*, Proc. Camb. Phil. Soc., **66** (1969), 251-264.
5. C.T.C. Wall, *Surgery on Compact Manifolds*, Academic Press, 1970.
6. ———, *Finiteness conditions for CW complexes*, I, Ann. of Math., **81** (1965), 56-69.

Received November 1976 and in revised form May 16, 1978. The author is partially supported by an NSF grant.

UNIVERSITY OF CALIFORNIA, SAN DIEGO
LA JOLLA, CA 92037