HARMONIC MAJORATION OF QUASI-BOUNDED TYPE

SHIGEO SEGAWA

Let $O_{AL}$ (resp. $O_{AS}$) be the class of open Riemann surfaces on which there exists no nonconstant analytic functions $f$ such that $\log^+ |f|$ have harmonic (resp. quasi-bounded harmonic) majorant. It is shown that $O_{AL} = O_{AS}$ for surfaces of finite genus.

1. An analytic function $f$ on an open Riemann surface $R$ is said to be Lindelöfian if $\log^+ |f|$ has a harmonic majorant ([2]). Denote by $AL(R)$ the class of Lindelöfian analytic functions on $R$. Relating to the class $AL(R)$, consider the class $AS(R)$ which consists of analytic functions $f$ on $R$ such that $\log^+ |f|$ has a quasi-bounded harmonic majorant. The class $AS(R)$ is referred to as the Smirnov class ([4] and [4]). Denote by $O_{AL}$ (resp. $O_{AS}$) the class of open Riemann surfaces $R$ such that $AL(R)$ (resp. $AS(R)$) consists of only constant functions. It is known that $O_{G} < O_{AL} < O_{AS}$ (strict inclusions) in general and that $O_{G} = O_{AL}$ for surfaces of finite genus ([2] and [5]). In this paper, it is shown that $O_{G} = O_{AS}$, and therefore $O_{G} = O_{AL} = O_{AS}$, for surfaces of finite genus (cf. [3]).

2. Let $s$ be a superharmonic function on a hyperbolic Riemann surface $R$ and $e$ be a compact subset of $R$ such that $R - e$ is connected. Denote by $\Phi(s, e)$ the class of superharmonic functions $v$ on $R$ such that $v \geq s$ on $e$ except for a polar set. Consider the function $(s, e)(p) = \inf_{v \in \Phi(s, e)} v(p)$ on $R$. Then $(s, e)$ has following properties (see [1]):

**Lemma.** $(s, e)$ is superharmonic on $R$, $(s, e) = H_{R - e}^{s}$ (the solution of the Dirichlet problem with boundary values $s$ on $\partial e$ and $0$ on $\partial R$) on $R - e$, and $(s, e) = s$ on $e$ except for a polar set.

3. **Theorem.** The relation $O_{G} = O_{AS}$ is valid for surfaces of finite genus.

**Proof.** We only have to show that $O_{G} \supset O_{AS}$. Let $F$ be of finite genus not belonging to $O_{G}$ and $S$ be a compact surface such that $F \subset S$. In order to show that $F \in O_{AS}$, we may assume that $K = F^c = S - F$ is totally disconnected. Hence we can decompose $K$ into two compact sets $E$ and $e$ such that $E$ and $e$ have positive capacity. Set $R = E^c = S - E$ and choose a point $x \in e$ which is a regular boundary point for $R - e$. Let $e_n = e \cap \{z \in R; G_R(z, x) \leq n\} (n \in N)$, where $G_R(\cdot, x)$ is the Green's function on $R$ with pole at $x$. Set $h_n =$
Then it is easily seen that \( \{h_n\} \) is increasing and \( h_n \in HB(R - e) \) (the class of bounded harmonic functions on \( R - e \)).

Here and hereafter, the lemma in no. 2 will be used repeatedly without referring to it. Let \( y \) be an arbitrarily fixed point in \( R - e \).

Again, we set \( u_n = (G_R(\cdot, y), e_n)(n \in N) \) and \( u = (G_R(\cdot, y), e) \). Then, since \( \{u_n\} \) is increasing and \( u_n \leq u \), the limit function \( U \) of \( \{u_n\} \) exists, is superharmonic on \( R \), and \( U \leq u \). On the other hand, since \( u_n \leq U \leq G_R(\cdot, y) \) and \( u_n = G_R(\cdot, y) \) on \( e \) except for a polar set for every \( n \in N \), \( U = G_R(\cdot, y) \) on \( e \) except for a polar set by the fact that the union of countably many polar sets is also polar, and a fortiori \( U \geq u \), which implies that \( U = u \). Observe that

\[
    h_n(y) = H^{R-e}_{G_R(\cdot, y)}(y) = G_R(y, x) - G_{R-e}(y, x) \\
    = G_R(x, y) - G_{R-e}(x, y) = H^{R-e}_{G_R(\cdot, y)}(x) \\
    = u_n(x) \uparrow u(x) = (G_R(\cdot, y), e)(x) \quad (n \to \infty) \\
    = G_R(x, y).
\]

Here the regularity of \( x \) is used in the last equality. Consequently we see that the increasing sequence \( \{h_n\} \) with \( h_n \in HB(R - e) \) converges to \( G_R(\cdot, x) \), i.e., \( G_R(\cdot, x) \) is quasi-bounded on \( R - e \).

Consider a meromorphic function \( f \) on \( S \) with a single pole of order \( k \) at \( x \). Then \( \log^+ |f| \leq kG_R(\cdot, x) + C \) for a sufficiently large constant \( C \). Therefore \( f \in AS(R - e) = AS(F) \), i.e., \( F \in O_{AS} \). This completes the proof.

**References**


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**Daido Institute of Technology**

**Daido, Minami, Nagoya 457**

**Japan**