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THE SPLITTING OF OPERATOR ALGEBRAS

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We say the singly generated C^*-algebra, $C^*(T_1 \oplus T_2)$, splits if $C^*(T_1 \oplus T_2) = C^*(T_1) \oplus C^*(T_2)$. A necessary and sufficient condition is derived for the splitting of $C^*(T_1 \oplus T_2)$ in terms of the topological structure of the primitive ideal space of $C^*(T_1 \oplus T_2)$. In particular, when $C^*(T_1 \oplus T_2)$ is strongly amenable, the necessary and sufficient condition can be simplified and does not depend on the topology of the primitive ideal space of $C^*(T_1 \oplus T_2)$. Several applications of this theorem, such as the cases, among others, where T_1, T_2 are compact operators, and $C^*(T_1)$, $C^*(T_2)$ have only finite-dimensional irreducible representations, are discussed. For the splitting of the W^*-algebra, $W^*(T_1 \oplus T_2)$, two equivalent conditions are derived which are quite different in nature. It is also shown that $W^*(T_1 \oplus T_2)$ splits if either $W^*(\mathrm{Re}\, T_1 \oplus \mathrm{Re}\, T_2)$ or $W^*(\mathrm{Im}\, T_1 \oplus \mathrm{Im}\, T_2)$ splits, but the converse is false. An example is given to show that $W^*(T_1 \oplus T_2)$ splits whereas $C^*(T_1 \oplus T_2)$ does not.

1. Introduction. Let $\mathscr{A}$ be a C^*-algebra. If $\mathscr{A}$ has an identity element and T is in $\mathscr{A}$, $C^*(T)$ will denote the C^*-subalgebra of $\mathscr{A}$ generated by T and the identity element; if $\mathscr{A}$ has no identity element, $C^*(T)$ will denote the C^*-subalgebra of $\mathscr{A}$ generated by T alone. If $\mathscr{B}$ is another C^*-algebra and $\mathscr{A} \oplus \mathscr{B}$ is the C^*-direct sum of $\mathscr{A}$ and $\mathscr{B}$, one can ask the following question: Given $T_1 \oplus T_2$ in $\mathscr{A} \oplus \mathscr{B}$, when does $C^*(T_1 \oplus T_2) = C^*(T_1) \oplus C^*(T_2)$? One always has $C^*(T_1 \oplus T_2) \subseteq C^*(T_1) \oplus C^*(T_2)$, and if equality holds, we say $C^*(T_1 \oplus T_2)$ *splits*. A similar question can be posed in the context of W^*-algebras. Given W^*-algebras $\mathscr{R}$, $\mathscr{S}$ and $T_1 \oplus T_2$ in $\mathscr{R} \oplus \mathscr{S}$, when does $W^*(T_1 \oplus T_2) = W^*(T_1) \oplus W^*(T_2)$ ($W^*(T) =$ the W^*-algebra generated by T)? As in the C^*-algebra case, $W^*(T_1 \oplus T_2)$ is said to *split* if equality holds.

In this paper, necessary and sufficient conditions are derived for the splitting of $C^*(T_1 \oplus T_2)$ and $W^*(T_1 \oplus T_2)$. These results should be compared with theorems in [2], [7], [5], and [6], where the splitting problem for various functors involving the direct sum is treated. Indeed, the results in the present paper can be viewed as "self-adjoint" analogs of the non-self-adjoint situations of this previous work.

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2. The splitting of $C^*(T_1 \oplus T_2)$. If $\mathscr{A}$ is a C^*-algebra, Prim $(\mathscr{A})$ will denote the primitive ideal space of $\mathscr{A}$ equipped with the hull-kernel topology, and $\mathrm{Irr}(\mathscr{A})$ will stand for the set of all irreducible representations of $\mathscr{A}$. If E is a central projection in $\mathscr{A}$ and π is a representation of $\mathscr{A}$, π_E is the representation of $\mathscr{A}$ defined by $\pi_E(T) = \pi(TE)$ $(T \in A)$.

We denote by $\mathscr{M}(\mathscr{A})$ the multiplier algebra of $\mathscr{A}$; $\mathscr{M}(\mathscr{A})$ can be characterized as the largest C^*-subalgebra of $\mathscr{A}^{**}$, the enveloping von Neumann algebra of $\mathscr{A}$, which contains $\mathscr{A}$ as a closed, two-sided ideal. If π is in $\mathrm{Irr}(\mathscr{A})$, then π' denotes the unique extension of π to an irreducible representation of $\mathscr{M}(\mathscr{A})$ (since $\mathscr{A}$ is a closed, two-sided ideal of $\mathscr{M}(\mathscr{A})$, π' exists for each π in $\mathrm{Irr}(\mathscr{A})$).

We begin by stating a noncommutative C^*-algebra analog of the Silov idempotent theorem ([15], Theorem 8.6). Its proof is obtained from a straightforward application of the Dauns-Hofmann theorem ([10], Theorem 3; [13] Corollary 4.7), and is therefore left to the reader.

PROPOSITION. *Let $\{\Sigma_1, \Sigma_2\}$ be a disconnection of* $\mathrm{Prim}(\mathscr{A})$, $\mathscr{A}$ *a C^*-algebra. Then there exists a unique central projection E of $\mathscr{M}(\mathscr{A})$ such that*

$$\Sigma_1 = \{\ker \pi : \pi \in \mathrm{Irr}(\mathscr{A}), \pi' = \pi'_E\} ,$$
$$\Sigma_2 = \{\ker \pi : \pi \in \mathrm{Irr}(\mathscr{A}), \pi' = \pi'_{I-E}\} .$$

Conversely, any nontrivial central projection E of $\mathscr{M}(\mathscr{A})$ induces a disconnection of $\mathrm{Prim}(\mathscr{A})$ *in this way.*

Now let $\mathscr{A}_i$, $i = 1, 2$, be C^*-algebras and let π_i be a representation of $\mathscr{A}_i$, $i = 1, 2$. We define a representation $\tilde{\pi}_i$ of $\mathscr{A}_1 \oplus \mathscr{A}_2$ by "evaluation at coordinates", i.e.,

$$\tilde{\pi}_i : A_1 \oplus A_2 \longrightarrow \pi_i(A_i) \quad (A_1 \oplus A_2 \in \mathscr{A}_1 \oplus \mathscr{A}_2) .$$

In particular, if $T_1 \oplus T_2$ is a fixed element in $\mathscr{A}_1 \oplus \mathscr{A}_2$ and $\sigma \in \mathrm{Irr}(C^*(T_i))$, $i = 1, 2$, then $\tilde{\sigma}$ is an irreducible representation of $C^*(T_1 \oplus T_2)$. With this in mind, we now state and prove the main theorem of this section.

THEOREM 2.1. *Let $\mathscr{B}_i$, $i = 1, 2$, be C^*-algebras with $T_1 \oplus T_2$ a fixed element in $\mathscr{B}_1 \oplus \mathscr{B}_2$. Then $C^*(T_1 \oplus T_2)$ splits if and only if* *the sets*

$$\Sigma_i = \{\ker \tilde{\sigma}_i\colon \sigma_i \in \mathrm{Irr}(C^*(T_i))\}\ , \quad i = 1, 2\ ,$$

disconnect $\mathrm{Prim}(C^*(T_1 \oplus T_2))$.

Proof. ($\Rightarrow$). Let $\mathscr{A} = C^*(T_1) \oplus C^*(T_2)$, $\mathscr{A}_i = C^*(T_i)$, $i = 1, 2$. Since $\mathscr{A}^{**} = \mathscr{A}_1{}^{**} \oplus \mathscr{A}_2{}^{**}$, there exist orthogonal central projections E_1, E_2 in $\mathscr{A}^{**}$ with $I = E_1 + E_2$, $E_1 = I \oplus 0$, $E_2 = 0 \oplus I$. Thus $E_i\mathscr{A} \subseteq \mathscr{A}$, $i = 1, 2$. Since, for π in $\mathrm{Irr}(\mathscr{A})$, $\pi' = \pi'_{E_1} \Leftrightarrow \pi$ vanishes on $0 \oplus \mathscr{A}_2 \Leftrightarrow \pi = \tilde{\sigma}$ for some σ in $\mathrm{Irr}(\mathscr{A}_1)$, we conclude that $\Sigma_1 = \{\ker \pi\colon \pi \in \mathrm{Irr}(\mathscr{A}), \pi' = \pi'_{E_1}\}$, and similarly $\Sigma_2 = \{\ker \pi\colon \pi \in \mathrm{Irr}(\mathscr{A}), \pi' = \pi'_{1-E_1}\}$. By the previous proposition we have that $\{\Sigma_1, \Sigma_2\}$ disconnects $\mathrm{Prim}(\mathscr{A})$.
($\Leftarrow$). Let $\mathscr{A} = C^*(T_1 \oplus T_2)$ and $\mathscr{A}_i = C^*(T_i)$ for $i = 1, 2$. Due to the above proposition, there exists a central projection E of $\mathscr{M}(\mathscr{A})$ such that

$$\Sigma_1 = \{\ker \pi\colon \pi \in \mathrm{Irr}(\mathscr{A}), \pi' = \pi'_E\} \tag{2.1}$$

$$\Sigma_2 = \{\ker \pi\colon \pi \in \mathrm{Irr}(\mathscr{A})\ , \quad \pi' = \pi'_{1-E}\}\ . \tag{2.2}$$

Let $A_1 \oplus A_2$ be a fixed element in $\mathscr{A}$, and σ in $\mathrm{Irr}(\mathscr{A}_1)$. By (2.1) there exists π in $\mathrm{Irr}(\mathscr{A})$ such that $\ker \pi = \ker \tilde{\sigma}, \pi = \pi'_E$. Thus $1 - E$ is in $\ker \pi'$, and so

$$0 = \pi'((1 - E)(A_1 + A_2)) = \pi((1 - E)(A_1 + A_2))\ .$$

Hence $(1 - E)(A_1 \oplus A_2) \in \ker \pi = \ker \tilde{\sigma}$, i.e.,

$$\begin{aligned} 0 &= \tilde{\sigma}((1 - E)(A_1 \oplus A_2)) \\ &= \sigma([(1 - E)(A_1 \oplus A_2)]_1)\ . \end{aligned}$$

Since σ is arbitrary in $\mathrm{Irr}(\mathscr{A}_1)$ and $\mathrm{Irr}(\mathscr{A}_1)$ separates points of $\mathscr{A}_1$, we conclude that

$$0 = [(1 - E)(A_1 \oplus A_2)]_1\ . \tag{2.3}$$

Similarly

$$0 = [E(A_1 \oplus A_2)]_2\ . \tag{2.4}$$

Thus,

$$E(A_1 \oplus A_2) = [E(A_1 \oplus A_2)]_2 \oplus 0 \tag{2.5}$$

$$(1 - E)(A_1 \oplus A_2) = 0 \oplus [(1 - E)(A_1 \oplus A_2)]_2\ . \tag{2.6}$$

Adding (2.5) and (2.6) yields

$$A_1 \oplus A_2 = [E(A_1 \oplus A_2)]_1 \oplus [(1 - E)(A_1 \oplus A_2)]_2\ .$$

Hence

$$A_1 = [E(A_1 \oplus A_2)]_1 \,, \quad A_2 = [(1 - E)(A_1 \oplus A_2)]_2 \,,$$

whence by (2.5) and (2.6),

$$E(A_1 \oplus A_2) = A_1 \oplus 0 \,, \quad (1 - E)(A_1 \oplus A_2) = 0 \oplus A_2 \,.$$

Since E multiplies $\mathscr{A}$, $A_1 \oplus 0$ and $0 \oplus A_2$ are both in $\mathscr{A}$. It follows that $C^*(T_1 \oplus T_2)$ splits.

Let T be a normal element of a C^*-algebra, $\mathscr{A}$. We identify the spectrum $\Lambda(T)$ with $\mathrm{Prim}(C^*(T))$, if $\mathscr{A}$ has an identity element. It is easy to see that $\Lambda(T_1 \oplus T_2) = \Lambda(T_1) \cup \Lambda(T_2)$ for T_1, T_2 in $\mathscr{A}$. We therefore deduce from Theorem 2.1:

COROLLARY 2.2. *Let T_1 and T_2 be normal elements in a C^*-algebra $\mathscr{A}$. If $\mathscr{A}$ has an identity, then $C^*(T_1 \oplus T_2)$ splits if and only if $\Lambda(T_1) \cap \Lambda(T_2) = \varnothing$. If $\mathscr{A}$ has no identity, then, $C^*(T_1 \oplus T_2)$ splits if and only if $\Lambda(T_1) \cap \Lambda(T_2) = \{0\}$.*

Of particular interest is the case $\mathscr{A}_1 = \mathscr{A}_2 = \mathscr{B}(\mathscr{H})$, where $\mathscr{B}(\mathscr{H})$ denotes the C^*-algebra of all bounded operators on the Hilbert space $\mathscr{H}$. The following results indicate the utility of Theorem 2.1.

COROLLARY 2.3. *Suppose T_1 and T_2 are irreducible operators on $\mathscr{H}$. Then $C^*(T_1 \oplus T_2)$ splits if and only if* $\mathrm{Prim}(C^*(T_1 \oplus T_2))$ *is disconnected.*

Proof. $I \oplus 0$ and $0 \oplus I$ are the only possible nontrivial central projections in $C^*(T_1 \oplus T_2)$. If $\mathrm{Prim}(C^*(T_1 \oplus T_2))$ is disconnected, we hence conclude by the proposition preceeding Theorem 2.1 that $C^*(T_1 \oplus T_2)$ contains $I \oplus 0$, and therefore splits.

Suppose that T_1 and T_2 are isometries on $\mathscr{H}$ with von Neumann-Wold decompositions $T_i = U_i \oplus S_i$, $i = 1, 2$, i.e., U_i is unitary and S_i is a (possibly trivial) unilateral shift. If either S_1 or S_2 is nonzero, it follows from [4] that $C^*(T_1 \oplus T_2)$ is isomorphic to $C^*(S)$, where S denotes the unilateral shift of multiplicity 1. Since S is irreducible, we conclude that $C^*(T_1 \oplus T_2)$ does not splits. Hence we have:

COROLLARY 2.4. *Let T_1 and T_2 be isometries with von Neumann-Wold decompositions $U_i \oplus S_i$, $i = 1, 2$. Then $C^*(T_1 \oplus T_2)$ splits if and only if $S_1 = S_2 = 0$ and $\Lambda(U_1) \cap \Lambda(U_2) = \varnothing$.*

Let T_1 and T_2 be two compact operators acting on a Hilbert space $\mathscr{H}$. If $\mathscr{H}$ is infinite-dimensional, then it is easy to see that the C^*-algebra generated by $T_1 \oplus T_2$ and the identity operator on $\mathscr{H} \oplus \mathscr{H}$ never splits. However, if we consider $C^*(T_1 \oplus T_2)$ in $\mathscr{K}(\mathscr{H} \oplus \mathscr{H})$ (the C^*-algebra of all compact operators on $\mathscr{H} \oplus \mathscr{H}$), then the splitting of $C^*(T_1 \oplus T_2)$ can be characterized as follows:

COROLLARY 2.5. *Let T_1 and T_2 be compact operators on $\mathscr{H}$. Then $C^*(T_1 \oplus T_2)$ (generated as a C^*-subalgebra of $\mathscr{K}(\mathscr{H} \oplus \mathscr{H})$) splits if and only if every minimal projection in $C^*(T_1 \oplus T_2)$ is of the form $P_1 \oplus 0$ or $0 \oplus P_2$, where P_i is a minimal projection in $C^*(T_i)$, $i = 1, 2$.*

Proof. ($\Rightarrow$). Clear.

($\Leftarrow$). Let $\mathscr{A} = C^*(T_1 \oplus T_2)$, $\mathscr{A}_i = C^*(T_i)$, $i = 1, 2$. Let $\mathscr{M}_1$ (resp. $\mathscr{M}_2$) denote the set of minimal projections of $\mathscr{A}$ of the form $P_1 \oplus 0$ (resp. $0 \oplus P_2$), where P_i is a minimal projection in $\mathscr{A}_i$, $i = 1, 2$. Then by hypothesis,

$$\{\text{minimal projections in } \mathscr{A}\} = \mathscr{M}_1 \cup \mathscr{M}_2 . \tag{2.7}$$

Let $\pi \in \mathrm{Irr}(\mathscr{A})$. Then ([1], Theorem 1.4.4) there exists a minimal projection $P = P_\pi \in \mathscr{A}$, a nonzero vector $\xi = \xi_\pi \in P(\mathscr{H} \oplus \mathscr{H})$, and a unitary operator $U = U_\pi \colon [\mathscr{A}\xi] \to \mathscr{H}$ such that

(i) $\pi(P) \neq 0$,

(ii) $\pi(T_1 \oplus T_2) = U(T_1 \oplus T_2)QU^*$, where Q = projection of $\mathscr{H} \oplus \mathscr{H}$ onto $[\mathscr{A}\xi]$.

We denote this by writing $\pi \sim \mathrm{id}_P$. By (2.7), P must be in either $\mathscr{M}_1$ or $\mathscr{M}_2$; suppose $P \in \mathscr{M}_1$, i.e., $P = R \oplus 0$, R a minimal projection in $\mathscr{A}_1$. Then $\xi = (\xi', 0)$, ξ' a nonzero vector in $R(\mathscr{H})$, and so

$$[\mathscr{A}\xi] = [\mathscr{A}_1\xi'] \oplus (0) .$$

Therefore, there exists a unitary $U' \colon [\mathscr{A}_1\xi'] \to \mathscr{H}_\pi$ such that $U \colon (x, 0) \to U'x$, $x \in [\mathscr{A}_1\xi']$. Thus by (ii),

$$\pi(T_1 \oplus T_2) = U'T_1Q'(U')^* , \quad Q' = \text{projection of } \mathscr{H} \text{ onto } [\mathscr{A}_1\xi'] . \tag{2.8}$$

But by ([1], Proposition 1.4.3), the right side of (2.8) defines an irreducible representation σ of $\mathscr{A}_1$. Thus $\pi = \tilde{\sigma}$. If $P \in \mathscr{M}_2$, the same argument shows that $\pi = \tilde{\tau}$ for some irreducible representation τ of $\mathscr{A}_2$. If Σ_1 and Σ_2 are as defined in Theorem 2.1, we conclude that

$$\mathrm{Prim}(\mathscr{A}) = \Sigma_1 \cup \Sigma_2 . \tag{2.9}$$

We now assert that

$$(2.10)\qquad \Sigma_1 = \text{hull}\,\mathcal{M}_2 = \{\ker\pi : \pi\in \text{Irr}(\mathcal{A}),\ \mathcal{M}_2 \subseteq \ker\pi\}\ ,$$

$$(2.11)\qquad \Sigma_2 = \text{hull}\,\mathcal{M}_1 = \{\ker\pi : \pi\in \text{Irr}(\mathcal{A}),\ \mathcal{M}_1 \subseteq \ker\pi\}\ .$$

It is clear from the definition of $\mathcal{M}_2$ that $\Sigma_1 \subseteq$ hull $\mathcal{M}_2$. Suppose $\ker\pi \in \text{hull}(\mathcal{M}_2)$. Now $\pi \sim \text{id}_P$, with $P\in\mathcal{M}_1\cup\mathcal{M}_2$. If $P\in\mathcal{M}_2$, then $\pi(P)=0$, which by (i) is contrary to the choice of P. Thus $P\in\mathcal{M}_1$ whence by the previous reasoning, $\pi\in\Sigma_1$. This verifies (2.10), and (2.11) follows similarly.

Suppose $\ker\pi\in\Sigma_1\cap\Sigma_2$. Then $\mathcal{M}_1\cup\mathcal{M}_2\subseteq\ker\pi$. But $\pi\sim\text{id}_P$, for some $P\in\mathcal{M}_1\cup\mathcal{M}_2$ with $\pi(P)\neq 0$, a contradiction. Thus

$$(2.12)\qquad \Sigma_1\cap\Sigma_2=\varnothing\ .$$

It follows from (2.9)-(2.12) that $\{\Sigma_1, \Sigma_2\}$ disconnects $\text{Prim}(\mathcal{A})$, whence by Theorem 2.1, $C^*(T_1\oplus T_2)$ splits.

Let ρ be natural map from $\mathcal{B}(\mathcal{H})$ onto the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. The following concept is also seen in [12].

DEFINITION. Let T be an element in $\mathcal{B}(\mathcal{H})$. A projection P in $\mathcal{B}(\mathcal{H})$ is *fully n-reducing* for T if $TP=PT$, rank $(P)<\infty$, and $C^*(T)P\cong M_n$, the $n\times n$ matrix algebra. A projection P in $\mathcal{B}(\mathcal{H})$ is *essentially fully n-reducing* for T if $\rho(P)\rho(T)=\rho(T)\rho(P)$, P has infinite rank and nullity, and $\rho(C^*(T))\rho(P)\cong M_n$. We denote the set of all fully (essentially fully) n-reducing projections for T by $R_n(T)(R_n^e(T))$, and let $R(T)=\bigcup_n R_n(T)$, $R^e(T)=\bigcup_n R_n^e(T)$, where n ranges through all positive integers. Each P in $R^e(T)$ (or in $R(T)$) induces an irreducible representation, π_P, of $C^*(T)$ in a natural way as:

$$(2.13)\qquad \pi_P(A)=\rho(A)\rho(P)\quad(\pi_P(A)=AP)\quad \text{for all } A \text{ in } C^*(T)\ .$$

DEFINITION. Let T and S be elements in C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ respectively. T is *algebraically equivalent* to S, if there exists a *-isomorphism φ of $C^*(T)$ onto $C^*(S)$ with $\varphi(T)=S$.

PROPOSITION 2.6. *Let T_i, $i=1,2$, be two operators in $\mathcal{B}(\mathcal{H})$ such that every irreducible representation of $C^*(T_i)$, $i=1,2$, has a finite-demensional representation space. $C^*(T_1\oplus T_2)$, a C^*-subalgebra of $\mathcal{B}(\mathcal{H}\oplus\mathcal{H})$, splits if and only if the following two conditions hold*:

(i) *If an operator in $C^*(T_1)\oplus C^*(T_2)$ is of the form $P_1\oplus 0$ or $0\oplus P_2$, where P_i is in $R(T_i)\cap C^*(T_i)$, $i=1,2$, then it is in $C^*(T_1\oplus T_2)$.*

(ii) *If $P_i\in R^e(T_i)$, $i=1,2$, then $\rho(P_1T_1)$ is not algebraically* *equivalent to* $\rho(P_2T_2)$.

Proof. Let $\mathscr{A}$ be $C^*(T_1 \oplus T_2)$, $\mathscr{A}_i$ be $C^*(T_i)$, $i = 1, 2$, and Σ_i be as in Theorem 2.1, $i = 1, 2$.

($\Leftarrow$) Condition (i) follows from the fact $C^*(T_1) \oplus 0$ and $0 \oplus C^*(T_2)$ are contained in $\mathscr{A}_1 \oplus \mathscr{A}_2 = \mathscr{A}$.

(ii) Let P_i be in $R^e(T_i)$, $i = 1, 2$. If there exists a *-isomorphism φ of $C^*(\rho(T_1P_1))$ onto $C^*(\rho(T_2P_2))$, with $\varphi(\rho(T_1P_1)) = \rho(T_2P_2)$, then the kernels of the two irreducible representations $\tilde{\pi}_1, \tilde{\pi}_2$ of $\mathscr{A}$ induced by P_1, P_2 ($\pi_i = \pi_{P_i}$ as in (2.13)) are equal. Since $\ker \tilde{\pi}_i$ is in Σ_i, $i = 1, 2$, this contradicts the fact that $\Sigma_1 \cap \Sigma_2 = \varnothing$ by Theorem 2.1.

($\Rightarrow$) Any π in $\mathrm{Irr}(\mathscr{A}_1 \oplus \mathscr{A}_2)$ is of the form $\pi = \tilde{\sigma}_i$ for some σ_i in $\mathrm{Irr}(\mathscr{A}_i)$, and hence is finite-dimensional. Since $\mathscr{A}_1 \oplus \mathscr{A}_2$ is CCR, any two irreducible representations π_1, π_2 of $\mathscr{A}_1 \oplus \mathscr{A}_2$ are unitarily equivalent if and only if $\ker \pi_1 = \ker \pi_2$ ([8], 4.3.7). $\mathscr{A}$, a C^*-subalgebra of $\mathscr{A}_1 \oplus \mathscr{A}_2$, is also CCR, and also has the above property. Next, we state a proposition ([8], 11.1.6), and then use the proposition to show that $\mathscr{A}$ splits.

PROPOSITION. *Let* $\mathscr{B}$ *be a* C^**-algebra, and* $\mathscr{B}_1$ *a* C^**-subalgebra of* $\mathscr{B}$. *If* $\mathscr{B}_1$ *satisfies the following two conditions*:

(i) $\pi|_{\mathscr{B}_1}$ *is in* $\mathrm{Irr}(\mathscr{B}_1)$, *if* π *is in* $\mathrm{Irr}(\mathscr{B})$;

(ii) $\pi|_{\mathscr{B}_1}$ *is not unitarily equivalent to* $\pi'|_{\mathscr{B}_1}$, *if* π *is not unitarily equivalent to* π' *in* $\mathrm{Irr}(\mathscr{B})$, *then* $\mathscr{B}_1 = \mathscr{B}$.

Let π be in $\mathrm{Irr}(\mathscr{A}_1 \oplus \mathscr{A}_2)$ and of the form $\pi = \tilde{\sigma}_i$ for some σ_i in $\mathrm{Irr}(\mathscr{A}_i)$. So $\pi(T_1 \oplus T_2) = \tilde{\sigma}_i(T_1 \oplus T_2) = \sigma_i(T_i)$, whence $\pi(\mathscr{A}) = \sigma_i(\mathscr{A}_i)$ on $\mathscr{H}_\pi$, and $\pi|_{\mathscr{A}}$ is irreducible. Let π be an n-dimensional irreducible representation of $C^*(T)$ for some T in $\mathscr{B}(\mathscr{H})$. Theorem 1.1 in [12] implies that either (a) $\exists P \in C^*(T) \cap R(T)$ such that $\pi(P) = 1$ and the restriction of π to $C^*(T)P$ is a *-isomorphism of $C^*(T)P$ onto M_n, or (b) $\exists P \in R^e(T)$ and a *-isomorphism φ of $\rho(C^*(T))\rho(P)$ onto M_n such that $\pi(A) = \varphi(\rho(A)\rho(P))$ $(A \in C^*(T))$.

Suppose π_1, π_2 are two unitarily inequivalent elements in $\mathrm{Irr}(\mathscr{A}_1 \oplus \mathscr{A}_2)$, and $\pi_i = \tilde{\sigma}_i$ for σ_i in $\mathrm{Irr}(\mathscr{A}_{j(i)})$, $i = 1, 2$.

Case 1. $j(1) = j(2)$. We note $\pi_i(T_1 \oplus T_2) = \tilde{\sigma}_i(T_1 \oplus T_2) = \sigma_i(T_{j(i)})$, $i = 1, 2$, and unitary equivalence between $\pi_1|_{\mathscr{A}}$ and $\pi_2|_{\mathscr{A}}$ implies that there exists a *-isomorphism φ of $\pi_1(\mathscr{A})$ onto $\pi_2(\mathscr{A})$ with $\varphi(\sigma_1(T_{j(i)}) = \sigma(\pi_1(T_1 \oplus T_2)) = \pi_2(T_1 \oplus T_2) = \sigma_2(T_{j(2)})$. This φ induces a unitary equivalence between $\sigma_1(\mathscr{A}_{j(1)})$ and $\sigma_2(\mathscr{A}_{j(2)})$. If $A_1 \oplus A_2$ are in $\mathscr{A}_1 \oplus \mathscr{A}_2$, $\varphi(\tilde{\sigma}_1(A_1 \oplus A_2)) = \varphi(\sigma_1(A_{j(1)})) = \sigma_2(A_{j(1)}) = \sigma_2(A_{j(2)}) = \tilde{\sigma}_2(A_1 \oplus A_2)$. The second equality in this equation is due to a property of φ, which is illustrated in the following commutative diagram:

$$\begin{array}{ccc} & \overset{\rho_1}{\nearrow} & M_n \\ \mathscr{A}_{j(1)} & & \downarrow \varphi \\ & \underset{\rho_2}{\searrow} & M_n \end{array}$$

It follows that π_1 and π_2 are unitarily equivalent on $\mathscr{A}_1 \oplus \mathscr{A}_2$, which is a contradiction. Therefore $\pi_1|_{\mathscr{A}}$ is not unitarily equivalent to $\pi_2|_{\mathscr{A}}$.

Case 2. $j(1) \neq j(2)$. Let $j(1) = 1$, $j(2) = 2$. If σ_1 is of form (a) relative to P in $R(T_1) \cap \mathscr{A}_1$ with $P \oplus 0$ in $\mathscr{A}$, then $\pi_1(P \oplus 0) = \tilde{\sigma}_1(P \oplus 0) = \pi_1(P) = I$ and $\pi_2(P \oplus 0) = \tilde{\sigma}_2(P \oplus 0) = \sigma_2(0) = 0$. It follows that $\pi_1|_{\mathscr{A}}$ is not unitarily equivalent to $\pi_2|_{\mathscr{A}}$. Similary $\pi_1|_{\mathscr{A}}$ is not unitarily equivalent to $\pi_2|_{\mathscr{A}}$ if π_2 is of form (a). Suppose both π_1 and π_2 are of form (b), i.e., $\exists P_i \in R^e(T_i)$ and a *-isomorphism φ_i of $\rho(\mathscr{A}_i)\rho(P_i)$ onto M_{n_i} such that $\sigma_i(A) = \varphi_i(\rho(A)\rho(P_i))$, $(A \in \mathscr{A}_i)$, $i = 1, 2$. We note that

$$\pi_1(T_1 \oplus T_2) = \tilde{\sigma}_1(T_1 \oplus T_2) = \tilde{\sigma}_1(T_1) = \varphi_1(\rho(T_1)\rho(P_1))$$
$$\pi_2(T_1 \oplus T_2) = \tilde{\sigma}_2(T_1 \oplus T_2) = \sigma_2(T_2) = \varphi_2(\rho(T_2)\rho(P_2)) \,.$$

Since $\rho(T_1)\rho(P_1)$ and $\rho(T_2)\rho(P_2)$ are not algebraically equivalent, there exists no *-isomorphism of $\varphi_1(\rho(\mathscr{A}_1)\rho(P_1))$ onto $\varphi_2(\rho(\mathscr{A}_2)\rho(P_2))$, which maps $\varphi_1(\rho(T_1)\rho(P_1))$ to $\varphi_2(\rho(T_2)\rho(P_2))$. This implies that there exists no *-isomorphism of $\pi_1(\mathscr{A})$ onto $\pi_2(\mathscr{A})$ which maps $\pi_1(T_1 \oplus T_2)$ to $\pi_2(T_1 \oplus T_2)$. Hence $\pi_1|_{\mathscr{A}}$ is not unitarily equivalent to $\pi_2|_{\mathscr{A}}$.

In the following we use a Stone-Weierstrass theorem for C^*-algebras to obtain a significant improvement of Theorem 2.1 in an important special case.

Recall that a subset $\mathscr{B}$ containing the identity of a unital C^*-algebra $\mathscr{A}$ *separates the pure states of* $\mathscr{A}$ if to each pair ρ_1 and ρ_2 of distinct pure states of $\mathscr{A}$, there corresponds a $B \in \mathscr{B}$ such that $\rho_1(B) \neq \rho_2(B)$.

We fix a unital C^*-algebra $\mathscr{A}$ and elements T_1, T_2 of $\mathscr{A}$. Σ_1 and Σ_2 are defined relative to $C^*(T_1 \oplus T_2)$ as in Theorem 2.1.

LEMMA 2.7. *If* $\Sigma_1 \cap \Sigma_2 = \emptyset$, *then* $C^*(T_1 \oplus T_2)$ *separates the pure states of* $C^*(T_1) \oplus C^*(T_2)$.

Proof. Let $\mathscr{A} = C^*(T_1) \oplus C^*(T_2)$, $\mathscr{B} = C^*(T_1 \oplus T_2)$, $\mathscr{A}_i = C^*(T_i)$, $i = 1, 2$.

Suppose ρ_1 and ρ_2 are pure states of $\mathscr{A}$ such that $\rho_1|_{\mathscr{B}} = \rho_2|_{\mathscr{B}}$. For $i = 1, 2$, there is an irreducible representation π_i of $\mathscr{A}$ and a unit vector $\xi_i \in \mathscr{H}_{\pi_i}$ for which $\rho_i(\cdot) = (\pi_i(\cdot)\xi_i, \xi_i)$. Now π_i is of the form $\tilde{\sigma}$ for $\sigma \in \operatorname{Irr} \mathscr{A}_1 \cup \operatorname{Irr} \mathscr{A}_2$. If $\sigma \in \operatorname{Irr} \mathscr{A}_1$, then

$$\rho_i(A \oplus B) = (\sigma(A)\xi_i, \xi_i) , \quad \forall A \oplus B \in \mathscr{A} ,$$

and so $\rho_i = \tilde{f}$ for some pure state f on $\mathscr{A}_1$. Similarly, $\rho_i = \tilde{g}$ for some pure state g on $\mathscr{A}_2$ if $\sigma \in \operatorname{Irr} \mathscr{A}_2$.

Suppose $\rho_i = \tilde{f}_i$, f_i a pure state on $\mathscr{A}_i$, $i = 1, 2$. We denote by $\mathscr{P}$ the set of all polynomials in two noncommutating variables and for $p \in \mathscr{P}$, we set $p(T_i) = p(T_i, T_i^*)$, $i = 1, 2$. Since $\rho_1|_{\mathscr{B}} = \rho_2|_{\mathscr{B}}$, it follows that

$$f_1(p(T_1)) = f_2(p(T_2)) , \quad \forall p \in \mathscr{P} .$$

Let $\mathscr{H}_i =$ GNS Hilbert space corresponding to f_i, and set $\ker f_i = \{A \in \mathscr{A}_i : f_i(A^*A) = 0\}$. Define the mapping $U: \mathscr{P}(T_1)/\ker f_1 \to \mathscr{P}(T_2)/\ker f_2$ by $U: p(T_1) + \ker f_1 \to p(T_2) + \ker f_2$, $p \in \mathscr{P}$. Then by (1),

$$\begin{aligned} \|p(T_1) + \ker f_1\|^2_{\mathscr{H}_1} &= f_1(p(T_1)^* p(T_1)) \\ &= f_2(p(T_2)^* p(T_2)) \\ &= \|p(T_2) + \ker f_2\|^2_{\mathscr{H}_2} , \quad \forall p \in \mathscr{P} , \end{aligned}$$

and so U extends to a unitary transformation of $\mathscr{H}_1$ onto $\mathscr{H}_2$. Also, if $p, q \in \mathscr{P}$ and π_{f_i} is the GNS representation corresponding to f_i, then

$$\begin{aligned} \pi_{f_2}(p(T_2))U(q(T_1) + \ker f_1) &= \pi_{f_2}(p(T_2))(q(T_2) + \ker f_2) \\ &= p(T_2)q(T_2) + \ker f_2 \\ &= U(p(T_1)q(T_1) + \ker f_1) \\ &= U\pi_{f_1}(p(T_1))(q(T_1) + \ker f_1) . \end{aligned}$$

Since p and q are arbitrary, it follows that $\tilde{\pi}_{f_1}|_{\mathscr{B}}$ is unitarily equivalent to $\tilde{\pi}_{f_2}|_{\mathscr{B}}$, and so $\ker(\tilde{\pi}_{f_1}|_{\mathscr{B}}) = \ker(\tilde{\pi}_{f_2}|_{\mathscr{B}}) \in \Sigma_1 \cap \Sigma_2$, contrary to assumption.

We conclude that either

(a) $\rho_i = \tilde{\sigma}_i$, σ_i a pure state on $\mathscr{A}_1$, $i = 1, 2$,

or

(b) $\rho_i = \tilde{\sigma}_i$, σ_i a pure state on $\mathscr{A}_2$, $i = 1, 2$.

Suppose (a) holds. Let $p, q \in \mathscr{P}$. We have

$$\rho_1(p(T_1) \oplus q(T_2)) = \sigma_1(p(T_1)) , \tag{2.14}$$

$$\rho_2(p(T_1) \oplus q(T_2)) = \sigma_2(p(T_1)) . \tag{2.15}$$

Now $p(T_1) \oplus p(T_2) \in \mathscr{B}$, and so since $\rho_1|_{\mathscr{B}} = \rho_2|_{\mathscr{B}}$,

$$\sigma_1(p(T_1)) = \sigma_2(p(T_1)) . \tag{2.16}$$

Thus by (2.14), (2.15), (2.16), and the arbitrariness of p and q, $\rho_1 = \rho_2$. For case (b), argue similarly.

THEOREM 2.8. *Suppose* $C^*(T_1 \oplus T_2)$ *is strongly amenable (consult* ([11], *definition*, p. 70). *Then* $C^*(T_1 \oplus T_2)$ *splits if and only if* $\Sigma_1 \cap \Sigma_2 = \emptyset$.

Proof. We need only verify the "if" part. By Lemma 2.7, $C^*(T_1 \oplus T_2)$ separates the pure states of $C^*(T_1) \oplus C^*(T_2)$. Thus by Proposition 3.3 in [3], $C^*(T_1 \oplus T_2) = C^*(T_2) \oplus C^*(T_1)$.

COROLLARY 2.9. *Suppose* T_1 *and* T_2 *are* GCR *elements* (i.e., $C^*(T_i)$ *is a* GCR *algebra*, $i = 1, 2$). *Then* $C^*(T_1 \oplus T_2)$ *splits if and only if* $\Sigma_1 \cap \Sigma_2 = \emptyset$.

Proof. Since all GCR algebras are strongly amenable ([11], Theorem 7.9, p. 78), this corollary is evident from the above theorem.

REMARK 2.10. Theorem 2.8 (and hence Corollary 2.9) also holds in the nonunital case. One need only check that there can exist no nonzero pure state of $C^*(T_1) \oplus C^*(T_2)$ which vanishes on $C^*(T_1 \oplus T_2)$, and this follows from the fact that each pure state of $C^*(T_1) \oplus C^*(T_2)$ is "evaluation at coordinates" of a pure state of either $C^*(T_1)$ or $C^*(T_2)$ (see the beginning of the proof of Lemma 2.7).

3. The splitting of $W^*(T_1 \oplus T_2)$.

In this section necessary and sufficient conditions for the splitting of $W^*(T_1 \oplus T_2)$ are given, where $T_i \in \mathscr{B}(\mathscr{H}_i)$ for Hilbert spaces $\mathscr{H}_i$, $i = 1, 2$.

We begin by considering a slightly more general problem. Let

$$\begin{aligned} \mathscr{S} &= \{T_1 \oplus 0, 0 \oplus T_2, T_1^* \oplus 0, 0 \oplus T_2^*\}\,, \\ \mathscr{T} &= \{T_1 \oplus T_2, T_1^* \oplus T_2^*\}\,, \\ \mathscr{V} &= \mathscr{S} \cup \{I \oplus 0\}\,. \end{aligned}$$

We are interested in deriving conditions under which the W^*-algebras generated by $\mathscr{S}$, $\mathscr{T}$, and $\mathscr{V}$ coincide. By the double commutant theorem, it suffices to consider $\mathscr{S}''$, $\mathscr{T}''$, and $\mathscr{V}''$ (′ denotes commutant), and we easily see that $\mathscr{T}'' \subseteqq \mathscr{S}'' \subseteqq \mathscr{V}''$.

Let $S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$ be in $\mathscr{T}'$, with $S^* = S$, i.e., $S_{ii}^* = S_{ii}$, $i = 1, 2$, and $S_{12} = S_{21}^*$. From

$$S \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} S\,,$$

it follows that

$$\begin{pmatrix} S_{11}\ T_1 & S_{12}\ T_2 \\ S_{22}\ T_1 & S_{22}\ T_2 \end{pmatrix} = \begin{pmatrix} T_1\ S_{11} & T_1\ S_{12} \\ T_2\ S_{21} & T_2\ S_{22} \end{pmatrix}.$$

Thus we have

$$\begin{aligned} S_{ii}T_i &= T_iS_{ii},\ i = 1, 2 \\ S_{12}T_2 &= T_1S_{12}\,, \\ S_{12}^*T_1 &= T_2S_{12}^*\,. \end{aligned} \tag{3.1}$$

Similarly from

$$S\begin{pmatrix} T_1^* & 0 \\ 0 & T_2^* \end{pmatrix} = \begin{pmatrix} T_1^* & 0 \\ 0 & T_2^* \end{pmatrix} S, \text{ we obtain}$$

$$\begin{aligned} S_{ii}T_i^* &= T_i^*S_{ii},\ i = 1, 2 \\ S_{12}T_2^* &= T_1^*S_{12} \\ S_{12}^*T_1^* &= T_2^*S_{12}^*\,. \end{aligned} \tag{3.1*}$$

Since $(3.1)^*$ is just the "adjoint" version of (3.1), we have the following lemma:

LEMMA 3.1. *Let* $S^* = S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$ *be in* $\mathscr{B}(\mathscr{H}_1 \oplus \mathscr{H}_2)$. *Then* $S \in \mathscr{T}'$ *if and only if*

$$\begin{aligned} S_{ii}T_i &= T_iS_{ii},\ i = 1, 2\,, \\ S_{12}T_2 &= T_1S_{12} \\ S_{12}^*T_1 &= T_2S_{12}^*\,. \end{aligned}$$

Now suppose $S \in \mathscr{S}'$ and $S = S^*$. From

$$S\begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} S \quad\text{and}\quad S\begin{pmatrix} 0 & 0 \\ 0 & T_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & T_2 \end{pmatrix} S\,,$$

we get

$$\begin{aligned} T_iS_{ii} &= S_{ii}T_i,\ i = 1, 2\,, \\ S_{12}^*T_1 &= T_1S_{12} = S_{12}T_2 = T_2S_{12}^* = 0\,. \end{aligned}$$

Similarly from

$$S\begin{pmatrix} T_1^* & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} T_1^* & 0 \\ 0 & 0 \end{pmatrix} S \quad\text{and}\quad S\begin{pmatrix} 0 & 0 \\ 0 & T_2^* \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & T_2^* \end{pmatrix} S\,,$$

we get

$$T_i^* S_{ii} = S_{ii} T_i^*,\ i = 1, 2\ ,$$
$$S_{12}^* T_1^* = T_1^* S_{12} = S_{12} T_2^* = T_2^* S_{12}^* = 0\ .$$

Therefore we have

LEMMA 3.2. *Let* $S^* = S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$ *be in* $\mathscr{B}(\mathscr{H}_1 \oplus \mathscr{H}_2)$. *Then* $S \in \mathscr{S}'$ *if and only if*

$$S_{ii} T_i = T_i S_{ii},\ i = 1, 2\ ,$$
$$S_{12} T_2 = T_1 S_{12} = S_{12}^* T_1 = T_2 S_{12}^* = 0\ .$$

Finally if $S \in \mathscr{V}'$ if follows from

$$S \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} S \text{ that } \begin{pmatrix} S_{11} & S_{12} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} S_{11} & 0 \\ S_{12}^* & 0 \end{pmatrix}, \quad \text{where } S_{12} = 0\ .$$

The following theorem is an immediate consequence of Lemmas 3.1 and 3.2.

THEOREM 3.3. (1). $\mathscr{T}'' = \mathscr{S}''$ *if and only if for any bounded linear operator* S *from* $\mathscr{H}_2$ *into* $\mathscr{H}_1$ *we have* $S\ T_2 = S^* T_1 = 0$ *whenever* $ST_2 = T_1 S$ *and* $S^* T_1 = T_2 S^*$.

(2) $\mathscr{S}'' = \mathscr{V}''$ *if and only if for any bounded linear operator* S *from* $\mathscr{H}_2$ *into* $\mathscr{H}_1$ *we have* $S = 0$ *whenever* $ST_2 = T_1 S = S^* T_1 = T_2 S^* = 0$.

(3) $\mathscr{T}'' = \mathscr{V}''$ *if and only if for any bounded linear operator* S *from* $\mathscr{H}_2$ *into* $\mathscr{H}_1$ *we have* $S = 0$ *whenever* $ST_2 = T_1 S$ *and* $S^* T_1 = T_2 S^*$.

Let $\mathscr{N}$ be a W^*-algebra, $\mathscr{N}_*$ its predual, and let $\mathrm{Rep}_\sigma(\mathscr{N})$ denote the family of all $\sigma(\mathscr{N}, \mathscr{N}_*)$-continuous representations of $\mathscr{N}$. Each point of the positive part of the unit ball of $\mathscr{N}_*$ gives rise to an element of $\mathrm{Rep}_\sigma(\mathscr{N})$ via the Gelfand-Naimark-Segal construction, and therefore $\mathrm{Rep}_\sigma(\mathscr{N})$ separates points in $\mathscr{N}$.

Now, let $T_i \in \mathscr{B}(\mathscr{H}_i)$, $i = 1, 2$, and set $\mathscr{N} = W^*(T_1 \oplus T_2)$, $\mathscr{N}_i = W^*(T_i)$, $i = 1, 2$. For $\pi \in \mathrm{Rep}_\sigma(\mathscr{N}_i)$, defined as in §2, $\tilde{\pi}(T_1 \oplus T_2) = \pi(T_i)$. Then $\tilde{\pi} \in \mathrm{Rep}_\sigma(\mathscr{N})$. There hence exists a central projection $P = P_{\tilde{\pi}} \in \mathscr{N}$ such that $\ker \tilde{\pi} = \mathscr{N} P$. Let $\mathrm{supp}\, \tilde{\pi} = I - P$, and let

$$\Pi_i = \{\mathrm{supp}\, \tilde{\pi} : \pi \in \mathrm{Rep}_\sigma(\mathscr{N}_i)\}\ , \quad i = 1, 2\ .$$

Suppose that

$$(*) \qquad \Pi_1 \perp \Pi_2 \text{ (i.e., } S_1 S_2 = 0\ ,\quad S_i \in \Pi_i\ ,\quad i = 1, 2)\ ,$$

$\sup(\Pi_1 \cup \Pi_2) \equiv \sup\{P: P \in \Pi_1 \cup \Pi_2\} = I =$ identity on $\mathscr{H}_1 \oplus \mathscr{H}_2$. Let $P_i = \sup\{P: P \in \Pi_i\}$, $i = 1, 2$. P_i is a central projection in $\mathscr{N}$, and by (*), $P_1 \perp P_2$, $P_1 + P_2 = I$. Let $Q = P_1$, so that $I - Q = P_2$. Let $Q = Q_1 \oplus Q_2$.

Since $Q = P_1$, $P_1 \perp P_2$, and $\pi(Q_2) = \tilde{\pi}(Q) = \tilde{\pi}(P_1) = 0$ for all π in $\mathrm{Rep}_\sigma(\mathscr{N}_2)$, we conclude that $Q_2 = 0$. Similarly, for all π in $\mathrm{Rep}_\sigma(\mathscr{N}_1,)$ we have

$$\begin{aligned}\pi(I_1 - Q_1) &= \pi(I_1) - \pi(Q_1)\\ &= I - \tilde{\pi}(Q)\\ &= I - \tilde{\pi}(P_1)\\ &= I - I = 0\ .\end{aligned}$$

Hence $I_1 = Q_1$. Therefore $Q = I_1 \oplus 0$, and $W^*(T_1 \oplus T_2)$ splits.

From the preceding discussion and Theorem 3.3, we may hence deduce the following result, which gives spatial and space-free criteria for the splitting of $W^*(T_1 \oplus T_2)$.

THEOREM 3.4. *Let* $T_i \in \mathscr{B}(\mathscr{H}_i)$, $i = 1, 2$. *The following are equivalent*:

(a) $W^*(T_1 \oplus T_2)$ *splits.*

(b) $\Pi_1 \perp \Pi_2$ *and* $\sup(\Pi_1 \cup \Pi_2) = I$.

(c) *For any bounded linear operator* S *from* $\mathscr{H}_2$ *into* $\mathscr{H}_1$, *we have* $S = 0$ *whenever* $ST_2 = T_1S$ *and* $S^*T_1 = T_2S^*$.

Furthermore, $W^*(T_1 \oplus T_2)$ *splits if either* $W^*(\mathrm{Re}\, T_1 \oplus \mathrm{Re}\, T_2)$ *or* $W^*(\mathrm{Im}\, T_1 \oplus \mathrm{Im}\, T_2)$ *splits.*

Proof. (a) $\Leftrightarrow$ (b). This follows immediately from the discussion following Theorem 3.3.

(a) $\Leftrightarrow$ (c). Notice first that by the double commutant theorem, $W^*(T_1 \oplus T_2)$ splits precisely when $\mathscr{T}'' = \mathscr{Y}''$. Now apply Theorem 3.3(3).

Suppose $W^*(\mathrm{Re}\, T_1 \oplus \mathrm{Re}\, T_2)$ splits. Let S be a bounded linear operator from H_2 into H_1 such that $ST_2 = T_1S$ and $S^*T_1 = T_2S^*$. Then $T_1S = ST_2^*$, so

$$(\mathrm{Re}\, T_1)S = \frac{T_1 + T_1^*}{2}S = S\frac{T_2 + T_2^*}{2} = S(\mathrm{Re}\, T_2)\ .$$

Thus from Theorem 3.3 (3) and the fact that $W^*(\mathrm{Re}\, T_1 \oplus \mathrm{Re}\, T_2)$ splits, we conclude that $S = 0$. This verifies (c), and so $W^*(T_1 \oplus T_2)$ splits. Argue similarly if $W^*(\mathrm{Im}\, T_1 \oplus \mathrm{Im}\, T_2)$ splits.

REMARK 3.5. We now show by example that $W^*(T_1 \oplus T_2)$ can split with neither $W^*(\mathrm{Re}\, T_1 \oplus \mathrm{Re}\, T_2)$ nor $W^*(\mathrm{Im}\, T_1 \oplus \mathrm{Im}\, T_2)$ splitting.

Let $\alpha_n = 1/n$, $\beta_n = 1/n + i$, $n = 1, 2, 3, \cdots$. Let T_1 (resp. T_2) be the diagonal operator with diagonal $\{\alpha_1, \beta_2, \alpha_3, \beta_4, \cdots\}$ (resp. $\{\beta_1, \alpha_2, \beta_3, \alpha_4, \cdots\}$), acting on the separable Hilbert space H. We have

$$\Lambda(T_1) = \{0, i\} \cup \{\alpha_1, \beta_2, \alpha_3, \cdots\},$$
$$\Lambda(T_2) = \{0, i\} \cup \{\beta_1, \alpha_2, \beta_2, \cdots\}.$$

If A and B are normal operators, it follows from Theorem 3.4 (b) or ([9], Theorem 4.71) that $W^*(A \oplus B)$ splits if and only if a scalar spectral measure of A is orthogonal to a scalar spectral measure of B. Let E_k denote the projection-valued spectral measure of T_k, $k = 1, 2$. If $\{\chi_n\}$ is a countable dense subset of the unit ball of $\mathscr{H}$, then

$$\mu_k(\cdot) = \sum_{n=1}^{\infty} 2^{-n} \| E_k(\cdot)\chi_n \|^2$$

is a scalar spectral measure for T_k, $k = 1, 2$. Since 0 and i are not eigenvalues of T_k, $k = 1, 2$, it follows by ([14], Theorem 12.29) that $\mu_k(\{0, i\}) = 0$, $k = 1, 2$. Since μ_k is supported on $\Lambda(T_k)$, $k = 1, 2$, we conclude that μ_1 and μ_2 are orthogonal, and so $W^*(T_1 \oplus T_2)$ splits. But one easily checks that $\Lambda(\mathrm{Re}\, T_1) = \Lambda(\mathrm{Re}\, T_2)$, $\Lambda(\mathrm{Im}\, T_1) = \Lambda(\mathrm{Im}\, T_2)$ and therefore neither $W^*(\mathrm{Re}\, T_1 \oplus \mathrm{Re}\, T_2)$ nor $W^*(\mathrm{Im}\, T_1 \oplus \mathrm{Im}\, T_2)$ splits. This also provides an example of operators T_1 and T_2 such that $W^*(T_1 \oplus T_2)$ splits, but $C^*(T_1 \oplus T_2)$ does not.

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