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PEIRCE IDEALS IN JORDAN TRIPLE SYSTEMS

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We show that an ideal in a Peirce space $J_i (i = 1, 1/2, 0)$ of a Jordan triple system J is the Peirce i-component of a global ideal precisely when it is invariant under the multiplications $L(J_{1/2}, J_{1/2}), P(J_{1/2})P(J_{1/2})$ (for $i=1$); under $L(J_{1/2}, J_{1/2})$, $P(J_{1/2})P(J_{1/2})$, $P(J_{1/2})P(e)P(J_{1/2})$, $L(J_{1/2}, e)P(J_0, J_{1/2})$ (for $i = 0$); under $L(J_1)$, $L(J_0)$, $L(J_{1/2}, e)L(e, J_{1/2})$, $L(J_{1/2}, e)P(e, J_{1/2})$ (for $i = 1/2$). We use this to show that the sub triple systems J_1 and J_0 are simple when J is. The method of proof closely follows that for Jordan algebras, but requires a detailed development of Peirce relations in Jordan triple systems.

Throughout we consider Jordan triple systems (henceforth abbreviated JTS) with basic product $P(x)y$ linear in y and quadratic in x, with derived trilinear product $\{xyz\} = P(x, z)y = L(x, y)z$, over an arbitrary ring Φ of scalars. Because we are already overburdened with subscripts and indices, we prefer not to treat the general case of Jordan pairs directly, but rather derive it via hermitian JTS. For basic facts about JTS and Jordan pairs we refer to [1], [3], [6]. Our analysis of Peirce ideals will closely follow that for Jordan algebras; although the basic lines of our treatment are the same as in [4], the triple system case requires such horrible computations that we do not carry out so fine an analysis, but concentrate just on the main simplicity theorem.

1. Peirce relations in Jordan triple systems.

Any Jordan triple system satisfies the general identities

$$\text{(JT1)} \quad L(x, y)P(x) = P(x)L(y, x)$$

$$\text{(JT2)} \quad L(x, P(y)x) = L(P(x)y, y)$$

$$\text{(JT3)} \quad P(P(x)y) = P(x)P(y)P(x)$$

and the linearization

$$\text{(JT3}'\text{)} \quad P(\{xyz\}) + P(P(x)y, P(z)y) = P(x)P(y)P(z) + P(z)P(y)P(x) + P(x, z)P(y)P(x, z) .$$

A more useful version of this is the identity

$$\text{(JT4)} \quad P(\{xyz\}) = P(x)P(y)P(z) + P(z)P(y)P(x) + L(x, y)P(z)L(y, x) - P(P(x)P(y)z, z) .$$

Other basic identities we require are

(JT5) $L(x, y)P(z) + P(z)L(y, x) = P(L(x, y)z, z)$

(JT6) $P(x)P(y, z) = L(x, y)L(x, z) - L(P(x)y, z)$

(JT7) $P(y, z)P(x) = L(z, x)L(y, x) - L(z, P(x)y)$

(JT8) $2P(x)P(y) = L(x, y)^2 - L(P(x)y, y)$

(JT9) $[L(x, y), L(z, w)] = L(L(x, y)z, w) - L(z, L(y, x)w)$.

(See for example JP1-3, 20, 21, 12-13, 9 in [1, pp. 13, 14, 19, 20].)

PEIRCE DECOMPOSITIONS. Now let e be a *tripotent*, $P(e)e = e$. Then J decomposes into a direct sum of Peirce spaces

$$J = J_1 \oplus J_{1/2} \oplus J_0$$

relative to e, where the *Peirce projections* are

$$\begin{aligned} E_1 &= P(e)P(e)\ , \quad E_{1/2} = L(e, e) - 2P(e)P(e)\ , \\ E_0 &= B(e, e) = I - L(e, e) + P(e)P(e)\ . \end{aligned} \tag{1.1}$$

We have

$$L(e, e) = 2iI \quad \text{on} \quad J_i\ , \qquad P(e) = 0 \quad \text{on} \quad J_{1/2} + J_0\ . \tag{1.2}$$

Note that $P(e)$ is not the identity on J_1, though $J_1 = P(e)J$: it induces a map of period 2 which is an involution of the triple structure and is denoted by $x \to x^*(x \in J_1)$. For reasons of symmetry we introduce a trivial involution $x \to x$ on J_0, so $*$ is defined on $J_1 + J_0$:

$$x_1^* = P(e)x_1\ , \quad x_0^* = x_0\ . \tag{1.3}$$

Note that if J is a Jordan algebra and e is actually an idempotent, then $x_1^* = x_1$ too.

The Peirce relations describe how the Peirce spaces multiply. Let i be either 1 or 0, and $j = 1 - i$ its complement. Then just as in Jordan algebras we have

$$\begin{array}{ll} \text{(PD1)} & P(J_i)J_i \subset J_i\ , \quad P(J_i)J_j = P(J_i)J_{1/2} = 0 \\ \text{(PD2)} & P(J_{1/2})J_{1/2} \subset J_{1/2}\ , \quad P(J_{1/2})J_i \subset J_j \\ \text{(PD3)} & \{J_iJ_iJ_{1/2}\} \subset J_{1/2}\ , \quad \{J_{1/2}J_{1/2}J_i\} \subset J_i \\ \text{(PD4)} & \{J_iJ_{1/2}J_j\} \subset J_{1/2} \\ \text{(PD5)} & \{J_iJ_jJ\} = 0\ . \end{array} \tag{1.4}$$

(For all this see [6] and [1, p. 44].) These show that the Peirce spaces are invariant under the multiplications mentioned in the introduction.

PEIRCE IDENTITIES. For a finer description of multiplication between Peirce spaces it is useful to reduce Jordan triple products to bilinear products whenever possible. We introduce a dot operation $x \cdot y$ (corresponding to $x \circ y$ in Jordan algebras) for elements a_k in Peirce spaces J_k, and a component product $E_i(x_{1/2}, y_{1/2})$ (corresponding to the J_i-component of $x_{1/2} \circ y_{1/2}$) as follows:

$$
(1.5)\quad
\begin{array}{ll}
\text{(B1)} & x_1 \cdot y_{1/2} = y_{1/2} \cdot x_1 = \{x_1 e y_{1/2}\} \quad L(x_1) = L(x_1, e): J_{1/2} \longrightarrow J_{1/2} \\
\text{(B2)} & x_0 \cdot y_{1/2} = y_{1/2} \cdot x_0 = \{x_0 y_{1/2} e\} \quad L(x_0) = P(x_0, e): J_{1/2} \longrightarrow J_{1/2} \\
\text{(B3)} & x_1^2 = P(x_1)e,\ x_1 \cdot y_1 = \{x_1 e y_1\} \quad L(x_1) = L(x_1, e): J_1 \longrightarrow J_1 \\
\text{(B4)} & E_1(x_{1/2}, y_{1/2}) = \{x_{1/2} y_{1/2} e\} \qquad J_{1/2} \times J_{1/2} \longrightarrow J_1 \\
\text{(B5)} & E_0(x_{1/2}, y_{1/2}) = \{x_{1/2} e y_{1/2}\},\ E_0(x_{1/2}) = P(x_{1/2})e: J_{1/2} \times J_{1/2} \longrightarrow J_0 \\
\text{(B6)} & L_1(x_{1/2}) = L(x_{1/2}, e),\ L_0(x_{1/2}) = L(e, x_{1/2}) \text{ so that} \\
 & L_i(x_{1/2})a_i = a_i \cdot x_{1/2},\ L_i(x_{1/2})a_j = 0,\ L_i(x_{1/2})y_{1/2} = E_j(y_{1/2}, x_{1/2})\ .
\end{array}
$$

It turns out that the only Jordan products x^2 or $x \circ y$ which are not expressible in triple terms are

$$x_0^2,\ x_0 \circ y_0,\ E_1(x_{1/2})\ .$$

The need to avoid these products causes many complications when passing from Jordan algebra results to triple system results.

For example, let e be an ordinary symmetric idempotent in an associative algebra A with involution, made into a triple system $J = JT(A, {}^*)$ via $P(x)y = xy^*x$. Then the Peirce spaces are the usual ones, $J_1 = A_{11}$, $J_{1/2} = A_{10} + A_{01}$, $J_0 = A_{00}$. The bilinear products we have introduced take the form

$$
\begin{aligned}
x_1 \cdot y_{1/2} &= x_1 y_{1/2} + y_{1/2} x_1 \\
x_0 \cdot y_{1/2} &= x_0 y_{1/2}^* + y_{1/2}^* x_1 \\
E_1(x_{1/2}, y_{1/2}) &= E_1(x_{1/2} y_{1/2}^* + y_{1/2}^* x_{1/2}) \\
E_0(x_{1/2}, y_{1/2}) &= E_0(x_{1/2} y_{1/2} + y_{1/2} x_{1/2})\ .
\end{aligned}
$$

This suggests that because of the * the products $x_0 \cdot y_{1/2}$ and $E_1(x_{1/2}, y_{1/2})$ are going to behave anomalously.

1.6. PROPOSITION. *The triple products of Peirce elements are expressed in terms of bilinear products by*

$$
\begin{array}{ll}
\text{(P1)} & P(x_{1/2})y_{1/2} = x_{1/2} \cdot E_1(x_{1/2}, y_{1/2}) - y_{1/2} \cdot E_0(x_{1/2}) \\
\text{(P2)} & \{x_{1/2} y_{1/2} z_{1/2}\} = x_{1/2} \cdot E_1(z_{1/2}, y_{1/2}) + z_{1/2} \cdot E_1(x_{1/2}, y_{1/2}) \\
 & \qquad - y_{1/2} \cdot E_0(x_{1/2}, z_{1/2}) \\
\text{(P3)} & \{x_{1/2} a_i y_{1/2}\} = E_j(x_{1/2}, a_i^* \cdot y_{1/2}) = E_j(y_{1/2}, a_i^* \cdot x_{1/2}) \\
\text{(P4)} & \{x_{1/2} y_{1/2} a_i\} = E_i(x_{1/2}, a_i^* \cdot y_{1/2}) \\
\text{(P5)} & \{a_i b_i z_{1/2}\} = a_i \cdot (b_i^* \cdot z_{1/2})
\end{array}
$$

(P6) $\{a_i z_{1/2} b_j\} = a_i \cdot (z_{1/2} \cdot b_j^*) = (a_i^* \cdot z_{1/2}) \cdot b_j$

(P7) $e \cdot z_{1/2} = z_{1/2}$

(P8) $E_i(x_{1/2}, y_{1/2})^* = E_i(y_{1/2}, x_{1/2})$

and we can write

$$\text{(P9)}\quad L(x_{1/2}, a_i) = L_i(x_{1/2} \cdot a_i^*),\ L(a_i, x_{1/2}) = L_j(a_i^* \cdot x_{1/2}) .$$

The triple product of elements $x = x_1 + x_{1/2} + x_0$, $y = y_1 + y_{1/2} + y_0$ *may be written as*

$$\begin{aligned} P(x)y &= P(x_1)y_1 + P(x_0)y_0 + P(x_{1/2})y_{1/2} + P(x_{1/2})(y_1 + y_0) + \{x_1 y_{1/2} x_0\} \\ &\quad + \{x_1 y_1 x_{1/2}\} + \{x_0 y_0 x_{1/2}\} + \{x_1 y_{1/2} x_{1/2}\} + \{x_0 y_{1/2} x_{1/2}\} \\ &= P(x_1)y_1 + P(x_0)y_0 + \{x_{1/2} \cdot E_1(x_{1/2}, y_{1/2}) - y_{1/2} \cdot E_0(x_{1/2})\} \\ &\quad + P(x_{1/2})(y_1 + y_0) + x_1 \cdot (x_0 \cdot y_{1/2}) + x_1 \cdot (y_1^* \cdot x_{1/2}) + x_0 \cdot (y_0 \cdot x_{1/2}) \\ &\quad + E_1(x_{1/2}, x_1^* \cdot y_{1/2}) + E_0(x_{1/2}, x_0 \cdot y_{1/2}) . \end{aligned} \tag{1.7}$$

Proof. Most of these product rules can be established either by using JT5 to move $L(x, y)$ inside a triple product $P(z)w$, or by using the linearization of JT2 to interchange x and z in a product $\{x(P(y)z)w\}$. Thus (P1) is $P(x)y = P(x)\{yee\}$ (by 1.2)) $= \{\{eyx\}ex\} - \{ey(P(x)e)\}$ (by JT5) $= E_1(x, y) \cdot x - y \cdot E_0(x)$, and (P2) is its linearization. (P7) follows from PD2, $\{eez_{1/2}\} = z_{1/2}$, and (P8) is vacuous for $i = 0$ by triviality of * and symmetry of E_0, while for $i = 1$ $P(e)\{xye\} = P(e)L(e, y)x = -L(y, e)P(e)x + P(\{yee\}, e)x = -0 + \{yxe\}$ by JT5. For (P3)-(P6) we will need (P9),

$$\begin{aligned} L(x_{1/2}, a_1) &= L(x_{1/2} \cdot a_1^*, e) \qquad & L(a_1, x_{1/2}) &= L(e, a_1^* \cdot x_{1/2}) \\ L(x_{1/2}, a_0) &= L(e, x_{1/2} \cdot a_0) \qquad & L(a_0, x_{1/2}) &= L(a_0 \cdot x_{1/2}, e) . \end{aligned}$$

To establish this for a_1 we note $L(x_{1/2}, a_1) = L(x_{1/2}, P(e)a_1^*) = -L(a_1^*, P(e)x_{12}) + L(\{x_{1/2} e a_1^*\}, e)$ (linearized JT2) $= L(x_{1/2} \cdot a_1^*, e)$ and dually for $L(a_1, x_{1/2})$; for a_0 we have $L(x_{1/2}, a_0) = L(\{x_{1/2} ee\}, a_0) = -L(\{x_{1/2} a_0 e\}, e) + L(x_{1/2}, \{eea_0\}) + L(e, \{ex_{1/2} a_0\}) = -0 + 0 + L(e, x_{1/2} \cdot a_0)$ and dually for $L(a_0, x_{1/2})$. By B6 we can write these in the uniform manner (P9). Applying these to $x_{1/2}$ yields (P3) and (P4) respectively, and applying them to a_i, b_j respectively yields (P5) and (P6). □

Even in a Jordan algebra the products $P(x_i)y_i$ and $P(x_{1/2})y_i$ cannot be reduced to bilinear products if there is no scalar $1/2 \in \Phi$ (though $2P(x_{1/2})y_i$, and more generally $P(x_{1/2}, y_{1/2})a_i$, can be reduced by (P3)).

It will be convenient to introduce the abbreviation

$$\begin{aligned} &P^*(x_{1/2}) = {}^* \circ P(x_{1/2}) \circ {}^* \quad (\text{i.e., } P^*(x_{1/2})a_1 = P(x_{1/2})a_1^* , \\ &P^*(x_{1/2})a_0 = (P(x_{1/2})a_0)^*, \text{ so } P(P^*(x_{1/2})a_i) = P^*(x_{1/2})P(a_i)P^*(x_{1/2})) . \end{aligned} \tag{1.8}$$

We now list the basic Peirce identities. Many of these have appeared in [6], or in [1], [2] disguised as alternative triple identities.

1.9. PEIRCE IDENTITIES. *The following identities hold for elements* $a_i, b_i, c_i \in J_i (i = 1, 0, j = 1 - i)$ *and* $x, y, z \in J_{1/2}$:

(PI1) *we have a Peirce specialization* $a_i \to L(a_i)$ *of* J_i *in* $\mathrm{End}(J_{1/2})$:

(i) $P(a_i)b_i \cdot z = a_i \cdot (b_i^* \cdot (a_i \cdot z))$ $\quad L(P(a_i)b_i^*) = L(a_i)L(b_i)L(a_i)$

(ii) $e \cdot z = z$ $\quad L(e) = Id$

(iii) $a_1^2 \cdot z = a_1 \cdot (a_1 \cdot z)$ $\quad L(a_1^2) = L(a_1)^2$

(iv) $(a_1 \cdot b_1) \cdot z = a_1 \cdot (b_1 \cdot z) + b_1 \cdot (a_1 \cdot z)$

$$L(a_1 \cdot b_1) = L(a_1)L(b_1) + L(b_1)L(a_1)$$

(PI2) $P(a_i)E_i(x, y)^* = E_i(a_i \cdot x, a_i^* \cdot y)$

(PI3) $L(a_i, b_i)E_i(x, y) = E_i(a_i \cdot (b_i^* \cdot x), y) + E_i(x, a_i^* \cdot (b_i \cdot y))$

(PI4) $a_1 \cdot E_1(x, y) = E_1(a_1 \cdot x, y) + E_1(x, a_1^* \cdot y)$

(PI5) $P(z)E_i(x, y) = E_j(z, E_j(y, z) \cdot x) - E_j(P(z)x, y)$

(PI6) $P(E_i(x, y))a_i = P(x)P^*(y)a_i + P^*(y)P(x)a_i + E_i(x, P(y)(a_i^* \cdot x))$

(PI7) $\{P(x)a_i\} \cdot y + P(x)(a_i \cdot y) = E_i(x, y) \cdot (a_i^* \cdot x)$

(PI8) $\{P^*(x)a_i\} \cdot y + a_i \cdot P(x)y = E_i(a_i \cdot x, y) \cdot x$

(PI9) $P(x)\{a_1 x b_0\} = P(x)a_1 \cdot (b_0 \cdot x) = P(x)b_0 \cdot (a_1^* \cdot x)$

(PI10) $P(a_i \cdot x)b_j = P(a_i)P^*(x)b_j, P(a_i \cdot x)b_i = P^*(x)P(a_i)b_i$

(PI11) $P(a_i)P(x)b_j = P^*(a_i^* \cdot x)b_j, P(x)P(a_i)b_i = P^*(a_i^* \cdot x)b_i$

(PI12) $L(a_i, b_i)P(x)c_j = P(a_i \cdot (b_i^* \cdot x), x)c_j = E_i(a_i \cdot (b_i^* \cdot x), c_j^* \cdot x)$

(PI13) $L(a_i, b_i)P^*(x)c_j = P^*(a_i^* \cdot (b_i \cdot x), x)c_j = E_i(c_j \cdot x, a_i \cdot (b_i^* \cdot x))$

(PI14) $P(x)\{a_i b_i c_i\} = P(x, b_i \cdot (a_i^* \cdot x))c_i = E_j(x, c_i^* \cdot (b_i \cdot (a_i^* \cdot x)))$

(PI15) $E_0(a_0 \cdot x) = P(a_0)E_0(x), E_0(a_1 \cdot x) = P^*(x)a_1^2$

(PI16) $P(a_i \cdot x)y = a_i \cdot P(x)(a_i^* \cdot y)$

(PI17) $P(a_1 \cdot x, x)y = a_1 \cdot P(x)y + P(x)(a_1^* \cdot y)$.

Proof. The Peirce specialization relation PI1(i) follows from JT5, using B6: $P(a_i)b_i \cdot z = L_i(z)P(a_i)b_i = \{-P(a_i)L_j(z) + P(L_i(z)a_i, a_i)\}b_i = -0 + \{(z \cdot a_i)b_i a_i\}$(by PD1) $= a_i \cdot (b_i^* \cdot (a_i \cdot z))$ by P5. We have already noted $e \cdot z_{1/2} = z_{1/2}$, whence (ii). Setting $b_1 = e$ in (i) yields (iii), and linearization yields (iv).

The identities involving the E_i follow from JT5 and JT4. For PI2 and PI5 we have B6 $P(u)E_i(x, y) = P(u)L_j(y)x = -L_i(y)P(u)x + \{(L_i(y)u)xu\}$ (by JT5); when $u = a_i$ we get $-0 + \{(a_i \cdot y)xa_i\} = E_i(a_i \cdot y, a_i^* \cdot x)$(by P4) as in PI2, and when $u = z$ we get $-E_j(P(z)x, y) + E_j(z, x \cdot E_j(z, y)^*)$ (by P4) $= E_j(z, E_j(y, z) \cdot x) - E_j(P(z)x, y)$ (by P8) as in PI5. For PI3, $L(a_i, b_i)E_i(x, y) = L(a_i, b_i)L_j(y)x = L_j(y)L(a_i, b_i)x -$

$[L_j(y), L(a_i, b_i)]x = E_i(L(a_i, b_i)x, y) - L(L_j(y)a_i, b_i)x + L(a_i, L_i(y)b_i)x$ (by JT9) $= E_i(a_i \cdot (b_i^* \cdot x), y) - 0 + \{a_i(b_i \cdot y)x\} = E_i(a_i \cdot (b_i^* \cdot x), y) + E_i(x, a_i^* \cdot (b_i \cdot y))$ (by P4). PI4 is the special case $b_1 = e$ of PI3. For PI6 we use JT3′ for $i = 1$: $P(\{xye\})a_1 = \{P(x)P(y)P(e) + P(e)P(y)P(x) - P(P(x)y, P(e)y) + P(e, x)P(y)P(e, x)\}a_1 = P(x)P(y)a_1^* + (P(y)P(x)a_1)^* - 0 + E_1(x, P(y)(a_1^* \cdot x))$, while for $i = 0$ we use JT4: $P(\{xey\})a_0 = \{P(x)P(e)P(y) + P(y)P(e)P(x) + L(x, e)P(y)L(e, x) - P(P(x)P(e)y, y)\}a_0 = P(x)(P(y)a_0)^* + P(y)(P(x)a_0)^* + E_0(x, P(y)(a_0 \cdot x)) - 0$.

The identities involving $P(x)a_i$ are established in the same ways. For (PI7), $P(x)a_i \cdot y + P(x)(a_i \cdot y) = \{L_j(y)P(x) + P(x)L_i(y)\}a_i = P(L_j(y)x, x)a_i = P(E_i(x, y), x)a_i$ (by JT5) $= E_i(x, y) \cdot (a_i^* \cdot x)$ (by P5). For (PI8) we use linearized JT1: for $i = 1$, $\{(P(x)a_1^*)ye\} + \{(P(x)y)a_1^* e\} = \{x\{a_1^* xy\}e\}$, for $i = 0$ $\{(yP(x)a_0)e\} + \{a_0(P(x)y)e\} = \{\{a_0 xy\}xe\}$, and we use P8. For (PI9), $P(x)\{a_i x a_j\} = P(x)L(a_i, x)a_j = L(x, a_i)P(x)a_j$ (by JT1) $= \{xa_i P(x)a_j\} = P(x)a_j \cdot (a_i^* \cdot x)$. For (PI10) with $i = 1$ we have by JT4 that $P(\{a_1 ex\})b_k = \{P(a_1)P(e)P(x) + P(x)P(e)P(a_1) - P(P(a_1)P(e)x, x) + L(a_1, e)P(x)L(e, a_1)\}b_k = \{P(a_1)P(e)P(x) + P(x)P(e)P(a_1)\}b_k$. If $k = 0$ this becomes $P(a_1)P(e)P(x)b_0 = P(a_1)(P(x)b_0)^* = P(a_1)P^*(x)b_0$, while for $k = 1$ becomes $P(x)P(e)P(a_1)b_1 = P(x)(P(a_1)b_1)^* = P^*(x)P(a_1)b_1$ by (1.8). Similarly if $i = 0$ we have $P(\{a_0 xe\})b_k = \{P(a_0)P(x)P(e) + P(e)P(x)P(a_0) - P(P(a_0)P(x)e, e) + L(a_0, x)P(e)L(x, a_0)\}b_k = \{P(a_0)P(x)P(e) + P(e)P(x)P(a_0)\}b_k$, reducing if $k = 0$ to $P(e)P(x)P(a_0)b_0 = P^*(x)P(a_0)b_0$ and if $k = 1$ to $P(a_0)P(x)P(e)b_1 = P(a_0)P^*(x)b_1$. Since $*$ is an involution on J_i, J_j, (PI11) follows by applying $*$ to (PI10) (with a_i, b_k replaced by a_i^*, b_k^*). Similarly (PI13) follows by applying $*$ to (PI12) (with a_i, b_i replaced by a_i^*, b_i^*), where (PI12) follows from JT5: $L(a_i, b_i)P(x)c_j = \{-P(x)L(b_i, a_i) + P(\{a_i b_i x\}, x)\}c_j = P(a_i \cdot (b_i^* \cdot x), x)c_j$ (by P5) $= E_i(a_i \cdot (b_i^* \cdot x), c_j^* \cdot x)$ (by P3). For (PI14), $P(x)\{a_i b_i c_i\} = -L(b_i, a_i)P(x)c_i + P(\{b_i a_i x\}, x)c_i$ (by JT5) $= -0 + \{(b_i \cdot (a_i^* \cdot x))c_i x\} = E_j(x, c_i^* \cdot (b_i \cdot (a_i^* \cdot x)))$ (by P3). (PI15) is just the particular case $b = e$ of (PI10). For (PI16) with $i = 0$, $P(a_0 \cdot x)y = E_1(a_0 \cdot x \cdot y) \cdot (a_0 \cdot x) - E_0(a_0 \cdot x) \cdot y = a_0 \cdot \{E_1(a_0 \cdot x, y)^* \cdot x\} - P(a_0)E_0(x) \cdot y$ (by PI15) $= a_0 \cdot \{E_1(y, a_0 \cdot x) \cdot x\} - a_0 \cdot \{E_0(x) \cdot (a_0 \cdot y)\}$ (by PI1i) $= a_0 \cdot \{E_1(x, a_0 \cdot y) \cdot x - E_0(x) \cdot (a_0 \cdot y)\}$ (by symmetry of P3) $= a_0 \cdot \{P(x)(a_0 \cdot y)\}$. For $i = 1$, $P(a_1 \cdot x)y = E_1(a_1 \cdot x, y) \cdot (a_1 \cdot x) - E_0(a_1 \cdot x) \cdot y = \{-a_1 \cdot (E_1(a_1 \cdot x, y) \cdot x)\} + \{E_1(a_1^2 \cdot x, y) + E_1(a_1 \cdot x, a_1^* \cdot y)\} \cdot x - P^*(x)a_1^2 \cdot y$ (by (PI1iv), (PI4), (PI15)) $= -a_1 \cdot (E_1(a_1 \cdot x, y) \cdot x) + P(a_1)E_1(x, y)^* \cdot x + E_1(a_1^2 \cdot x, y) \cdot x + \{a_1^2 \cdot P(x)y - E_1(a_1^2 \cdot x, y) \cdot x\}$ (by (PI2), (PI8)) $= a_1 \cdot \{-E_1(a_1 \cdot x, y) \cdot x + E_1(x, y) \cdot (a_1 \cdot x) + a_1 \cdot [E_1(x, y) \cdot x - E_0(x) \cdot y]\}$ (by PI1i, iii) $= a_1 \cdot \{E_1(x, a_1^* \cdot y) - E_0(x) \cdot (a_1^* \cdot y)\}$ (by (PI4), (P6)) $= a_1 \cdot \mathrm{P}(x)(a_1^* \cdot y)$. (PI17) is just the linearization $a_1 \to a_1, e$ of PI16, or it follows from JT5. □

Observe that the proof of PI16 depended only on PI1, 2, 4, 8, 15. Note also that there is no analogue of PI1iv for J_0, so we cannot commute an $L(a_0)$ past an $L(b_0)$ at the expense of an $L(a_0 \cdot b_0)$, which means that if K_0 is an ideal in J_0 we do not have $L(J_0)L(K_0) \subset L(K_0)N(J_0)$ as we do for an ideal K_1 in J_1. Similarly there is no analogue of PI4 or PI17 for $i = 0$.

THE BRACKET PRODUCT ON $J_{1/2}$. Even more basic than the inherited triple product $P(x)y$ on $J_{1/2}$ are the *bracket products*

$$\langle xyz\rangle_i = E_i(x, y)\cdot z,\ \langle x; z\rangle_0 = E_0(x)\cdot z\ . \tag{1.10}$$

This gives two trilinear compositions on $J_{1/2}$, the one for $i = 0$ being symmetric in the first two variables

$$\langle xyz\rangle_0 = \langle yxz\rangle_0\ .$$

Formulas P1, P2 show

$$\begin{aligned} P(x)y &= \langle xyx\rangle_1 - \langle x; y\rangle_0 \\ \{xyz\} &= \langle xyz\rangle_1 + \langle zyx\rangle_1 - \langle xzy\rangle_0\ . \end{aligned} \tag{1.11}$$

In the special case of a maximal idempotent where $J_0 = 0$ we see $P(x)y = \langle xyx\rangle_1$, so the bracket product coincides with the triple product; Loos [1, 2] has abstractly characterized such products $\langle, ,\rangle$ on such $J_{1/2}$ as *alternative triple systems.* We will show that in general even if $J_0 \neq 0$ the product $\langle xyz\rangle_1$ still behaves somewhat like an alternative triple product.

The interaction of the bracket with multiplications from the diagonal Peirce spaces is given by

$$\begin{aligned} L(a_i, b_i)\langle xyz\rangle_i = \langle L(a_i, b_i)x, y, z\rangle_i &+ \langle x, L(a_i^*, b_i^*)y, z\rangle_i \\ &- \langle x, y, L(b_i^*, a_i^*)\rangle_i \end{aligned} \tag{1.12}$$

$$a_1\cdot\langle xyz\rangle_1 = \langle a_1\cdot x, y, z\rangle_1 + \langle x, a_1^*\cdot y, z\rangle_1 - \langle x, y, a_1\cdot z\rangle_1 \tag{1.13}$$

$$L(a_i, b_i)\langle xyz\rangle_j = \langle x, y, L(a_i^*, b_i^*)z\rangle_j \tag{1.14}$$

$$L(a_i)\langle xyz\rangle_j = \langle y, x, L(a_i^*)z\rangle_j \tag{1.15}$$

$$a_1\cdot\langle xyx\rangle_1 - \langle a_1\cdot x, y, x\rangle_1 = E_0(x)\cdot(a_1^*\cdot y) - P(x)a_1^*\cdot y\ . \tag{1.16}$$

Unfortunately (1.13) with 1 replaced by 0 is false (even in triple systems $JT(A, {}^*)$ derived from associative algebras), and there does not seem to be any analogous identity for the interaction of $\langle\ ,\ ,\ \rangle_0$ with J_0.

To verify these identities, note for (1.12) $L(a_i, b_i)E_i(x, y)\cdot z = a_i\cdot(b_i^*\cdot(E_i(x, y)\cdot z))$ (by P5) $= \{a_ib_iE_i(x, y)\}\cdot z - E_i(x, y)\cdot(b_i^*\cdot(a_i\cdot z))$ (by linearized PI1i) $= \{E_i(a_i\cdot(b_i^*\cdot x), y) + E_i(x, a_i^*\cdot(b_i\cdot y))\}\cdot z - E_i(x, y)\cdot\{b_i^*a_i^*z\}$ (by PI3, P5) $= \langle L(a_i, b_i)x, y, z\rangle_i + \langle x, L(a_i^*, b_i^*)y, z\rangle_i - \langle x, y, L(b_i^*, a_i^*)z\rangle_i$ (by P5). We obtain (1.13) by setting $b_1 = e$ in (1.12). For (1.14),

$L(a_i, b_i)E_j(x, y)\cdot z = L(a_i)L(b_i^*)L(E_j(x, y))z = L(E_j(x, y))L(a_i^*)L(b_i)z$ (using P6 twice) $= \langle x, y, L(a_i^*, b_i^*)z\rangle_j$ (using P8). When $i = 1$ (1.15) follows from (1.14) by setting $b_i = e$; in general we argue as before $L(a_i)L(E_j(x, y))z = L(E_j(x, y)^*)L(a_i^*)z = \langle y, x, a_i^*\cdot z\rangle_j$. For (1.16), $a_1\cdot\langle xyx\rangle = a_1\cdot\{P(x)y + E_0(x)\cdot y\}$ (by (1.10), P1) $= \{-P^*(x)a_1\cdot y + E_1(a_1\cdot x, y)\cdot x\} + E_0(x)\cdot(a_1^*\cdot y)$ (by PI8, P6) $= E_0(x)\cdot(a_1^*\cdot y) - P(x)a_1^*\cdot y + \langle a_1\cdot x, y, x\rangle_1$.

Next we have some intrinsic bracket relations for the more important bracket $\langle x, y, z\rangle = \langle x, y, z\rangle_1$:

$$\text{(1.17)}\qquad \langle uv\langle xyz\rangle\rangle + \langle xy\langle uvz\rangle\rangle = \langle\langle uvx\rangle yz\rangle + \langle x\langle vuy\rangle z\rangle$$

$$\text{(1.18)}\qquad \begin{aligned}\langle uv\langle xyx\rangle\rangle - \langle\langle uvx\rangle yx\rangle &= \langle x\langle vuy\rangle x\rangle - \langle xy\langle uvx\rangle\rangle\\ &= E_0(x)\cdot\langle vuy\rangle - E_0(E_0(x)\cdot v, u)\cdot y\\ &\quad + E_0(x, [E_1(x, v)\cdot u - E_0(x, u)\cdot v])\cdot y\end{aligned}$$

$$\text{(1.19)}\qquad \langle\langle xyx\rangle yw\rangle - \langle xy\langle xyw\rangle\rangle = \{P(e)P(y)P(x) - P(x)P(y)\}e\cdot w$$

$$\text{(1.20)}\qquad \langle x\langle yxy\rangle w\rangle - \langle xy\langle xyw\rangle\rangle = \{P(x)P(y) - P(e)P(y)P(x)\}e\cdot w$$

$$\text{(1.21)}\qquad \langle\langle xyx\rangle vw\rangle - \langle x\langle vxy\rangle w\rangle = \{P(e)P(y, v)P(x) - P(x)P(y, v)\}e\cdot w$$

$$\text{(1.22)}\qquad \langle\langle xyz\rangle yw\rangle - \langle x\langle yzy\rangle w\rangle = \{P(e)P(y)P(x, z) - P(x, z)P(y)\}e\cdot w$$

$$\text{(1.23)}\qquad \langle\langle uvx\rangle yw\rangle + \langle x\langle vuy\rangle w\rangle = \langle\langle xyu\rangle vw\rangle + \langle u\langle yxv\rangle w\rangle\,.$$

Here (1.17) is just (1.13) for $a_1 = E_1(u, v)$, $a_1^* = E_1(v, u)$, while (1.23) is a consequence of the symmetry in uv, xy on the left side of (1.17). Setting $a_1 = E_1(u, v)$ in (1.16) yields $\langle uv\langle xyx\rangle\rangle - \langle\langle uvx\rangle yx\rangle (= \langle x\langle vuy\rangle x\rangle - \langle xy\langle uvx\rangle\rangle$ by (1.17)) $= E_0(x)\cdot(E_1(v, u)\cdot y) - P(x)E_1(v, u)\cdot y = E_0(x)\cdot(E_1(v, u)\cdot y) - E_0(x, E_0(u, x)\cdot v)\cdot y + E_0(P(x)v, u)\cdot y$ (by PI5) $= E_0(x)\cdot(E_1(v, u)\cdot y) - E_0(x, E_0(u, x)\cdot v)\cdot y + E_0(E_1(x, v)\cdot x, u)\cdot y - E_0(E_0(x)\cdot v, u)\cdot y$ (by P1) $= E_0(x)\cdot(E_1(v, u)\cdot y) - E_0(E_0(x)\cdot v, u)\cdot y + E_0(x, [E_1(x, v)\cdot u - E_0(x, u)\cdot v])\cdot y$ (by P3 and symmetry of E_0), which is (1.18). The formulas (1.19), (1.20), (1.21), (1.22) are respectively

$$\text{(1.19}'\text{)}\qquad E_1(\langle xyx\rangle, y) - E_1(x, y)^2 = \{P(e)P(y)P(x) - P(x)P(y)\}e$$

$$\text{(1.20}'\text{)}\qquad E_1(x, \langle yxy\rangle) - E_1(x, y)^2 = \{P(x)P(y) - P(e)P(y)P(x)\}e$$

$$\text{(1.21}'\text{)}\qquad E_1(\langle xyx\rangle, v) - E_1(x, \langle vxy\rangle) = \{P(e)P(y, v)P(x) - P(x)P(y, v)\}e$$

$$\text{(1.22}'\text{)}\qquad E_1(\langle xyz\rangle, y) - E_1(x, \langle yzy\rangle) = \{P(e)P(y)P(x, z) - P(x, z)P(y)\}e\,.$$

Here (1.19′) will follow by setting $v = y$ in (1.21′) (or $z = x$ in (1.22′)) and using (1.20′). For (1.20′) note $E_1(x, y)^2 = P(E_1(x, y))e = P(x)P^*(y)e + P^*(y)P(x)e + E_1(x, P(y)(x\cdot e))$ (by PI6) $= P(x)P(y)e + (P(y)P(x)e)^* + E_1(x, P(y)x) = E_1(x, \langle yxy\rangle - P(y)e\cdot x) + P(x)P(y)e + P(e)P(y)P(x)e =$

$E_1(x, \langle yxy\rangle) - \{x(P(y)e)x\} + P(x)P(y)e + P(e)P(y)P(x)e = E_1(x, \langle yxy\rangle) + P(e)P(y)P(x)e - P(x)P(y)e$. For (1.21′) note that $E_1(P(x)y + E_0(x)\cdot y, v) - E_1(x, E_1(v, x)\cdot y) = \{(P(x)y)ve\} + \{yE_0(x)v\}^* - \{xyE_1(v, x)^*\}$ (by P1, P3, P4) $-$ $\{L(P(x)y, v) + P(e)P(y, v)P(x) - L(x, y)L(x, v)\}e = \{P(e)P(y, v)P(x) - P(x)P(y, v)\}e$ by JT6. Finally, for (1.22′) we have $E_1(y, E_1(x, y)\cdot z)^* - E_1(x, E_1(y, z)\cdot y) = \{yzE_1(x, y)^*\}^* - \{xyE_1(y, z)^*\} = P(e)L(y, z)L(y, x)e - L(x,y)L(z,y)e = P(e)\{L(P(y)z,x) + P(y)P(x,z)\}e - \{L(x,P(y)z) + P(x,z)P(y)\}e$ (by JT6, JT7) $= E_1(P(y)z, x)^* - E_1(x, P(y)z) + \{P(e)P(y)P(x, z) - P(x, z)P(y)\}e = \{P(e)P(y)P(x, z) - P(x, z)P(y)\}e$ (by P8).

In the special case that $J_0 = 0$ we obtain the easy half of Loos′ characterization [1, p. 76] of alternative triple systems.

1.24. PROPOSITION. *If* $K_{1/2} \subset J_{1/2}$ *is a bracket subalgebra* ($\langle K_{1/2}K_{1/2}K_{1/2}\rangle \subset K_{1/2}$) *with* $E_0(K_{1/2}) = P(K_{1/2})e = 0$ *(for example,* $K_{1/2} = J_{1/2}$ *if* $J_0 = 0$, *or* $K_{1/2} = P(x)J_{1/2}$ *or* $K_{1/2} = P(x)J_{1/2} + \Phi x$ *principal inner ideals determined by an* $x \in J_{1/2}$ *with* $P(x)e = 0$), *then* $K_{1/2}$ *becomes an alternative triple system under the bracket*

$$\langle xyz\rangle = E_1(x, y)\cdot z = \{\{xye\}ez\} \qquad (x, y, z \in K_{1/2}) .$$

The Jordan triple product on $K_{1/2}$ *is then* $P(x)y = \langle xyx\rangle$.

Proof. The axioms for an alternative triple system are

$$\text{(AT1)} \quad \langle uv\langle xyz\rangle\rangle + \langle xy\langle uvz\rangle\rangle = \langle\langle uvx\rangle yz\rangle + \langle x\langle vuy\rangle z\rangle$$

$$\text{(AT2)} \quad \langle uv\langle xyx\rangle\rangle = \langle\langle uvx\rangle yx\rangle$$

$$\text{(AT3)} \quad \langle xy\langle xyz\rangle\rangle = \langle\langle xyx\rangle yz\rangle .$$

Here (AT1) follows from (1.17), and (AT2), (AT3) from (1.18), (1.19) since $E_0(K_{1/2}) = P(K_{1/2})e = 0$. By (P1) we have $P(x)y = E_1(x, y)\cdot x = \langle xyx\rangle$ in this case.

If x has $P(x)e = 0$ then the inner ideals $K_{1/2} = P(x)J_{1/2} \subset P(x)J_{1/2} + \Phi x = K'_{1/2}$ kill e, $P(K_{1/2})e = P(K'_{1/2})e = 0$. Indeed, by JT3 we have $P(K_{1/2}) = P(x)P(J_{1/2})P(x)$, and by JT1 $P(K'_{1/2}) = P(P(x)J_{1/2}) + P(P(x)J_{1/2}, x) + \Phi P(x) = \{P(x)P(J_{1/2}) + L(x, J_{1/2}) + \Phi\}P(x)$. To see next that these inner ideals are bracket-closed subalgebras, first note that since $P(K'_{1/2})J_{1/2} \subset K_{1/2} \subset K'_{1/2}$ by innerness we have $\langle xyx\rangle = P(x)y \in K_{1/2}$, hence by linearization $\langle xyz\rangle + \langle zyx\rangle \in K_{1/2}$, for any $x, z \in K'_{1/2}$ and any $y \in J_{1/2}$. Next we show $\langle K_{1/2}J_{1/2}x\rangle$ and $\langle xJ_{1/2}K_{1/2}\rangle$ are contained in $K_{1/2}$; by skewness it suffices to prove the latter, where $\langle xJ_{1/2}K_{1/2}\rangle = E_1(x, J_{1/2})\cdot P(x)J_{1/2} \subset -P(x)(E_1(x, J_{1/2})^* \cdot J_{1/2}) + P(E_1(x, J_{1/2})\cdot x, x)J_{1/2}$ (by PI17) $\subset P(x)J_{1/2} + P(\langle xJ_{1/2}x\rangle, x)J_{1/2} \subset P(K'_{1/2})J_{1/2} \subset K_{1/2}$. Finally, $\langle K_{1/2}J_{1/2}K_{1/2}\rangle = E_1(K_{1/2}, J_{1/2})\cdot K_{1/2} \subset -P(x)(E_1(K_{1/2}, J_{1/2})^* \cdot J_{1/2}) + P(E_1(K_{1/2}, J_{1/2})\cdot x, x)J_{1/2} \subset P(x)J_{1/2} + P(\langle K_{1/2}J_{1/2}x\rangle, x)J_{1/2} \subset P(K'_{1/2})J_{1/2}$ (by the previous case) $\subset K_{1/2}$. Thus in fact we have the stronger closure $\langle K'_{1/2} J_{1/2} K'_{1/2} \rangle \subset K_{1/2}$. $\square$

In any alternative triple system we obtain an ordinary bilinear alternative multiplication by fixing the middle factor: the homotopes $A^{(u)}$ with products $x \cdot_u y = \langle xuy \rangle$ are alternative.

1.25. PROPOSITION. *If $K_{1/2}$ is a bracket-closed subspace of $J_{1/2}$ with $P(K_{1/2})e = 0$, then for any $u \in K_{1/2}$ the homotope $K_{1/2}^{(u)}$ with product*

$$x \cdot_u y = \langle xuy \rangle$$

is an alternative algebra. If u is a tripotent with $P(u)e = 0$ then we have an involutory map $x \to P(u)x = \bar{x}$ on $K_{1/2} = J_{1/2}(e) \cap J_1(u) = P(u)J_{1/2}(e)$, and the bracket can be recovered as

$$\langle xyz \rangle = (x \cdot_u \bar{y}) \cdot_u z \; . \tag{1.26}$$

If in addition $E_1(u, u) = \{uue\} = e$ then u acts as unit for $P(u)J_{1/2}(e)$, and $x \to \bar{x}$ is an involution of the multiplicative structure.

Proof. By 1.24 we know $K_{1/2}$ is an alternative triple system under the bracket, hence the homotope $K_{1/2}^{(u)}$ is an alternative algebra [1, p. 64]. When u is tripotent $P(u)^3 = P(u)$, so $P(u)$ is involutory on $P(u)J_1$, and furthermore for $x, y, z \in P(u)J_{1/2}$ we have $(x \cdot_u y) \cdot_u z - \langle x\bar{y}z \rangle = \langle\!\langle xuy \rangle uz \rangle - \langle x \langle uyu \rangle z \rangle = \{P(e)P(u)P(x, y) - P(x, y)P(u)\}e \cdot z$ (by 1.22) $= 0$ since $P(K_{1/2})e = P(u)P(J_{1/2})P(u)e = 0$. Thus we recover the bracket on $P(u)J_{1/2}$ from the bilinear product and the involution.

When $\{uue\} = E_1(u, u) = e$ in addition then u is a left unit, $u \cdot_u y = E_1(u, u) \cdot y = e \cdot y = y$. If we knew $x \to \bar{x}$ reversed multiplication this would imply $\bar{u} = u$ was also a right unit; we can also argue directly, $x \cdot_u u = \langle xuu \rangle = E_1(x, u) \cdot u = \{xuu\} - E_1(u, u) \cdot x + E_0(x, u) \cdot u = L(u, u)(P(u)^2 x) - e \cdot x + 0$ (since $E_0(K_{1/2}) = 0$) $= P(P(u)u, u)P(u)x - x$ (using JT1) $= 2P(u)^2 x - x = x$.

To see $x \to \bar{x}$ is indeed an involution, first use the right unit to see $x \cdot_u y = (x \cdot_u y) \cdot_u u = \langle x\bar{y}u \rangle$,

$$x \cdot_u y = \langle xuy \rangle = \langle x\bar{y}u \rangle \qquad (\text{when } \{uue\} = e) \; . \tag{1.27}$$

Then

$$\begin{aligned} \overline{x \cdot_u y} &= \langle u \langle xuy \rangle u \rangle \\ &= \langle\!\langle uxu \rangle yu \rangle - \{P(e)P(x, y)P(u) - P(u)P(x, y)\}e \cdot u \text{ (by 1.27)} \\ &= \langle \bar{x}yu \rangle - 0 \qquad (\text{again } P(K_{1/2})e = 0) \\ &= \bar{x} \cdot_u \bar{y} \qquad (\text{above}) \; . \end{aligned}$$

Thus the involution condition is precisely (1.27).

The condition $E_1(u, u)\cdot y = y$ is necessary well as sufficient for (1.27) to hold. Indeed, using (1.21), (1.18) and $P(K_{1/2})e = 0$ one can show in general that $P(u)\{\langle xuy\rangle - \langle x\bar{y}u\rangle\} = \langle u\langle xuy\rangle u\rangle - \langle u\langle x\bar{y}u\rangle u\rangle = \langle\langle uyu\rangle xu\rangle - \langle uu\langle\bar{y}xu\rangle\rangle = \{\mathrm{Id} - L(E_1(u, u))\}\langle\bar{y}xu\rangle$, which again establishes sufficiency; for necessity set $x = u$, so $\langle uuy\rangle - \langle u\bar{y}u\rangle = E_1(u, u)\cdot y - P(u)\bar{y} = E_1(u, u)\cdot y - y$. $\square$

These alternative structures on the subsystems $P(u)J_{1/2}$ are important for the study of collinear idempotents [5]. These are families of tripotents $\{e_1, \cdots, e_n\}$ with $P(e_i)e_j = 0$, $\{e_ie_ie_j\} = e_j$ for $i \neq j$, and the $P(e_j)J_{1/2}(e_i) = J_{1/2}(e_i) \cap J_1(e_j)$ carry isomorphic alternative structures. (The motivating example is the collinear matrix units $\{e_{11}, e_{12}, \cdots, e_{1n}\}$ in $M_n(\Phi)$ under xy^tx.)

2. Ideal-building. A subspace $K \subset J$ is an *ideal* if it is both an *outer ideal*

$$P(J)K \subset K \tag{2.1}$$

$$L(J, J)K \subset K \tag{2.2}$$

and an *inner ideal*

$$P(K)J \subset K\,. \tag{2.3}$$

If K is already an outer ideal, the inner condition (2.3) reduces to

$$P(k_i)J \subset K \text{ for some spanning set } \{k_i\} \text{ for } K\,. \tag{2.3$'$}$$

Note that the operators $L(y, z)$ cannot be derived from the $P(x)$'s.

From now on we fix a tripotent e with corresponding Peirce decomposition

$$J = J_1 \oplus J_{1/2} \oplus J_0\,.$$

Since the Peirce projections (1.1) are multiplication operators, any ideal $K \lhd J$ breaks into Peirce pieces

$$K = K_1 \oplus K_{1/2} \oplus K_0 \qquad (K_i = K \cap J_i)\,.$$

Using the expression (1.7) for the product $P(x)y$ in terms of bilinear products, we obtain a componentwise criterion for K to be an ideal (exactly like that in Jordan algebras).

2.4. IDEAL CRITERION. *A subspace $K = K_1 \oplus K_{1/2} \oplus K_0$ is an ideal in the* JTS *$J = J_1 \oplus J_{1/2} \oplus J_0$ iff for $i = 1, 0$ and $j = 1 - i$ we have*

(C1) K_i *is an ideal in* J_i

(C2) $E_i(J_{1/2}, K_{1/2}) \subset K_i$

(C3) $J_i \cdot K_{1/2} \subset K_{1/2}$

(C4) $K_i \cdot J_{1/2} \subset K_{1/2}$

(C5) $P(J_{1/2})K_i \subset K_j$

(C6) $P(k_{1/2})J_i \subset K_j$ *for some spanning set* $\{k_{1/2}\}$ *for* $K_{1/2}$.

If $1/2 \in \Phi$ *then* (C5) *and* (C6) *are superfluous.*

Proof. Clearly the conditions are necessary, since any product with a factor in K must fall back in K. Just as in the Jordan algebra case, they also suffice. Outerness (2.1) $P(J)K \subset K$ follows by (1.7) since $P(J_i)K_i \supset K_i$ (by (C1)), $P(J_{1/2})K_i \subset K_j$ (by (C5)), $J_{1/2} \cdot E_1(J_{1/2}, K_{1/2}) \subset K_{1/2}$ (by (C2), (C4)), $K_{1/2} \cdot J_0 \subset K_{1/2}$ (by (C3)), $J_1 \cdot (J_0 \cdot K_{1/2}) \subset K_{1/2}$ (by (C3)), $J_i \cdot (K_i^* \cdot J_{1/2}) \subset K_{1/2}$ (by (C4), (C3) — note that $K_i^* = K_i$ for any ideal $K_i \lhd J_i$ since the involution is given by a multiplication), and $E_i(J_{1/2}, J_i^* \cdot K_{1/2}) \subset K_i$ (by (C3), (C2)).

Outerness (2.2) $L(J, J)K = P(J, K)J \subset K$ follows by the linearization of (1.7). First note

$$\text{(C2}'\text{)} \quad E_i(K_{1/2}, J_{1/i}) \subset K_i$$

since $E_i(K_{1/2}, J_{1/2}) = E_i(J_{1/2}, K_{1/2})^* \subset K_i^* \subset K_i$. We have $\{J_iJ_iK_i\} \subset K_i$ (by (C1)), $\{J_{1/2}J_iK_{1/2}\} \subset E_j(J_{1/2}, J_i^* \cdot K_{1/2}) \subset K_j$ (by P3, (C3), (C2)), $K_{1/2} \cdot E_1(J_{1/2}, J_{1/2}) \subset K_{1/2}$ (by (C3)), $J_{1/2} \cdot E_1(K_{1/2}, J_{1/2}) + J_{1/2} \cdot E_1(J_{1/2}, K_{1/2}) \subset K_{1/2}$ (by (C2′), (C2), (C4)), $J_{1/2} \cdot P(J_{1/2}, K_{1/2})e = J_{1/2} \cdot E_0(J_{1/2}, K_{1/2}) \subset K_{1/2}$ (by (C2), (C4)), $J_i \cdot (K_i^* \cdot J_{12}) + K_i \cdot (J_i^* \cdot J_{1/2}) \subset K_{1/2}$ (by (C4), (C3)), $E_i(K_{1/2}, J_i^* \cdot J_{1/2}) \subset E_i(K_{1/2}, J_{1/2}) \subset K_i$ (by (C2′)), and $E_i(J_{1/2}, K_i^* \cdot J_{1/2}) = E_i(J_{1/2} \cdot K_i \cdot J_{1/2}) \subset K_i$ (by (C4), (C2)).

Once K is outer we can apply (2.3′) to obtain innerness: for the spanning elements $k_r \in K_r$ we have $P(k_i)J = P(k_i)J_i \subset K_i$ by (C1) if $i = 1, 0$, while $P(k_{1/2})J_i \subset K_j$ by (C6) and $P(k_{1/2})J_{1/2} = k_{1/2} \cdot E_1(k_{1/2}, J_{1/2}) - J_{1/2} \cdot P(k_{1/2})e \subset K_{1/2} \cdot J_1 - J_{1/2} \cdot K_0 \subset K_{1/2}$ by P1, (C5), (C3), (C4). Thus K is an ideal.

When $1/2 \in \Phi$, (C5) and (C6) follow from (C2–C4) since $P(x) = 1/2P(x, x)$ where $P(J_{1/2}, J_{1/2})K_i = E_j(J_{1/2}, K_i^* \cdot J_{1/2}) \subset K_j$ by (C4), (C2), and $P(J_{1/2}, K_{1/2})J_i \subset E_j(J_{1/2}, J_i^* \cdot K_{1/2}) + E_j(K_{1/2}, J_i^* \cdot J_{1/2}) \subset K_j$ by (C3), (C2), (C2′). □

An ideal K_i in a diagonal Peirce space J_i is *invariant* if it is both *L-invariant*

$$L(J_{1/2}, J_{1/2})K_i = E_i(J_{1/2}, K_i^* \cdot J_{1/2}) \subset K_i \tag{2.5}$$

and if $i = 0$, also

$$L(J_{1/2}, e)P(J_0, J_{1/2})K_0 = E_0(J_{1/2}, J_0\cdot(K_0\cdot J_{1/2})) \subset K_0 \,, \tag{2.6}$$

and *P-invariant*

$$P(J_{1/2})P(J_{1/2})K_i \subset K_i \tag{2.7}$$

and again if $i = 0$ also

$$P^*(J_{1/2})P(J_{1/2})K_0 = P(J_{1/2})P^*(J_{1/2})K_0 = P(J_{1/2})P(e)P(J_{1/2})K_0 \subset K_0 \,. \tag{2.8}$$

Note that the maps $L(J_{1/2}, J_{1/2})$ and $P(J_{1/2})P(J_{1/2})$ automatically send J_i into itself (and $L(J_{1/2}, e)P(J_0, J_{1/2})$ and $P(J_{1/2})P(e)P(J_{1/2})$ send J_0 into itself).

An ideal $K_{1/2} \lhd J_{1/2}$ in the off-diagonal Peirce space is *invariant* if

$$L(J_i)K_{1/2} = J_i\cdot K_{1/2} \subset K_{1/2} \tag{2.9}$$

$$\begin{aligned} L_1(J_{1/2})L_0(J_{1/2})K_{1/2} &= L(J_{1/2}, e)L(e, J_{1/2})K_{1/2} = \langle K_{1/2}J_{1/2}J_{1/2}\rangle \subset K_{1/2} \\ L_1(J_{1/2})L_0(K_{1/2})J_{1/2} &= L(J_{1/2}, e)P(e, J_{1/2})K_{1/2} = \langle J_{1/2}K_{1/2}J_{1/2}\rangle \subset K_{1/2} \,. \end{aligned} \tag{2.10}$$

Note that these maps do send $J_{1/2}$ back into itself.

An alternate characterization of invariance in terms of the bracket products is that $K_{1/2}$ be a subspace satisfying

$$J_i\cdot K_{1/2} \subset K_{1/2} \tag{2.9$'$}$$

$$\langle J_{1/2}J_{1/2}K_{1/2}\rangle_1 + \langle J_{1/2}K_{1/2}J_{1/2}\rangle_1 + \langle K_{1/2}J_{1/2}J_{1/2}\rangle_1 \subset K_{1/2} \tag{2.10$'$}$$

$$\langle J_{1/2}K_{1/2}J_{1/2}\rangle_0 + \langle K_{1/2}; J_{1/2}\rangle_0 \subset K_{1/2} \,, \tag{2.10$''$}$$

i.e., that $K_{1/2}$ be an ideal of the bracket algebra $J_{1/2}$. Clearly any invariant bracket ideal (2.9′)-(2.10″) is invariant in the sense of (2.9)-(2.10) and is an ordinary ideal by (1.11). Conversely, if $K_{1/2}$ is an invariant ordinary ideal it must be a bracket ideal: $\langle K_{1/2}J_{1/2}J_{1/2}\rangle_1 + \langle J_{1/2}K_{12}J_{1/2}\rangle_1$ is contained in $K_{1/2}$ by invariance (2.10), $\langle J_{1/2}J_{1/2}K_{1/2}\rangle_1 \subset J_1\cdot K_{1/2} \subset K_{1/2}$ by invariance (2.9), similarly $\langle J_{1/2}J_{1/2}K_{1/2}\rangle_0 \subset J_0\cdot K_{1/2} \subset K_{1/2}$ by (2.9), while $\langle J_{1/2}K_{1/2}J_{1/2}\rangle_0 = \langle K_{1/2}J_{1/2}J_{1/2}\rangle_0 \subset -\{J_{1/2}J_{1/2}K_{1/2}\} + \langle J_{1/2}J_{1/2}K_{1/2}\rangle_1 + \langle K_{1/2}J_{1/2}J_{1/2}\rangle_1 \subset K_{1/2}$ by ordinary idealness and closure under $\langle\ ,\ ,\ \rangle_1$, also $\langle K_{1/2}; J_{1/2}\rangle_0 = \langle K_{1/2}J_{1/2}K_{1/2}\rangle_1 - P(K_{1/2})J_{1/2} \subset K_{1/2}$ for the same reason, with $\langle J_{1/2}; K_{1/2}\rangle_0 \subset J_0\cdot K_{1/2} \subset K_{1/2}$ by (2.9).

If $1/2 \in \Phi$ then L-invariance (2.5) of $K_i \lhd J_i$ implies P-invariance (2.7) in view of JT8. It is not clear whether (2.5), (2.6) imply (2.8) when $1/2 \in \Phi$.

An important tool is the ability to flip an ideal from one diagonal Peirce space to another.

2.11. FLIPPING LEMMA. *If K_1 is an ideal in J_1 then*

$$K_0 = P(J_{1/2})K_1$$

is an ideal in J_0, *which is invariant if* K_1 *is. If* K_0 *is an ideal in* J_0 *then*

$$K_1 = P(J_{1/2})K_0 + P^*(J_{1/2})K_0$$

is an ideal in J_1, *which again is invariant if* K_0 *is.*

Proof. We handle both cases at once by proving

$$K_j = P(J_{1/2})K_i + P^*(J_{1/2})K_i$$

is an ideal inheriting invariance from K_i. Note again that $K_i^* = K_i$ for any ideal $K_i \lhd J_i$.

Outerness (2.1) follows from (PI11, 10):

$$P(a_j)P(x_{1/2})k_i = P^*(a_j^* \cdot x_{1/2})k_i \in P^*(J_{1/2})K_i$$
$$P(a_j)P^*(x_{1/2})k_i = P(a_j \cdot x_{1/2})k_i \in P(J_{1/2})K_i \, .$$

Outerness (2.2) follows from (PI12, 13):

$$L(a_j, b_j)P(x_{1/2})k_i = P(a_j \cdot (b_j^* \cdot x_{1/2}), x_{1/2})k_i \in P(J_{1/2})K_i$$
$$L(a_j, b_j)P^*(x_{1/2})k_i = P(a_j^* \cdot (b_j \cdot x_{1/2}), x_{1/2})k_i \in P^*(J_{1/2})K_i \, .$$

To see that K_j is inner (2.3′), for the spanning elements $P(x_{1/2})k_i$ and $P^*(x_{1/2})k_i$ we have

$$P(P(x_{1/2})k_i)J_j = P(x_{1/2})P(k_i)P(x_{1/2})J_j \subset P(x_{1/2})P(k_i)J_i \subset P(x_{1/2})K_i$$
$$P(P^*(x_{1/2})k_i)J_j = P^*(x_{1/2})P(k_i)P^*(x_{1/2})J_j \subset P^*(x_{1/2})P(k_i)J_i \subset P^*(x_{1/2})K_i$$

using (1.8) and innerness of K_i in J_i. Thus K_j is inner as well as outer, hence is an ideal in J_j.

If K_i is L-invariant (2.5) to begin with, then K_j will be L-invariant too:

$$\begin{aligned} L(x_{1/2}, y_{1/2})P(z_{1/2})k_i &= \{P(\{x_{1/2}y_{1/2}z_{1/2}\}, z_{1/2}) - P(z_{1/2})L(y_{1/2}, x_{1/2})\}k_i \quad \text{(by JT5)} \\ &\in P(J_{1/2})K_i + P(J_{1/2})L(J_{1/2}, J_{1/2})K_i \subset P(J_{1/2})K_i \\ &\qquad \text{(by } L\text{-invariance)} \end{aligned}$$

$$\begin{aligned} L(x_{1/2}, y_{1/2})P^*(z_{1/2})k_0 &= L(x_{1/2}, y_{1/2})P(e)P(z_{1/2})k_0 \\ &= \{P(\{x_{1/2}y_{1/2}e\}, e) - P(e)L(y_{1/2}, x_{1/2})\}P(z_{1/2})k_0 \quad \text{(by JT5)} \\ &\in P(J_1)P(J_{1/2})K_0 - (L(J_{1/2}, J_{1/2})P(J_{1/2})K_0)^* \\ &\subset P^*(J_{1/2})K_0 \quad \text{(by PI11, above, and } L\text{-invariance)} \, . \end{aligned}$$

L-invariance (2.6) only applies when $i = 1$. In this case it follows from L-invariance (2.5) of K_1: we have $E_0(J_{1/2}, K_1 \cdot J_{1/2}) = \{J_{1/2}K_1J_{1/2}\} \subset K_0$ by definition, and $J_0 \cdot (K_0 \cdot J_{1/2}) \subset K_1 \cdot J_{1/2}$ because $\{J_0(P(J_{1/2})K_1)J_{1/2}\} = -\{J_0(P(J_{1/2})J_{1/2})K_1\} + \{J_0J_{1/2}\{K_1J_{1/2}J_{1/2}\}\}$ (by JT2) $\subset \{J_0J_{1/2}K_1\}$ (by L-invariance of $K_1) = K_1\cdot(J_0\cdot J_{1/2})\subset K_1\cdot J_{1/2}$.

If in addition K_i is P-invariant (2.7) the same is true of K_j:

$$\begin{aligned}
&P(x_{1/2})P(y_{1/2})(P(z_{1/2})k_i) = P(x_{1/2})(P(y_{1/2})P(z_{1/2})k_i)\in P(J_{1/2})K_i\\
&P(x_{1/2})P(y_{1/2})(P^*(z_{1/2})k_0) = P(x_{1/2})P(y_{1/2})P(e)P(z_{1/2})k_0\\
&\quad = \{P(\{x_{1/2}y_{1/2}e\}) + P(P(x_{1/2})P(y_{1/2})e,\, e) - P(e)P(y_{1/2})P(x_{1/2})\\
&\qquad - L(x_{1/2},\, y_{1/2})P(e)L(y_{1/2},\, x_{1/2})\}P(z_{1/2})k_0 \quad (\text{by JT4})\\
&\quad \subset \{P(J_1) - P(e)P(J_{1/2})P(J_{1/2}) - L(J_{1/2},\, J_{1/2})P(e)L(J_{1/2},\, J_{1/2})\}P(J_{1/2})K_0\\
&\quad \subset P^*(J_{1/2})K_0 - L(J_{1/2},\, J_{1/2})P^*(J_{1/2})K_0 \quad (\text{by } P, L\text{-invariance of } K_0)\\
&\quad \subset P^*(J_{1/2})K_0 \quad (\text{by above } L\text{-invariance of } K_1)\ .
\end{aligned}$$

P-invariance (2.8) applies only when $i=1$. In this case it follows from P-invariance (2.7) for K_1: $P^*(J_{1/2})P(J_{1/2})K_0 = P(J_{1/2})P(e)P(J_{1/2})\{P(J_{1/2})K_1\}\subset P(J_{1/2})P(e)K_1$ (by P-invariance of K_1) $= P(J_{1/2})K_1 = K_0$.

It is not clear whether $P(J_{1/2})K_0$ inherits P-invariance when K_0 is merely P-invariant (not also L-invariant). $\square$

We can now obtain the main result on Peirce ideals. Notice how much messier the formulation becomes for triple systems.

2.12. Proposition Theorem. *An ideal K_i in a Peirce subsystem J_i is the projection of a global ideal K in J iff K_i is invariant. In this case the ideal generated by K_i takes the form*

$$\begin{aligned}
&(i=1)\quad K = K_1\oplus K_1\cdot J_{1/2}\oplus P(J_{1/2})K_1\\
&(i=0)\quad K = K_0\oplus \{K_0\cdot J_{1/2} + J_0\cdot(K_0\cdot J_{1/2}) + P(J_{1/2})K_0\cdot J_{1/2}\}\\
&\qquad\qquad \oplus \{P(J_{1/2})K_0 + P^*(J_{1/2})K_0\}\\
&\left(i=\frac{1}{2}\right) K = \{E_0(J_{1/2},\, K_{1/2}) + P(K_{1/2})J_1 + P(J_{1/2})P(K_{1/2})J_0 + P^*(J_{1/2})P(K_{1/2})J_0\}\\
&\qquad\qquad \oplus K_{1/2}\oplus \{E_1(J_{1/2},\, K_{1/2}) + E_1(K_{1/2},\, J_{1/2}) + P(K_{1/2})J_0 + P^*(K_{1/2})J_0\\
&\qquad\qquad + P(J_{1/2})P(K_{1/2})J_1 + P^*(J_{1/2})P(K_{1/2})J_1\}\ .
\end{aligned}$$

If $1/2\in\Phi$ we have $P(J_{1/2})K_i = E_j(J_{1/2},\, K_i\cdot J_{1/2})$, $P(K_{1/2})J_j + P^*(K_{1/2})J_j\subset E_i(K_{1/2}, K_{1/2})$, $P(J_{1/2})P(K_{1/2})J_i + P^*(J_{1/2})P(K_{1/2})J_i\subset E_i(J_{1/2}, K_{1/2}) + E_i(J_{1/2}, K_{1/2})^*$ so the expressions for K reduce to

$$\begin{aligned}
&(i=1)\quad K = K_1\oplus K_1\cdot J_{1/2}\oplus E_0(J_{1/2},\, K_1\cdot J_{1/2})\\
&(i=0)\quad K = K_0\oplus \{K_0\cdot J_{1/2} + J_0\cdot(K_0\cdot J_{1/2} + E_1(J_{1/2},\, K_0\cdot J_{1/2})\cdot J_{1/2})\}\\
&\qquad\qquad \oplus \{E_1(J_{1/2},\, K_0\cdot J_{1/2}) + E_1(K_0\cdot J_{1/2},\, J_{1/2})\}\\
&\left(i=\frac{1}{2}\right)\quad K = E_0(J_{1/2},\, K_{1/2})\oplus K_{1/2}\oplus \{E_1(J_{1/2},\, K_{1/2}) + E_1(K_{1/2},\, J_{1/2})\}\ .
\end{aligned}$$

Proof. We have already noted that a Peirce component K_i must be invariant under global multiplications sending J_i into itself. Certainly the ideal generated by K_i contains all the above products; it remains only to show in each case K forms an ideal.

We begin with the easier diagonal cases $i = 1, 0$, where $K = K_i \oplus K_{1/2} \oplus K_j = K_i \oplus \{K_i \cdot J_{1/2} + J_i \cdot (K_i \cdot J_{1/2}) + P(J_{1/2})K_i \cdot J_{1/2}\} \oplus \{P(J_{1/2})K_i + P^*(J_{1/2})K_i\}$ (note for $i = 1$ that some of these products simplify: $J_1 \cdot (K_1 \cdot J_{1/2}) \subset (J_1 \cdot K_1) \cdot J_{1/2} - K_1 \cdot (J_1 \cdot J_{1/2}) \subset K_1 \cdot J_{1/2}$ by PIiv, $P^*(J_{1/2})K_1 = P(J_{1/2})K_1$ since $K_1^* = K_1$, and $P(J_{1/2})K_1 \cdot J_{1/2} \subset J_{1/2} \cdot L(J_{1/2}, J_{1/2})K_1 - K_1^* \cdot P(J_{1/2})J_{1/2} \subset J_{1/2} \cdot K_1$ by JT2).

We verify that the K_r satisfy the conditions (C1)-(C6) of (1.4). For (C1), K_i is an invariant ideal in J_i by hypothesis and $K_j = P(J_{1/2})K_i + P^*(J_{1/2})K_i$ is an invariant ideal in J_j by the Flipping Lemma 2.11. For (C5) we have $P(J_{1/2})K_i \subset K_j$ by construction, and $P(J_{1/2})K_j = P(J_{1/2})P(J_{1/2})K_i + P(J_{1/2})P^*(J_{1/2})K_i \subset K_i$ by P-invariance (2.7), (2.8). For (C2) we have $E_i(J_{1/2}, K_{1/2})$ the sum of $E_i(J_{1/2}, K_i \cdot J_{1/2})$ and $E_i(J_{1/2}, J_i \cdot (K_i \cdot J_{1/2}))$ and $E_i(J_{1/2}, P(J_{1/2})K_i \cdot J_{1/2})$ (the latter two only when $i = 0$). The first of these has $E_i(J_{1/2}, K_i \cdot J_{1/2}) = L(J_{1/2}, J_{1/2})K_i^* \subset K_i$ by (P4) and the L-invariance (2.5) of $K_i = K_i^*$. For $i = 0$ the second term $E_0(J_{1/2}, J_0 \cdot (K_0 \cdot J_{1/2}))$ falls in K_0 by the hypothesis of L-invariance (2.6). For $i = 0$ the third term becomes $E_0(J_{1/2}, P(J_{1/2})K_0 \cdot J_{1/2}) = \{J_{1/2}(P(J_{1/2})K_0)^* J_{1/2}\}$ (by P3)$\subset P(J_{1/2})P^*(J_{1/2})K_0$, which falls in K_0 by the hypothesis of P-invariance (2.8). Continuing with (C2), we examine $E_j(J_{1/2}, K_{1/2})$. By (P3) $E_j(J_{1/2}, K_i \cdot J_{1/2}) = \{J_{1/2}K_i^* J_{1/2}\} \subset P(J_{1/2})K_i \subset K_j$ by (C5). When $i = 0$ we must examine two other terms: $E_1(J_{1/2}, J_0 \cdot (K_0 \cdot J_{1/2})) = E_1(K_0 \cdot J_{1/2}, J_0 \cdot J_{1/2}) \subset E_1(K_0 \cdot J_{1/2}, J_{1/2}) = E_1(J_{1/2}, K_0 \cdot J_{1/2})^* \subset K_1^* = K_1$ as above, and $E_1(J_{1/2}, P(J_{1/2})K_0 \cdot J_{1/2}) = L(J_{1/2}, J_{1/2})(P(J_{1/2})K_0)^* = L(J_{1/2}, J_{1/2})P(e)P(J_{1/2})K_0$ where $L(x, y)P(e)P(z)k_0 = P(e)P(z)L(x, y)k_0 + P(\{xye\}, e)P(z)k_0 - P(e)P(\{yxz\}, z)k_0 \in P(e)P(J_{1/2})L(J_{1/2}, J_{1/2})K_0 + P(J_1)P(J_{1/2})K_0 - P(e)P(J_{1/2})K_0 \subset P(e)P(J_{1/2})K_0 + P^*(J_{1/2})K_0$ (by PI11 and L-invariance (2.5)) $\subset K_1$. This completes the verification of (C2). We have (C4) because $K_i \cdot J_{1/2} \subset K_{1/2}$ by construction and $K_j \cdot J_{1/2} = (P(J_{1/2})K_i) \cdot J_{1/2} + (P(J_{1/2})K_i)^* \cdot J_{1/2}$ (the two differing only when $i = 0$) where the latter is by PI8 contained in $E_i(J_{1/2}, K_i^* \cdot J_{1/2})^* \cdot J_{1/2} - K_i^* \cdot P(J_{1/2})J_{1/2} \subset K_i^* \cdot J_{1/2} - K_i^* \cdot J_{1/2}$ (by L-invariance (2.5)) $\subset K_i \cdot J_{1/2} \subset K_{1/2}$ and when $i = 0$ the former $(P(J_{1/2})K_0) \cdot J_{1/2}$ is contained in $K_{1/2}$ by construction. (There does not seem to be any way to show it falls into $K_0 \cdot J_{1/2} + J_0 \cdot (K_0 \cdot J_{1/2})$.) For (C3) note that $J_i \cdot (K_i \cdot J_{1/2}) \subset K_{1/2}$ by construction, $J_j \cdot (K_i \cdot J_{1/2}) = K_i^* \cdot (J_j^* \cdot J_{1/2}) \subset K_{1/2}$ by P6, and for $i = 0$ $J_1 \cdot [J_0 \cdot (K_0 \cdot J_{1/2})] \subset J_0 \cdot (K_0 \cdot (J_1 \cdot J_{1/2})) \subset K_{1/2}$ using P6 twice, and $J_0 \cdot [J_0 \cdot (K_0 \cdot J_{1/2})] \subset \{J_0 J_0 K_0\} \cdot J_{1/2} - K_0 \cdot (J_0 \cdot (J_0 \cdot J_{1/2}))$ (by PI1i) $\subset K_0 \cdot J_{1/2} \subset K_{1/2}$, and finally $J_r \cdot (P(J_{1/2})K_0 \cdot J_{1/2}) \subset J_r \cdot (K_1 \cdot J_{1/2}) \subset K_{1/2}$ by the above. For the last criterion (C6) we consider the spanning elements $k_i \cdot x_{1/2}$ (and, when $i = 0$, $a_0 \cdot (k_0 \cdot x_{1/2})$ and $P(x_{1/2})k_0 \cdot y_{1/2}$ as well). We observe by PI10, (C5), (C1) that $P(k_i \cdot x_{1/2})(J_i + J_j) = P^*(x_{1/2})P(k_i)J_i + P(k_i)P^*(x_{1/2})J_j \subset$

$P^*(J_{1/2})K_i + P(K_i)J_i \subset K_j + K_i$, also $P(a_0\cdot(k_0\cdot x_{1/2}))(J_1 + J_0) = P(a_0)P^*(k_0\cdot x_{1/2})J_1 + P^*(k_0\cdot x_{1/2})P(a_0)J_0 = P(a_0)P(k_0)P(x_{1/2})J_1 + P(x_{1/2})P(k_0)P(a_0)J_0 \subset P(J_0)K_0 + P(J_{1/2})K_0 \subset K_0 + K_1$, and also $P(P(x_{1/2})k_0\cdot y_{1/2})(J_1 + J_0) = P^*(y_{1/2})P(P(x_{1/2})k_0)J_1 + P(P(x_{1/2})k_0)P^*(y_{1/2})J_0 = P^*(y_{1/2})P(x_{1/2})P(k_0)P(x_{1/2})J_1 + P(x_{1/2})P(k_0)P(x_{1/2})P^*(y_{1/2})J_0 \subset P^*(J_{1/2})K_1 + P(J_{1/2})K_0 \subset K_0 + K_1$. Thus (C1)-(C6) hold, and K is an ideal.

The case $i = 1/2$ is even more tiresome. We must again verify (C1)-(C6). (C3) follows from invariance (2.9), and (C2) and (C6) follow by our construction of K_1, K_0. For the sake of symmetry we write the diagonal Peirce pieces as

$$K_i = E_i(J_{1/2}, K_{1/2}) + E_i(J_{1/2}, K_{1/2})^* + P(K_{1/2})J_j + P^*(K_{1/2})J_j + P(J_{1/2})P(K_{1/2})J_i + P^*(J_{1/2})P(K_{12})J_i\ .$$

As we remarked after (2.10), an invariant ideal is closed under all brackets:

$$(*)\qquad \{E_i(K_{1/2}, J_{1/2}) + E_i(J_{1/2}, K_{1/2})\}\cdot J_{1/2} \subset K_{1/2}\ .$$

We can now establish the rest of (C4), $K_i\cdot J_{1/2} \subset K_{1/2}$. Since $E_i(J_{1/2}, K_{1/2})^* = E_i(K_{1/2}, J_{1/2})$ by P8, we have so far that $\{E_i + E_i^*\}\cdot J_{1/2} \subset K_{1/2}$. Next, we observe $\{P(K_{1/2})J_j + P^*(K_{1/2})J_j\}\cdot J_{1/2} \subset E_j(K_{1/2}, J_{1/2})\cdot(J_j^*\cdot K_{1/2}) - P(K_{1/2})(J_j\cdot J_{1/2}) + E_j(J_{1/2}, J_j^*\cdot K_{1/2})^*\cdot K_{1/2} - J_j^*\cdot P(K_{1/2})J_{1/2}$ (by PI7, 8) $\subset J_j\cdot(J_j\cdot K_{1/2}) - P(K_{1/2})J_{1/2} + J_j^*\cdot K_{1/2} - J_j\cdot P(K_{1/2})J_{1/2} \subset K_{1/2}$ by invariance (2.9) and inner idealness $P(K_{1/2})J_{1/2} \subset J_{1/2}$. Finally, $\{P(J_{1/2})P(K_{1/2})J_i + P^*(J_{1/2})P(K_{1/2})J_i\}\cdot J_{1/2} \subset E_j(J_{1/2}, J_{1/2})\cdot[(P(K_{1/2})J_i)^*\cdot J_{1/2}] - P(J_{1/2})[P(K_{1/2})J_i\cdot J_{1/2}] + E_j(P(K_{1/2})J_i\cdot J_{1/2}, J_{1/2})\cdot J_{1/2} - P(K_{1/2})J_i\cdot P(J_{1/2})J_{1/2}$ (by PI7, 8 again) $\subset J_j\cdot K_{1/2} - P(J_{1/2})K_{1/2} + E_j(K_{1/2}, J_{1/2})\cdot J_{1/2} - K_{1/2}$ (by the previous case) $\subset K_{1/2}$ by invariance, outer idealness, and (*). Thus all 6 pieces of K_i send $J_{1/2}$ into $K_{1/2}$, completing (C4).

Next we check (C5), $P(J_{1/2})K_i \subset K_j$. We have $P(J_{1/2})\{E_i(J_{1/2}, K_{1/2}) + E_i(J_{1/2}, K_{1/2})^*\} = P(J_{1/2})\{E_i(J_{1/2}, K_{1/2}) + E_i(K_{1/2}, J_{1/2})\} \subset E_j(J_{1/2}, \langle K_{1/2}, J_{1/2}, J_{1/2}\rangle_j) - E_j(P(J_{1/2})J_{1/2}, K_{1/2}) + E_j(J_{1/2}, \langle J_{1/2}, J_{1/2}, K_{1/2}\rangle_j) - E_j(P(J_{1/2})K_{1/2}, J_{1/2})$ (by PI5) $\subset E_j(J_{1/2}, K_{1/2}) + E_j(K_{1/2}, J_{1/2}) \subset K_j$ by invariance and outer idealness. We have $P(J_{1/2})[P(K_{1/2})J_1] \subset K_1$ and $P(J_{1/2})[P(K_{1/2})J_0 + (P(K_{1/2})J_0)^*] \subset P(J_{1/2})P(K_{1/2})J_0 + P^*(J_{1/2})P(K_{1/2})J_0 \subset K_0$ by construction. For $P(J_{1/2})[P(J_{1/2})(P(K_{1/2})J_i) + P^*(J_{1/2})P(K_{1/2})J_i]$ we first have $P(J_{1/2})P(J_{1/2})P(K_{1/2})J_i = \{P(\{J_{1/2}J_{1/2}K_{1/2}\}) - P(K_{1/2})P(J_{1/2})P(J_{1/2}) + P(P(J_{1/2})P(J_{1/2})K_{1/2}, K_{1/2}) - L(J_{1/2}, J_{1/2})P(K_{1/2})L(J_{1/2}, J_{1/2})\}J_i$ (by JT4) $\subset P(K_{1/2})J_i - L(J_{1/2}, J_{1/2})P(K_{1/2})J_i \subset P(K_{1/2})J_i + \{P(K_{1/2})L(J_{1/2}, J_{1/2}) - P(\{J_{1/2}J_{1/2}K_{1/2}\}, K_{1/2})\}J_i$ (by JT5) $\subset P(K_{1/2})J_i \subset K_j$. With the *'s we consider the cases $i = 1$, $i = 0$ separately. For $i = 1$, $P(J_{1/2})P^*(J_{1/2})P(K_{1/2})J_1 = P(J_{1/2})P(e)P(J_{1/2})P(K_{1/2})J_1 \subset P(J_{1/2})\{P(\{eJ_{1/2}K_{1/2}\}) - P(K_{1/2})P(J_{1/2})P(e) + P(P(e)P(J_{1/2})K_{1/2}, K_{1/2}) - L(e, J_{1/2})P(K_{1/2})L(J_{1/2}, e)\}J_1 \subset P(J_{1/2})P(E_1(K_{1/2}, J_{1/2})J_1 + P(J_{1/2})P(K_{1/2})J_0 + 0 - P(J_{1/2})L(e, J_{1/2})P(K_{1/2})J_{1/2} \subset P^*(J_{1/2}\cdot E_1(K_{1/2}, J_{1/2})^*)J_1 + P(J_{1/2})P(K_{1/2})J_0 -$

$P(J_{1/2})E_1(K_{1/2}, J_{1/2})$ (by PI11, since $K_{1/2} \lhd J_{1/2}) \subset P^*(K_{1/2})J_1 + P(J_{1/2}) P(K_{1/2})J_0 - P(J_{1/2})E_1(K_{1/2}, J_{1/2})$ (by invariance (2.10)) $\subset K_0$ (using the above relation $P(J_{1/2})E_i \subset E_j$). For $i=0$ we have $P(J_{1/2})P^*(J_{1/2})P(K_{1/2})J_0 = P(J_{1/2})P(J_{1/2})P(e)P(K_{1/2})J_0 \subset \{P(\{J_{1/2}J_{1/2}e\}) - P(e)P(J_{1/2})P(J_{1/2}) + P(P(e)P(J_{1/2}) J_{1/2}, J_{1/2}) - L(e, J_{1/2})P(J_{1/2})L(J_{1/2},e)\}P(K_{1/2})J_0$ (by JT4) $\subset P(J_1)P(K_{1/2})J_0 - P(e)[P(J_{1/2})P(J_{1/2})P(K_{1/2})J_0] + 0 - L(e, J_{1/2})P(J_{1/2})(J_{1/2} \cdot P(K_{1/2})J_0) \subset P^*(J_1^* \cdot K_{1/2})J_0 - P(e)K_1 - L(e, J_{1/2})P(J_{1/2})K_{1/2}$ (by PI11, the above, and (C4)) $\subset P^*(K_{1/2})J_0 - K_1^* - L(e, J_{1/2})K_{1/2} \subset K_1^* - E_1(K_{1/2}, J_{1/2}) \subset K_1$. Finally, we check (C1): $K_i \lhd J_i$. By PI2, 3 and invariance (2.9) we have $E_i(J_{1/2}, K_{1/2}) + E_i(K_{1/2}, J_{1/2})$ is an outer ideal in J_i. $P(K_{1/2})J_j + P^*(K_{1/2})J_j$ is also an outer ideal by invariance and PI10, 11, 12, 13. In the same way $P(J_{1/2})P(K_{1/2})J_i + P^*(J_{1/2})P(K_{1/2})J_i$ is outer, since

$$P(J_i)[P(J_{1/2})P(K_{1/2})J_i] \subset P^*(J_i^* \cdot J_{1/2})P(K_{1/2})J_i \text{ (by PI11)} \subset P^*(J_{1/2})P(K_{1/2})J_i$$

and $P(J_i)P^*(J_{1/2})P(K_{1/2})J_i \subset P(J_i \cdot J_{1/2})P(K_{1/2})J_i$ (by PI10) $\subset P(J_{1/2})P(K_{1/2})J_i$, establishing P-outerness (2.1), while L-outerness (2.2) follows from $L(J_i, J_i)[P(J_{1/2})P(K_{1/2})J_i] \subset P(J_i \cdot (J_i^* \cdot J_{1/2}), J_{1/2})P(K_{1/2})J_i$ (by PI12) $\subset P(J_{1/2})P(K_{1/2})J_i$, and $L(J_i, J_i)[P^*(J_{1/2})P(K_{1/2})J_i] = P^*(J_i^* \cdot (J_i \cdot J_{1/2}), J_{1/2}) P(K_{1/2})J_i$ (by PI13) $\subset P^*(J_{1/2})P(K_{1/2})J_i$. Thus K_i is an outer ideal in J_i. For innerness (2.3′) we need only check the generators $E_i(x_{1/2}, k_{1/2})$, $E_i(x_{1/2}, k_{1/2})^*$, $P(k_{1/2})a_j$, $P^*(k_{1/2})a_j$, $P(x_{1/2})P(k_{1/2})a_i$ and $P^*(x_{1/2})P(k_{1/2})a_i$. Using (1.8) we have $P(P(k_{1/2})a_j)J_i = P(k_{1/2})P(a_j)P(k_{1/2})J_i \subset P(K_{1/2})J_j$, $P(P^*(k_{1/2})a_j)J_i = P^*(k_{1/2})P(a_j)P^*(k_{1/2})J_i \subset P^*(K_{1/2})J_j$, $P(P(x_{1/2})P(k_{1/2})a_i)J_i = P(x_{1/2})P(k_{1/2})P(a_i)P(k_{1/2})P(x_{1/2})J_i \subset P(J_{1/2})P(K_{1/2})J_i$, $P(P^*(x_{1/2})P(k_{1/2})a_i)J_i = P^*(x_{1/2})P(k_{1/2})P(a_i)P(k_{1/2})P^*(x_{1/2}) \subset P^*(J_{1/2})P(K_{1/2})J_i$, while by PI6, $P(E_i(K_{1/2}, J_{1/2}))J_i \subset P(K_{1/2})P^*(J_{1/2})J_i + P^*(J_{1/2})P(K_{1/2})J_i + E_i(K_{1/2}, K_{1/2}) \subset K_i$ and therefore $P(E_i(K_{1/2}, J_{1/2})^*)J_i^* = \{P(E_i(K_{1/2}, J_{1/2}))J_i\}^* \subset K_i^* = K_i$ as well. Thus $K_i \lhd J_i$, all conditions (C1)-(C6) are met, and $K \lhd J$.

If $1/2 \in \Phi$ the cases $i = 1, 0$ are simplified since $P(J_{1/2})K_i = 2P(J_{1/2})K_i = P(J_{1/2}, J_{1/2})K_i = E_j(J_{1/2}, K_i \cdot J_{1/2})$ (by P3 since $K_i^* = K_i$). The case $i = 1/2$ is simplified by $P(K_{1/2})J_j = P(K_{1/2}, K_{1/2})J_j = E_i(K_{1/2}, J_j^* \cdot K_{1/2}) \subset E_i(K_{1/2}, K_{1/2})$ by invariance, hence by P8 $(P(K_{1/2})J_j)^* \subset E_i(_{1/2}, K_{1/2})$ too, and so $P(J_{1/2})(P(K_{1/2})J_i) + P^*(J_{1/2})P(K_{1/2})J_i \subset P(J_{1/2})E_j(K_{1/2}, K_{1/2}) + (P(J_{1/2})E_j(K_{1/2}, K_{1/2}))^* \subset E_i(J_{1/2}, J_j \cdot K_{1/2}) - E_i(P(J_{1/2})K_{1/2}, K_{1/2}) + \{E_i(J_{1/2}, J_j \cdot K_{1/2}) - E_i(P(J_{1/2})K_{1/2}, K_{1/2})\}^*$ (by PI5) $\subset E_i(J_{1/2}, K_{1/2}) + E_i(J_{1/2}, K_{1/2})^*$. □

We can easily describe the global ideal generated by a Peirce space.

2.13. COROLLARY. *The ideal in J generated by a Peirce $J_i(e)$ is*

$$(i = 1) \quad I(J_1) = J_1 \oplus J_{1/2} \oplus P(J_{1/2})J_1$$

$$(i = 0) \quad I(J_0) = J_0 \oplus \{J_0 \cdot J_{1/2} + P(J_{1/2})J_0 \cdot J_{1/2}\} \oplus \{P(J_{1/2})J_0 + P^*(J_{1/2})J_0\}$$

$$\left(i = \frac{1}{2}\right) \quad I(J_{1/2}) = P(J_{1/2})J_1 \oplus J_{1/2} \oplus \{E_1(J_{1/2}, J_{1/2}) + P(J_{1/2})J_0 + P^*(J_{1/2})J_0\}.$$

Proof. In each case $K_i = J_i$ is trivially invariant, so we have the explicit expressions for K given by the Projection Theorem. In case $i = 1$ the $J_{1/2}$-component simplifies by $K_1 \cdot J_{1/2} = e \cdot J_{1/2} = J_{1/2}$. In case $i = 0$ we have $J_0 \cdot (J_0 \cdot J_{1/2}) \subset J_0 \cdot J_{1/2}$ for the $J_{1/2}$-component. In case $i = 1/2$ we have for the J_0-component $E_0(J_{1/2}, J_{1/2}) = P(J_{1/2}, J_{1/2})e \subset P(J_{1/2})J_1$, $P(J_{1/2})[P(J_{1/2})J_0 + P^*(J_{1/2})J_0] \subset P(J_{1/2})J_1$ and for the J_1-component $P(J_{1/2})P(J_{1/2})J_1 + P^*(J_{1/2})P(J_{1/2})J_1 \subset P(J_{1/2})J_0 + P^*(J_{1/2})J_0$. $\square$

When J is simple and $J_i \neq 0$ the ideal $I(J_i)$ must be all of J, leading to

2.14. PROPOSITION. *If J is simple and e a proper tripotent (nonzero and noninvertible) then*

(i) $P(J_{1/2})J_1 = J_0$,

(ii) $P(J_{1/2})J_0 + P^*(J_{1/2})J_0 + E_1(J_{1/2}, J_{1/2}) = J_1$.

If $J_0 \neq 0$ then

(iii) $P(J_{1/2})J_0 + P^*(J_{1/2})J_0 = J_1$, (iv) $J_0 \cdot J_{1/2} + P(J_{1/2})J_0 \cdot J_{1/2} = J_{1/2}$.

In characteristic $\neq 2$ we have

(v) $J_1 = E_1(J_{1/2}, J_{1/2})$, $J_0 = E_0(J_{1/2}, J_{1/2})$.

Proof. $e \neq 0$ implies $J_1 \neq 0$, so $I(J_1) = J$, yielding (i). If $J_{1/2} = 0$ then $J = J_1 \boxplus J_0$ forces either $J = J_1$ (e invertible) or $J = J_0(e = 0)$ by primeness, so we must have $J_{1/2} \neq 0$, and $I(J_{1/2}) = J$ yields (ii). We may well have $J_0 = 0$ with $J_1, J_{1/2} \neq 0$, but if $J_0 \neq 0$ then $I(J_0) = J$ yields (iii), (iv). For characteristic $\neq 2$, note $2P(J_{1/2})J_j = P(J_{1/2}, J_{1/2})J_j = E_i(J_{1/2}, J_j \cdot J_{1/2}) \subset E_i(J_{1/2}, J_{1/2}) = E_i(J_{1/2}, J_{1/2})^*$. $\square$

In case $J_0 = 0$ we can also recover some ideal-building lemmas of Loos.

2.15. COROLLARY [1, pp. 131–132]. *Let e be a tripotent in a Jordan triple system with $J_0(e) = 0$.* (i) *If $K_{1/2}$ is an invariant bracket ideal of $J_{1/2}$ such that*

$$J_1 \cdot K_{1/2} \subset K_{1/2} \quad \langle K_{1/2} J_{1/2} J_{1/2} \rangle_1 + \langle J_{1/2} K_{1/2} J_{1/2} \rangle_1 \subset K_{1/2}$$

then the ideal in J generated by $K_{1/2}$ is $K = K_{1/2} \oplus \{E_1(K_{1/2}, J_{1/2}) + E_1(J_{1/2}, K_{1/2})\}$.

(ii) *If K_1 is an ideal of J_1 such that $L(J_{1/2}, J_{1/2})K_1 \subset K_1$ then the ideal in J generated by K_1 is $K_1 \oplus K_1 \cdot J_{1/2}$.*

Proof. (i) Note that $K_{1/2}$ is an ideal in $J_{1/2}$: Since $P(x_{1/2})y_{1/2} = E_1(x_{1/2}, y_{1/2}) \cdot x_{1/2} = \langle x_{1/2} y_{1/2} x_{1/2} \rangle$ by P1 when $J_0 = 0$, the above conditions guarantees a bracket (hence a product $P(x_{1/2})y_{1/2}$ or $P(x_{1/2}, z_{1/2})y_{1/2}$) falls in $K_{1/2}$ as soon as one factor does. This $K_{1/2}$ is invariant in the sense of (2.9), (2.10) by hypothesis, so by the Projection Theorem $K = K_1 + K_{1/2}$ where $P(K_{1/2})J_0 = P^*(J_{1/2})J_0 = P(J_{1/2})P(J_{1/2})J_1 = P^*(J_{1/2})P(J_{1/2})J_1 = 0$ when $J_0 = 0$, so K_1 reduces to $E_1(J_{1/2}, K_{1/2}) + E_1(K_{1/2}, J_{1/2})$.

(ii) K_1 is invariant since $P(J_{1/2})P(J_{1/2})K_1 = 0$, so by the Projection Theorem $K = K_1 \oplus K_1 \cdot J_{1/2}$. $\square$

Since invariant Peirce ideals correspond to global ideals and simple JTS contain no proper global ideals, the Peirce subsystems contain no proper invariant ideals.

2.16. PROPOSITION. *If e is a tripotent in a simple Jordan triple system J, then then Peirce subsystems J_1, $J_{1/2}$, J_0 contain no proper invariant ideals.* $\square$

We can also recover a result of Loos [1] on alternative triple systems.

2.17. COROLLARY. *If e is an idempotent in a simple Jordan triple system J with $J_0(e) = 0$, then $J_{1/2}(e)$ is simple as an alternative triple system under the bracket.*

Proof. By (2.15) $J_{1/2}$ contains no proper invariant ideals $K_{1/2}$, where the invariant ideal conditions (2.9′–2.10″) reduce to

$$J_1 \cdot K_{1/2} \subset K_{1/2} \quad \langle J_{1/2} J_{1/2} K_{1/2} \rangle_1 + \langle J_{1/2} K_{1/2} J_{1/2} \rangle_1 + \langle K_{1/2} J_{1/2} J_{1/2} \rangle_1 \subset K_{1/2} .$$

We may as well assume $J_{1/2} \neq 0$, so by (2.14) $J_1 = E_1(J_{1/2}, J_{1/2})$. Thus $J_1 \cdot K_{1/2} = E_1(J_{1/2}, J_{1/2}) \cdot K_{1/2} = \langle J_{1/2} J_{1/2} K_{1/2} \rangle_1$, and invariance under J_1 is a consequence of bracket-invariance. Therefore the nonexistence of proper invariant ideals means nonexistence of proper bracket ideals, that is, simplicity as an alternative triple system (note $J_{1/2}$ is not trivial under brackets since $0 \neq J_{1/2} = e \cdot J_{1/2} \subset E_1(J_{1/2}, J_{1/2}) \cdot J_{1/2} = \langle J_{1/2} J_{1/2} J_{1/2} \rangle_1$). $\square$

3. Simplicity theorem.

As in the Jordan algebra case, we will quickly find J_1 inherits simplicity from J, then will use a flipping argument to establish simplicity of J_0. Before flipping we need to consider the case when the flipping process annihilates an ideal $K_0 \lhd J_0$.

3.1. KERNEL LEMMA. *The maximal ideal of J_0 annihilated by $P(J_{1/2})$ is* $\operatorname{Ker} P(J_{1/2}) = \{z_0 \in J_0 \mid P(J_{1/2})z_0 = P(J_{1/2})P(z_0)J_0 = 0\}$. *It is an invariant ideal.*

Proof. Clearly any ideal K_0 annihilated by $P(J_{1/2})$ lies in Ker $P(J_{1/2})$ since $P(K_0)J_0 \subset K_0$. It remains to show $K_0 = \operatorname{Ker} P(J_{1/2})$ is actually an invariant ideal.

K_0 is a linear subspace: it is clearly closed under scalars, and for sums $z_0 + w_0$ note

$$\begin{aligned} P(J_{1/2})P(z_0 + w_0)J_0 &= P(J_{1/2})P(z_0, w_0)J_0 = P(J_{1/2})L(w_0, J_0)z_0 \\ &= \{-L(J_0, w_0)P(J_{1/2}) + P(\{J_0 w_0 J_{1/2}\}, J_{1/2})\}z_0 \text{ (by JT5)} \\ &\subset -L(J_0, J_0)P(J_{1/2})z_0 + P(J_{1/2})z_0 = 0 \,. \end{aligned}$$

K_0 is P-outer, $P(J_0)K_0 \subset K_0$, since $P(J_{1/2})[P(a_0)z_0] = P^*(J_{1/2}\cdot a_0)z_0$ (by PI11) $\subset P^*(J_{1/2})z_0 = 0$ and $P(J_{1/2})[P(P(a_0)z_0)J_0] = P(J_{1/2})P(a_0)P(z_0)P(a_0)J_0 \subset P^*(J_{1/2}\cdot a_0)P(z_0)J_0 \subset P(e)P(J_{1/2})P(z_0)J_0 = 0$. It is L-outer, $L(J_0, J_0)K_0 \subset K_0$, since $P(J_{1/2})[L(a_0, b_0)z_0] \subset P(J_{1/2})z_0 = 0$ by PI14 and $P(J_{1/2})[P(L(a_0, b_0)z_0)J_0] \subset P(J_{1/2})\{P(a_0)P(b_0)P(z_0) + P(z_0)P(b_0)P(a_0) + L(a_0, b_0)P(z_0)L(b_0, a_0) - P(P(a_0)P(b_0)z_0, z_0)\}J_0$ (by JT4) $\subset P^*(J_{1/2}\cdot a_0)P(b_0)P(z_0)J_0 + P(J_{1/2})P(z_0)J_0 + P(J_{1/2})L(a_0, b_0)P(z_0)J_0 - P(J_{1/2})L(J_0, J_0)z_0$ (by PI11) $\subset P((J_{1/2}\cdot a_0)\cdot b_0)P(z_0)J_0 + 0 + P(J_{1/2}, J_{1/2})P(z_0)J_0 - P(J_{1/2}, J_{1/2})z_0$ (by PI10 and PI14) $\subset P(J_{1/2})P(z_0)J_0 + 0 - 0 = 0$.

K_0 is inner, $P(K_0)J_0 \subset K_0$, since $P(J_{1/2})[P(z_0)a_0] = 0$ by hypothesis and $P(J_{1/2})[P(P(z_0)a_0)J_0] = P(J_{1/2})P(z_0)P(a_0)P(z_0)J_0 \subset P(J_{1/2})P(z_0)J_0 = 0$.

K_0 is trivially P-invariant (2.7) and (2.8), $P(J_{1/2})P(J_{1/2})K_0 = P(J_{1/2})P(e)P(J_{1/2})K_0 = 0$. It is L-invariant (2.5), $L(J_{1/2}, J_{1/2})K_0 \subset K_0$, since $P(J_{1/2})[L(x_{1/2}, y_{1/2})z_0] = \{P(\{y_{1/2}x_{1/2}J_{1/2}\}, J_{1/2}) - L(y_{1/2}, x_{1/2})P(J_{1/2})\}z_0$ (by JT5) $= 0$ and

$$\begin{aligned} P(J_{1/2})[P(\{x_{1/2}y_{1/2}z_0\})J_0] &\subset P(J_{1/2})\{P(x_{1/2})P(y_{1/2})P(z_0) + P(z_0)P(y_{1/2})P(x_{1/2}) \\ &\quad + L(x_{1/2}, y_{1/2})P(z_0)L(y_{1/2}, x_{1/2}) - P(P(x_{1/2})P(y_{1/2})z_0, z_0)\}J_0 \text{ (by JT4)} \\ &\subset P(J_{1/2})P(J_{1/2})(P(y_{1/2})P(z_0)J_0) + P(J_{1/2})P(z_0)J_0 \\ &\quad + P(J_{1/2})L(J_{1/2}, J_{1/2})P(z_0)J_0 - P(J_{1/2})L(J_0, J_0)z_0 = 0 \end{aligned}$$

as above. The trickiest part is L-invariance (2.6), $E_0(J_{1/2}, J_0\cdot(K_0\cdot J_{1/2})) \subset K_0$. We first show this is killed by $P(J_{1/2})$. We have

$$\begin{aligned} &P(J_{1/2})[E_0(J_{1/2}, J_0\cdot(K_0\cdot J_{1/2}))] \\ &\quad = P(J_{1/2})\{J_{1/2}(K_0\cdot J_{1/2})J_0\} \text{ (by P4)} = P(J_{1/2})L(J_0, K_0\cdot J_{1/2})J_{1/2} \\ &\quad \subset \{-L(K_0\cdot J_{1/2}, J_0)P(J_{1/2}) + P(\{(K_0\cdot J_{1/2})J_0 J_{1/2}\}, J_{1/2})\}J_{1/2} \text{ (by JT5)} \\ &\quad \subset \{(K_0\cdot J_{1/2})J_0 J_{1/2}\} + L(J_{1/2}, J_{1/2})\{(K_0\cdot J_{1/2})J_0 J_{1/2}\} \end{aligned}$$

where $\{(K_0\cdot J_{1/2})J_0 J_{1/2}\} = E_1(K_0\cdot J_{1/2}, J_0\cdot J_{1/2})$ (by P3) $\subset E_1(K_0\cdot J_{1/2}, J_{1/2}) = E_1(J_{1/2}, K_0\cdot J_{1/2})^*$ (by P8) $= \{J_{1/2}K_0 J_{1/2}\}^*$ (by P3) $\subset (P(J_{1/2})K_0)^* = 0$.

To see $P(J_{1/2})$ also kills $P(E_0)J_0$ we use PI6 to write $P(E_0(x_{1/2}, a_0\cdot(z_0\cdot y_{1/2})))J_0 \subset P(x_{1/2})P^*(a_0\cdot(z_0\cdot y_{1/2}))J_0 + P^*(a_0\cdot(z_0\cdot y_{1/2}))P(x_{1/2})J_0 + E_0(x_{1/2}, P(a_0\cdot(z_0\cdot y_{1/2}))(J_0\cdot x_{1/2}))$. Here $P^*(a_0\cdot(z_0\cdot y_{1/2}))J_0 = P(z_0\cdot y_{1/2})P(a_0)J_0$ (by

PI11) $= P^*(y_{1/2})P(z_0)P(a_0)J_0 \subset P^*(J_{1/2})P(z_0)J_0 = 0$ by PI10, and $P^*(a_0 \cdot (z_0 \cdot y_{1/2}))J_1 = P(a_0)P(z_0 \cdot y_{1/2})J_1 = P(a_0)P(z_0)P^*(y_{1/2})J_1$ (by PI10, 11) $\subset P(a_0)P(z_0)J_0 \subset K_0$ since $K_0 \lhd J_0$, also $P(a_0 \cdot (z_0 \cdot y_{1/2}))(J_0 \cdot x_{1/2}) = a_0 \cdot \{z_0 \cdot P(y_{1/2})(z_0 \cdot (a_0 \cdot J_{1/2}))\}$ (using PI16 twice) $\subset J_0 \cdot (z_0 \cdot J_{1/2})$ so that $E_0(x_{1/2}, P) \subset E_0(J_{1/2}, J_0 \cdot (z_0 \cdot J_{1/2}))$ is killed by $P(J_{1/2})$ by the above. Thus $P(J_{1/2})$ does kill all three pieces of $P(E_0)J_0$, E_0 is contained in K_0, and K_0 is an invariant ideal. □

Next we establish that $L(J_{1/2}, J_{1/2})$ and $P(J_{1/2})P(J_{1/2})$ and $P^*(J_{1/2})P(J_{1/2})$ send an ideal into its "square root" or "fourth root".

3.2. LEMMA. *For any ideal $K_i \lhd J_i (i = 1, 0)$ we have*

$$L(J_{1/2}, J_{1/2})P(K_i)J_i \subset K_i \tag{3.3}$$

$$P(J_{1/2})P(J_{1/2})P(P(K_i)J_i)J_i \subset K_i \tag{3.4}$$

$$\text{if } i = 0,\ P^*(J_{1/2})P(J_{1/2})P(J_0)P(P(K_0)J_0)J_0 \subset K_0\ . \tag{3.5}$$

Proof. (3.3) $L(x_{1/2}, y_{1/2})P(z_i)a_i = -P(z_i)L(y_{1/2}, x_{1/2})a_i + P(\{x_{1/2}y_{1/2}z_i\}, z_i)a_i$ (by JT5) $\in -P(K_i)J_i + P(J_i, K_i)J_i \subset K_i$ since K_i is an ideal.

(3.4) For $w_i \in P(K_i)J_i$ we have $P(x_{1/2})P(y_{1/2})P(w_i)J_i = \{P(\{x_{1/2}y_{1/2}w_i\}) - P(w_i)P(y_{1/2})P(x_{1/2}) - L(x_{1/2}, y_{1/2})P(w_i)L(y_{1/2}, x_{1/2}) + P(P(x_{1/2})P(y_{1/2})w_i, w_i)\}J_i$ (by JT4) $\subset P(K_i)J_i - P(K_i)J_i - L(J_{1/2}, J_{1/2})P(K_i)J_i + P(J_i, K_i)J_i$ (using (3.3) for w_i) $\subset K_i$.

(3.5) $P(x_{1/2})P(e)P(y_{1/2})P(a_0)L_0 \subset P(x_{1/2})[P(\{ey_{1/2}a_0\}) - P(a_0)P(y_{1/2})P(e) - L(e, y_{1/2})P(a_0)L(y_{1/2}, e) + P(P(e)P(y_{1/2})a_0, a_0)]L_0$ (by JT4) $\subset P(J_{1/2})P(J_{1/2})L_0 - 0 - L(e, y_{1/2})P(a_0)\{J_{1/2}eL_0\} + \{J_1L_0J_0\} = P(J_{1/2})P(J_{1/2})L_0$, so if $L_0 = P(P(K_0)J_0)J_0$ we have $P(J_{1/2})P(J_{1/2})L_0 \subset K_0$ by (3.4). □

It is not clear whether (3.5) can be improved to assert $P^*(J_{1/2})P(J_{1/2})P(P(K_0)J_0)J_0 \subset K_0$.

Now we can describe a class of ideals which is guaranteed to be invariant.

3.6 PROPOSITION. *Any strongly semiprime ideal $K_1 \lhd J_1$ is invariant.*

Proof. We first prove that K_1 is L-invariant, i.e., $w_1 = L(x_{1/2}, y_{1/2})z_1 \in K_1$ for all $z_1 \in K_1$. By strong semiprimeness we will have $w_1 \in K_1$ if we can show $P(w_1)J_1 \subset K_1$. But

$$\begin{aligned} P(w_1)J_1 = \{&P(x_{1/2})P(y_{1/2})P(z_1) + P(z_1)P(y_{1/2})P(x_{1/2}) \\ &+ L(x_{1/2}, y_{1/2})P(z_1)L(y_{1/2}, x_{1/2}) - P(P(x_{1/2})P(y_{1/2})z_1, z_1)\}J_1 \text{ (by JT4)} \end{aligned}$$

$$\subset P(x_{1/2})P(y_{1/2})P(z_1)J_1 + P(K_1)J_1 + L(x_{1/2}, y_{1/2})P(K_1)J_1 - \{J_1J_1K_1\}$$
$$\subset P(x_{1/2})P(y_{1/2})P(z_1)J_1 + K_1 \text{ (using (3.3))},$$

so it suffices if all $u_1 = P(x_{1/2})P(y_{1/2})P(z_1)a_1$ fall in K_1. Here again it suffices if $P(u_1)J_1 \subset K_1$, and for this

$$P(u_1)J_1 = P(x_{1/2})P(y_{1/2})P(P(z_1)a_1)P(y_{1/2})P(x_{1/2})J_1$$
$$\subset P(J_{1/2})P(J_{1/2})P(P(K_1)J_1)J_1 \subset K_1 \text{ by (3.4)}.$$

Next we prove K_1 is P-invariant. Let $w_1 = P(x_{1/2})P(y_{1/2})z_1$; to show w_1 falls in K_1 it again suffices by strong semiprimeness if it pushes J_1 into K_1, i.e., if $P(w_1)J_1 = P(x_{1/2})P(y_{1/2})P(z_1)P(y_{1/2})P(x_{1/2})J_1 \subset P(x_{1/2})P(y_{1/2})P(z_1)J_1$ falls into K_1. But again this is in K_1 since it pushes J_1 into K_1, $P(P(x_{1/2})P(y_{1/2})P(z_1)a_1)J_1 \subset P(x_{1/2})P(y_{1/2})P(P(z_1)a_1)J_1 \subset K_1$ by (3.4). □

Because it is such a nuisance to verify the extra invariance needed when $i = 0$, and since we will not need the result, we do not establish the analogous result for $K_0 \lhd J_0$.

3.7. COROLLARY. *Any maxinal ideal $M_1 \lhd J_1$ is invariant.*

Proof. If M_1 is maximal then $\bar{J}_1 = J_1/M_1$ is simple with invertible element $\bar{e}$, hence the Jacobson and small radicals are zero and $\bar{J}_1$ is strongly semiprime (see [1, p. 38]), so M_1 is strongly semiprime in J_1. □

We now have the tools to establish our main result.

3.8. SIMPLICITY THEOREM. *If e is a tripotent in a simple Jordan triple system J, then the Peirce subsystems $J_1(e)$ and $J_0(e)$ are simple.*

Proof. We may as well assume e is proper, else the result is trivial. Then J_1 contains a nonzero tripotent and consequently is not trivial, and it has no proper ideals since any such could be enlarged to a maximal proper ideal $0 < M_1 < J_1$ (Zornifying and avoiding e), which would be invariant by 3.7, whereas by 2.15 J_i contains no proper invariant ideals.

Thus J_1 is simple. We may easily have $J_0 = 0$; we will show that if J_0 is nonzero then it must be simple. First, it is strongly semiprime: any element trivial in J_0 would be trivial in J ($P(z_0)J_0 = 0$ implies $P(z_0)J = 0$), whereas by simplicity and non-quasi-invertibility (thanks to $e \neq 0$) the system J is strongly semiprime (see [1, p. 38] again). In particular, J_0 is not trivial, and we need only show it contains no proper ideals $0 < K_0 < J_0$. Suppose on the contrary that such a K_0 exists. By (ordinary) semiprimeness we have successively $K_0' = P(K_0)K_0 \neq 0$, $K_0'' = P(K_0')K_0' \neq 0$, $K_0''' = P(K_0'')K_0'' \neq 0$. By the Flipping Lemma 2.11 $K_1''' = P(J_{1/2})K_0''' + P^*(J_{1/2})K_0'''$ is an ideal in J_1, so by simplicity of J_1 we have either $K_1''' = 0$ or $K_1''' = J_1$. In the first case K_0''' is an ideal annihilated by $P(J_{1/2})$, hence is contained in the invariant ideal Ker $P(J_{1/2})$ by 3.1; by (2.15) we know J_0 contains no proper invariant ideals, so Ker $P(J_{1/2}) \supset K_0''' > 0$ forces Ker $P(J_{1/2}) = J_0$, hence $P(J_{1/2})J_0 = 0$, contrary to (2.14iii) (assuming $J_0 \neq 0$). Thus the first case $K_1''' = 0$ is impossible.

On the other hand, consider the case $K_1''' = J_1$. Here (by (2.14i)) $J_0 = P(J_{1/2})J_1 = P(J_{1/2})K_1''' = P(J_{1/2})P(J_{1/2})K_0''' + P^*(J_{1/2})P(J_{1/2})K_0'''$ is contained in K_0 by (3.4) and (3.5) (noting $K_0'' = P(P(K_0)K_0)K_0' \subset P(P(K_0)J_0)J_0$ and $K_0''' = P(K_0'')K_0'' \subset P(J_0)(P(K_0')K_0') \subset P(J_0)P(P(K_0)J_0)J_0$ as required by (3.4) and (3.5)). But $J_0 = K_0$ contradicts propriety of K_0.

In either case the existence of a proper K_0 leads to a contradiction so no K_0 exists and J_0 too is simple. $\square$

This settles a question raised by Loos [1, p. 133] whether J_1 is simple in case J is simple and $J_0 = 0$. The result was known when J had d.c.c. on principal inner ideals. Of course, for the case $J_0 = 0$ we would not need the elaborate machinery of Peirce decompositions, since the Peirce relations and invariance are vastly simplified (for example $P(J_{1/2})P(J_{1/2})J_1 = 0$, so P-invariance is automatic).

The analogous simplicity result fails for $J_{1/2}$: $J_{1/2}$ need not inherit simplicity from J, since when $J = M_{p,q}(D)$ is the space of pxq matrices over D relative to $P(x)y = xy^*x$ $(y^* = {}^t\bar{y})$, then the diagonal idempotent $e = e_{11} + \cdots + e_{rr}$ $(1 \leqq r < p \leqq q)$ has $J_{1/2} = J_{10} \boxplus J_{01}$. In the simplest case $p = q = 2$, $r = 1$ we have $J_{1/2} = De_{12} \boxplus De_{21}$. Note, however, that these proper ideals $K_{1/2} = J_{10}$, $L_{1/2} = J_{01}$ are invariant under J_1 and J_0 but not under brackets. It is still an open question whether $J_{1/2}$ is simple as a bracket algebra (it is if $J_0 = 0$), or whether it is always simple or a direct sum of two ideals as a triple system.

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