PACIFIC JOURNAL OF MATHEMATICS
Vol. 83, No. 2, 1979

INNER FUNCTIONS INVARIANT CONNECTED COMPONENTS

V. NESTORIDIS

The inner functions $d = \exp\{(z+1)/(z-1)\}$ and zd belong to the same connected component in the space of inner functions under uniform topology. Therefore, simplification is not possible in general but it is always possible to simplify by a finite Blaschke product.

0. **Introduction.** This work deals with the inner functions of one variable. A complex, holomorphic function f, bounded on the open unit disk D of the complex plane is called inner if $|f(e^{i\theta})| = 1$ a.e.; where $f(e^{i\theta}) = \lim_{\rho\to 1} f(\rho e^{i\theta})$.

In the set F of the inner functions we consider the topology induced by the Banach space H^∞; that is, we consider F with the topology of uniform convergence.

In this work, related to a publication of D. Herrero [2], we are interested in the connected components of the space F, mainly with respect to multiplication of inner functions.

Let us denote by $f \sim g$ the fact that the inner functions f and g belong to the same connected component. The questions that motivate this work are the following:

(a) For the identity function z, is there an inner function f such that $f \sim zf$?

(b) Is simplification permitted? That is, does relation $f\omega \sim g\omega$ imply $f \sim g$ for any three inner functions f, g, ω?

The results of this work can be summarized as follow:

(1) "Simplification" by a finite Blaschke product is always possible.

(2) "Simplification" is not possible in general.

(3) If the singular measure μ associated with a singular function S contains at least one atom, then relation $S \sim zS$ holds.

(4) For any nonconstant inner function g, the inner functions $\exp\{(g+1)/(g-1)\}$ and $g\exp\{(g+1)/(g-1)\}$ belong to the same connected component.

(5) For any nonconstant singular function S, there exists a nonconstant inner function g such that: $S \sim gS$.

In order to prove that simplification by a finite Blaschke product is possible, we first show that the set $zF = \{zh: h \in F\} = \{x \in F: x(0) = 0\}$ is a retract of F.

In order to give an example of an inner function f such that $f \sim zf$, we shift the zeros of an infinite Blaschke product in such a way that the Blaschke product moves continuously with respect to the uniform topology.

The following problems seem to be open:

(1) Does relation $S \sim zS$ hold for any singular function?

(2) Find all inner functions such that $f \sim zf$.

(3) Characterize the inner functions ω such that $\omega f \sim \omega g \Rightarrow f \sim g$ for all $f, g \in F$.

(4) Find a necessary and sufficient condition for two inner functions f and g to belong to the same connected component.

1. Preliminaries. A complex, holomorphic function f, bounded on the open unit disk D of the complex plane is called inner if its boundary values have almost everywhere absolute volue one; that is, relation $|f(e^{i\theta})|=1$ holds almost everywhere (with $f(e^{i\theta})=\lim_{\rho\to 1} f(\rho e^{i\theta})$).

It is well-known that a function f is inner if and only if f is of the form:

$$f(z) = cz^k \prod_{i\in I} \frac{\bar{\alpha}_i}{|\alpha_i|} \frac{\alpha_i - z}{1-\bar{\alpha}_i z} \exp\left\{-\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} d\mu(\theta)\right\}$$

where c is a complex constant of modulus one ($|c|=1$), k is a nonnegative integer, μ is a positive singular measure on the unit circle and the points $\alpha_i \in D$ are such that $\sum_{i\in I} 1-|\alpha_i| < \infty$.

If $\mu = 0$, then f is a Blaschke product, finite if the set I is finite or infinite if the set I is infinite (countable).

In the case $I = \varnothing$ and $k=0$, the function f is called singular.

The topology of the uniform convergence on the set F of the inner functions is induced by the following metric:

$$d(f, g) = \|f-g\|_\infty = \sup_D |f(z) - g(z)| = \sup_{\theta\in R} \operatorname{ess} |f(e^{i\theta}) - g(e^{i\theta})| .$$

Let us denote by $f \sim g$ the fact that the inner functions f and g belong to the same connected component in the space F.

In what follows we make use of the well-known facts below:

(1) For any three inner functions f, g and ω the relation $f \sim g$ implies $\omega f \sim \omega g$. This is due to the continuity of the multiplication of inner functions.

(2) For any inner function f and any complex number α, with $|\alpha| < 1$, we have the relation:

$$f \sim f_\alpha = \frac{f-\alpha}{1-\bar{\alpha} f} ;$$

for the mapping $D \ni \alpha \to f_\alpha \in F$ is continuous.

(3) For every nonnegative integer n, the set of all finite Blaschke products with exactly n zeros forms a connected component

and an open and closed subset of F. In particular the set of the constant inner functions is connected and open and closed in F.

This fact is an easy application of Rouche's theorem.

2. Simplification by z. Let us begin with the question, does the relation $\omega f \sim \omega g$ implies $f \sim g$. This is the problem of "Simplification". In the case of a finite Blaschke product ω, the answer to this question is affirmative.

PROPOSITION 1. *Let ω be a finite Blaschke product. Then for any two inner functions f and g, the relation $\omega f \sim \omega g$ implies $f \sim g$.*

Proof. The general case easily follows from the case $\omega = z$, to which we will limit ourselves from now on.

Let us consider the set:

$$zF = \{zh: h \in F\} = \{x \in F: x(0) = 0\} .$$

The maps $z^*: zF \to F$ and $\Phi: F \to zF$, where $z^*(x) = x/z$, $\Phi(f) = (f - f(0))/(1 - \overline{f(0)}f)$ for $f \in F$ nonconstant and $\Phi(f) = z$ for $f \in F$ constant, are both continuous. (The set of the constant inner functions is, both, open and closed!).

Therefore the mapping $z^* \circ \Phi: F \to F$ is continuous and the relation $zf \sim zg$ implies: $f = z^* \circ \Phi(zf) \sim z^* \circ \Phi(zg) = g$, as $\Phi(x) = x$ for any $x \in zF$; that is, Φ is a retraction map and zF is a retract of F. The proof is complete now.

3. The main result. The following theorem implies in particular that we cannot "simplify" by any inner function.

THEOREM 1. *For any nonconstant inner function g, the inner functions* $\exp\{(g + 1)/(g - 1)\}$ *and* $g \exp\{(g + 1)/(g - 1)\}$ *belong to the same connected component.*

This theorem applied for the identity function $g = z(z(a) = a$ for all $a \in D)$ implies the following:

PROPOSITION 2. *The inner functions $d = \exp\{(z + 1)/(z - 1)\}$ and zd belong to the same connected component (that is: $d \sim zd$).*

Proposition 2 is equivalent to Theorem 1; for Proposition 2 implies also Theorem 1. The point is that the range of the continuous map $T_g: F \to H^\infty$, $T_g(f) = f \circ g$ is contained in F; that is, the composition of two inner functions is an inner function ([6] or [8]). Therefore relation $d \sim zd$ implies:

$$\exp\frac{g+1}{g-1} = T_g(d) \sim T_g(zd) = g\exp\frac{g+1}{g-1}\,.$$

Hence, it remains to prove Proposition 2, which will be a consequence of the following lemma, which is of a concrete geometric nature on the half-plane:

LEMMA 1. *Let*

$$K_1 = \prod_{n=1}^{\infty}\frac{\bar{\alpha}_n}{|\alpha_n|}\frac{\alpha_n - z}{1-\bar{\alpha}_n z} \quad \textit{and} \quad K_2 = \prod_{n=1}^{\infty}\frac{|\bar{\beta}_n|}{\bar{\beta}_n}\frac{\beta_n - z}{1-\bar{\beta}_n z}\,.$$

Be two infinite Blaschke products such that $K_1(0) > 0$ and $K_2(0) > 0$. If we denote $\varphi(z) = (1+z)/(1-z)$ then we have the following inequality:

$$\|K_1 - K_2\|_\infty \leqq \sum_{n=1}^{\infty}\left|\alpha\,\mathrm{rg}\,\frac{\alpha_n}{\beta_n}\right| + 2\sum_{n=1}^{\infty}\left|\alpha\,\mathrm{rg}\,\frac{1-\alpha_n}{1-\beta_n}\right|$$
$$+ 2\sup_{y\in R}\mathrm{ess}\sum_{n=1}^{\infty}\left|\alpha\,\mathrm{rg}\,\frac{\varphi(\alpha_n)-iy}{\varphi(\beta_n)-iy}\right|\,.$$

Proof of Lemma 1. The pointwise convergence $f_n \to f$ implies trivially the inequality:

$$\|f\|_\infty \leqq \liminf_n \|f_n\|_\infty$$

We have therefore:

$$\|K_1 - K_2\|_\infty \leqq \liminf_N\left\|\prod_{n=1}^{N}\frac{\bar{\alpha}_n}{|\alpha_n|}\frac{\alpha_n - z}{1-\bar{\alpha}_n z} - \prod_{n=1}^{N}\frac{\bar{\beta}_n}{|\beta_n|}\frac{\beta_n - z}{1-\bar{\beta}_n z}\right\|_\infty$$
$$= \liminf_N\left\|\prod_{n=1}^{N}\frac{\bar{\alpha}_n}{|\alpha_n|}\frac{1-\alpha_n}{1-\bar{\alpha}_n}\frac{\varphi(\alpha_n)-\varphi(z)}{\bar{\varphi}(\alpha_n)+\varphi(z)}\right.$$
$$\left. - \prod_{n=1}^{N}\frac{\bar{\beta}_n}{|\beta_n|}\frac{1-\beta_n}{1-\bar{\beta}_n}\frac{\varphi(\beta_n)-\varphi(z)}{\bar{\varphi}(\beta_n)+\varphi(z)}\right\|_\infty\,.$$

We notice that $|\alpha| = |\beta| = |\alpha'| = |\beta'| = 1 \Rightarrow |\alpha\beta - \alpha'\beta'| \leqq |\alpha-\alpha'| + |\beta-\beta'|$. Consequently, for almost every z, with $|z|=1$, we have:

$$\left|\prod_{n=1}^{N}\frac{\bar{\alpha}_n}{|\alpha_n|}\frac{1-\alpha_n}{1-\bar{\alpha}_n}\frac{\varphi(\alpha_n)-\varphi(z)}{\bar{\varphi}(\alpha_n)+\varphi(z)} - \prod_{n=1}^{N}\frac{\bar{\beta}_n}{|\beta_n|}\frac{1-\beta_n}{1-\bar{\beta}_n}\frac{\varphi(\beta_n)-\varphi(z)}{\bar{\varphi}(\beta_n)+\varphi(z)}\right|$$
$$\leqq \sum_{n=1}^{N}\left|\frac{\bar{\alpha}_n}{|\alpha_n|} - \frac{\bar{\beta}_n}{|\beta_n|}\right| + \sum_{n=1}^{N}\left|\frac{1-\alpha_n}{1-\bar{\alpha}_n} - \frac{1-\beta_n}{1-\bar{\beta}_n}\right|$$
$$+ \sum_{n=1}^{N}\left|\frac{\varphi(\alpha_n)-\varphi(z)}{\bar{\varphi}(\alpha_n)+\bar{\varphi}(z)} - \frac{\varphi(\beta_n)-\varphi(z)}{\varphi(\beta_n)+\varphi(z)}\right|$$

$$\leqq \sum_{n=1}^{N}\left|\alpha \text{ rg}\frac{\alpha_n}{\beta_n}\right| + 2\sum_{n=1}^{N}\left|\alpha \text{ rg}\frac{1-\alpha_n}{1-\beta_n}\right| + 2\sum_{n=1}^{N}\left|\alpha \text{ rg}\frac{\varphi(\alpha_n)-\varphi(z)}{\varphi(\beta_n)-\varphi(z)}\right|$$

$$\leqq \sum_{n=1}^{\infty}\left|\alpha \text{ rg}\frac{\alpha_n}{\beta_n}\right| + 2\sum_{n=1}^{\infty}\left|\alpha \text{ rg}\frac{1-\alpha_n}{1-\beta_n}\right| + 2\sup_{y\in R}\text{ess}\sum_{n=1}^{\infty}\left|\arg\frac{\varphi(\alpha_n)-iy}{\varphi(\beta_n)-iy}\right| .$$

The required inequality is now implied.

Proof of Proposition 2. Let $\alpha_n(t)$ be the unique point of D such that $\varphi(\alpha_n(t)) = 1 + i(n+t)\pi$, where $t \in [0, 1]$, $n \in N^* = \{1, 2, \cdots\}$ and $\varphi(z) = (1+z)/(1-z)$.

One, then, verifies easily that:

$$d_{1/e} = \frac{d - \dfrac{1}{e}}{1 - \dfrac{1}{e}d} = f\cdot\prod_{n=4}^{\infty}\frac{\overline{\alpha_n(0)}}{|\alpha_n(0)|}\frac{\alpha_n(0)-z}{1-\overline{\alpha_n(0)}z}\,, \quad \text{with} \quad f \in F\,.$$

It is enough to prove that

$$B_1 = \prod_{n=4}^{\infty}\frac{\overline{\alpha_n(0)}}{|\alpha_n(0)|}\frac{\alpha_n(0)-z}{1-\overline{\alpha_n(0)}z} \sim B_0 = \prod_{n=3}^{\infty}\frac{\overline{\alpha_n(0)}}{|\alpha_n(0)|}\frac{\alpha_n(0)-z}{1-\overline{\alpha_n(0)}z}\,;$$

for, then we have

$$d \sim d_{1/e} = fB_1 \sim fB_0 = fB_1\frac{\overline{\alpha_3(0)}}{|\alpha_3(0)|}\frac{\alpha_3(0)-1}{1-\overline{\alpha_3(0)}z} \sim fB_1 z = d_{1/e}\cdot z \sim dz\,,$$

and we obtain the result.

In order to prove $B_1 \sim B_0$, it is sufficient to prove the continuity of the following map:

$$[0, 1] \ni t \xmapsto{B} B_t = \prod_{n=3}^{\infty}\frac{\overline{\alpha_n(t)}}{|\alpha_n(t)|}\frac{\alpha_n(t)-z}{1-\overline{\alpha_n(t)}z} \in F\,;$$

that is, $\lim_{t\to t_0}\|B_t - B_{t_0}\|_\infty = 0$ for all $t_0 \in [0, 1]$. Using Lemma 1 we essentially have to prove the following fact:

$$\lim_{t\to t_0}\sup_{y\in R}\sum_{n=3}^{\infty}\left|\arg\frac{\varphi(\alpha_n(t))-iy}{\varphi(\alpha_n(t_0))-iy}\right| = 0\,.$$

This relation follows immediately from the observation that:

$$\sum_{n=3}^{\infty}\left|\arg\frac{\varphi(\alpha_n(t))-iy}{\varphi(\alpha_n(t_0))-iy}\right|$$
$$\leqq 2\sum_{n=3}^{\infty}\left|\arg\frac{1+2i\pi(n+t_0+|t-t_0|)-2i\pi(t_0+3)}{1+2i\pi(n+t_0-|t-t_0|)-2i\pi(t_0+3)}\right| \xrightarrow[t\to t_0]{} 0\,.$$

4. **Consequences.** Theorem 1 yields trivially the following:

COROLLARY 1. *For any inner function g, there exists an inner function f such that $f \sim gf$.*

Proposition 2 implies the following more general result:

COROLLARY 2. *Let f be an inner function whose singular measure μ contains at least one atom. Then $f \sim zf$.*

Proof of Corollary 2. We have $f = f_1 \exp K(z+\alpha)/(z-\alpha)$, with $f_1 \in F$, $|\alpha| = 1$ and $K > 0$. Thus, it is enough to establish the relation $\exp K(z+\alpha)/(z-\alpha) \sim z \exp K(z+\alpha)/(z-\alpha)$. By a rotation this becomes:

$$\exp K\frac{z+1}{z-1} \sim z \exp K\frac{z+1}{z-1}\,.$$

If $K \geqq 1$, using the known relation $d \sim zd$ (Proposition 2) we have

$$\begin{aligned}\exp K\frac{z+1}{z-1} &= d \cdot \exp\,(K-1)\frac{z+1}{z-1} \sim zd \exp\,(K-1)\frac{z+1}{z-1}\\ &= z \exp \frac{z+1}{z-1}K\,.\end{aligned}$$

If $0 < K < 1$, let us consider the transformation[1]:

$$-w(z) = \frac{\dfrac{1-K}{1+K} - z}{1 - \dfrac{1-K}{1+K}z}\,.$$

Evidently $w \in F$ and $w \sim z$. From the known relation $d \sim zd$ we obtain:

$$\exp K\frac{z+1}{z-1} = d\circ w \sim (zd)\circ w = w\cdot(d\circ w) \sim z\cdot(d\circ w) = z \exp K\frac{z+1}{z-1}\,.$$

REMARK. Corollary 2 implies that if the singular measure μ associated with a singular function S contains some atoms, then the relation $S \sim zS$ holds. If the measure $\mu \neq 0$ does not contain any atoms, then we do not know if the relation $S \sim zS$ is true. It seems that this problem (probably not difficult) is still open and we offer the following conjecture:

"Every nonconstant singular inner function S belongs to the same connected component as zS".

[1] This trick is found in [2].

In this direction we have the following proposition, which follows from Theorem 1 combined with a remark suggested to the author by K. Stephenson.

PROPOSITION 3. *For any nonconstant singular inner function S, there exists a nonconstant inner function g such that $S \sim gS$.*

Proof. The point is that any singular inner function S is of the form $S = \exp(g+1)/(g-1)$, with $g \in F$. Theorem 1 gives, then, the result.

In an obvious manner Proposition 3 implies the following:

COROLLARY 3. (i) *For every nonconstant singular inner function S, there exist inner functions f and g such that $fS \sim gS$ but $f \not\sim g$.*

(ii) *Let ω be an inner function such that the relation $f_1\omega \sim f_2\omega$ implies $f_1 \sim f_2$ for every couple (f_1, f_2) of inner functions f_1 and f_2. Then the connected component of ω contains only Blaschke products. In particular ω is a Blaschke product.*

(iii) *If the connected component of an inner function f does not contain any proper multiple of f, then this component contains only Blaschke products. In particular f is a Blaschke product.*

The existence of infinite Blaschke products satisfying the hypothesis of Corollary 3 (iii) follows from the proof of a theorem of D. Herrero ([3], Theorem 1.1). Later, the present author proved in [6] that if the zeros α_n, $n = 1, 2, \cdots$ of a Blaschke product B satisfy the condition

$$\lim_n \prod_{m \neq n} \left| \frac{\alpha_n - \alpha_m}{1 - \bar{\alpha}_n \alpha_m} \right| = 1$$

then, the connected component for B does not contain any proper multiple of B.

ACKNOWLEDGMENT. I wish to express my gratitude to A. Bernard for the direction and help he gave me.

REFERENCES

1. H. Helson, *Lectures on the invariant spaces.*

2. D. Herrero, *Inner functions under uniform topology*, Pacific J. Math., **51** (1974), 167–175.

3. ———, *Inner functions under uniform topology II*, Revista de la Unión Mathematica Argentina, Volumen 28, 1976.

4. K. Hoffman, *Banach spaces of analytic functions.*
5. D. Marshall, *Blaschke product generate H^∞*, Bull. Amer. Math. Soc., **82** (1976).
6. V. Nestoridis, *Fonctions intérieures: composantes connexes et multiplication par un produit de Blaschke,* Thèse de troisième cycle, Grenoble, (1977).
7. W. Rudin, *Real and complex analysis.*
8. K. Stephenson, *Isometries of the Nevanlina class,* Indiana Univ. Math. J., **26** (1977), 307-324.
9. ———, *Omitted values of singular inner functions,* to appear, Michigan Math. J.

Received April 7, 1978 and in revised form June 8, 1978.

KALAVRYTON 37
DASSOS HAIDARIOU
ATHENS, GREECE