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AMENABLE GROUPS FOR WHICH EVERY TOPOLOGICAL LEFT INVARIANT MEAN IS INVARIANT

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Let G be an amenable locally compact group. It is conjectured that every topological left invariant mean on $L_\infty(G)$ is (topologically) invariant if and only if $G \in [FC]^-$. This conjecture is shown to be true when G is discrete and when G is compactly generated.

1. Introduction. Let G be an amenable locally compact group and let $\mathfrak{L}_t(G)(\mathfrak{R}_t(G))$ be the set of topological left (right) invariant means on $L_\infty(G)$. A natural question to ask is: when does $\mathfrak{L}_t(G) = \mathfrak{R}_t(G)$? Obviously, $\mathfrak{L}_t(G) = \mathfrak{R}_t(G)$ if G is compact or abelian. The results of this paper strongly support the conjecture that $\mathfrak{L}_t(G) = \mathfrak{R}_t(G)$ if and only if $G \in [FC]^-$, the class of those locally compact groups each of whose conjugacy classes is relatively compact. Theorem 3.2 (Theorem 4.4) establishes this conjecture when G is discrete (compactly generated).

The present writer's interest in the above question arose from his inability to prove [1, Theorem 7]. The latter result asserts that if G is an exponentially bounded discrete group, then $\mathfrak{L}_t(G) = \mathfrak{R}_t(G)$. This result is false. (See (3.3).)

I am indebted to Dr F. W. Ponting for help in translating portions of [1].

2. Preliminaries. The cardinality of a set A is denoted $|A|$. Let G be a group. The identity of G will be denoted by e, and if $x \in G$, then $C_x = \{yxy^{-1}: y \in G\}$ is the conjugacy class of x in G. If $a, x \in G$, then

$$C(x) = \{y \in G: xy = yx\}\ , \quad C_a(x) = \{y \in G: yxy^{-1} = a\}\ .$$

Now let G be a locally compact group. The family of compact subsets of G is denoted by $\mathscr{C}(G)$ and the family of compact neighborhoods of e in G is denoted by $\mathscr{C}_e(G)$. The algebra of continuous, bounded, complex-valued functions on G is denoted by $C(G)$. Throughout the paper, λ will be a left Haar measure on G. The group G is called an $[FC]^-$ group if C_x is relatively compact for all $x \in G$. The class of discrete $[FC]^-$ groups is denoted by $[FC]$. The group G is called an $[IN]$ group if there exists $D \in \mathscr{C}_e(G)$ such that $xD = Dx$ for all $x \in G$. (For information about the classes $[FC]^-$ and $[IN]$, see [4].)

Let G be a locally compact group. For $\phi \in L_\infty(G)(=L_1(G)^*)$ and $\mu \in L_1(G)$, define $\phi\mu$, $\mu\phi \in L_\infty(G)$ by setting

$$\phi\mu(\nu) = \phi(\mu*\nu) , \quad \mu\phi(\nu) = \phi(\nu*\mu) \quad (\nu \in L_1(G)) .$$

Let $P(G)$ be the set of probability measures in $L_1(G)$. A mean M on $L_\infty(G)$ is said to be a topological left (right) invariant mean if

$$M(\phi\mu) = M(\phi) \quad (M(\mu\phi) = M(\phi))$$

for all $\phi \in L_\infty(G)$ and all $\mu \in P(G)$. The set of topological left (right) invariant means on G is denoted by $\mathfrak{L}_t(G)(\mathfrak{R}_t(G))$. A mean M on $L_\infty(G)$ is said to be a topological invariant mean if $M \in \mathfrak{L}_t(G) \cap \mathfrak{R}_t(G)$. The group G is amenable if and only if $\mathfrak{L}_t(G)(\mathfrak{R}_t(G))$ is not empty. If G is discrete, then $\mathfrak{L}_t(G)(\mathfrak{R}_t(G))$ coincides with $\mathfrak{L}(G)(\mathfrak{R}(G))$, the set of left (right) invariant means on $\ell_\infty(G)$. It is a simple consequence of the structure theory of $[FC]^-$ groups that every $[FC]^-$ group is amenable ([7], [5], [6]).

A measurable subset T of G is said to be topologically left (right) thick if

$$\sup_{x \in C} \lambda(C \cap Tx) = \lambda(C) \quad \left(\sup_{x \in C} \lambda(C \cap xT) = \lambda(C)\right)$$

for all $C \in \mathscr{C}(G)$. The subset T is topologically left (right) thick if and only if there exists $M \in \mathfrak{L}_t(G)(M \in \mathfrak{R}_t(G))$ such that $M(\chi_T) = 1$. (See [2, Theorem 7.8] and [12].) If G is discrete, then T is topologically left thick if and only if, for every finite subset F of G, there exists $x_F \in G$ such that $Fx_F \subset T$. In this case, T is said to be left thick ([10]).

3. The discrete case.

Lemma 3.1. *Let G be an amenable discrete group which is not an $[FC]$ group. Then $\mathfrak{L}(G) \neq \mathfrak{R}(G)$.*

Proof. The result will follow once we have constructed a left thick subset T of G which is not right thick: for then any left invariant mean M on G for which $M(\chi_T) = 1$ will not be right invariant.

To this end, let α be the smallest ordinal of cardinality $|G|$, and let $\{F_\beta : \beta \in \alpha\}$ be an enumeration of the family of finite subsets of G. Since $G \notin [FC]$, we can find $z \in G$ such that C_z is infinite. Choose z_1, z_2 in G such that $z_1^{-1}z_2 = z$. The lemma will be proved once we have constructed (by transfinite recursion) a subset $\{x_\beta : \beta \in \alpha\}$ of G such that for all $x \in G$ and all $\beta \in \alpha$,

$$(1) \qquad x\{z_1, z_2\} \not\subset \cup \{F_\delta x_\delta : \delta \in \beta\} .$$

(For then we can take $T = \cup\{F_\beta x_\beta : \beta \in \alpha\}$.) Suppose that $\beta \in \alpha$, and that elements $x_\delta(\delta \in \beta)$ have been constructed so that

$$x\{z_1, z_2\} \not\subset \cup \{F_\gamma x_\gamma : \gamma \in \delta\}$$

for all $x \in G$ and for all $\delta \in \beta$. Let $C = \cup\{F_\delta x_\delta : \delta \in \beta\}$. Note that $x\{z_1, z_2\} \not\subset C$ for all $x \in G$.

Let $y \in G$ and suppose that there exists $x \in G$ such that

$$(2) \qquad x\{z_1, z_2\} \subset C \cup F_\beta y .$$

Then either $xz_1 \in C$, $xz_2 \in F_\beta y$ or $xz_2 \in C$, $xz_1 \in F_\beta y$ or $xz_1 \in F_\beta y$, $xz_2 \in F_\beta y$. If $xz_1 \in C$ and $xz_2 \in F_\beta y$, then $z = (xz_1)^{-1}(xz_2) \in C^{-}F_\beta y$. Applying a similar argument to each of the other cases, we see that either $z \in C^{-1}F_\beta y$ or $z^{-1} \in C^{-1}F_\beta y$ or $z \in y^{-1}F_\beta^{-1}F_\beta y$. Let $A = F_\beta^{-1}Cz \cup F_\beta^{-1}Cz^{-1}$. Note that $|A| < |G|$. Let $B = \{u \in G: uzu^{-1} \in F_\beta^{-1}F_\beta\}$. Then $y \in A \cup B$. We now show that $|G \sim B| = |G|$. It is elementary that if $a \in G$ and if $x_a \in G$ is such that $x_a z x_a^{-1} = a$, then $C_a(z) = x_\alpha C(z)$. If follows that $|C_a(z)| = |C(z)|$ for all $a \in C_z$. If $|C(z)| = |G|$ and if $a \in C_z \sim F_\beta^{-1}F_\beta$, then $|G \sim B| \geqq |C_a(z)| = |G|$, and so $|G \sim B| = |G|$. If, on the other hand, $|C(z)| < |G|$, then $|B| \leqq |F_\beta^{-1}F_\beta|\,|C(z)| < |G|$, and again $|G \sim B| = |G|$.

Since $|A| < |G|$ and $|G \sim B| = |G|$, we can find $x_\beta \in G \sim (A \cup B)$. As $A \cup B$ is the set of elements y for which there exists x satisfying (2), it follows that $x\{z_1, z_2\} \not\subset C \cup F_\beta x_\beta$ for all $x \in G$. This completes the construction of $\{x_\beta : \beta \in \alpha\}$ and hence the proof of the lemma.

THEOREM 3.2. *Let G be an amenable discrete group. Then $\mathfrak{L}(G) = \mathfrak{R}(G)$ if and only if $G \in [FC]$.*

Proof. By (3.1), if $\mathfrak{L}(G) = \mathfrak{R}(G)$, then $G \in [FC]$. Conversely, suppose that $G \in [FC]$. We could appeal to the result mentioned in (4.5), but the following easy proof is available.

Let $M \in \mathfrak{L}(G)$, $x \in G$ and $E \subset G$. Since C_x is finite, we can find $x_1, \cdots, x_n$ in G such that G is the disjoint union of the sets $x_r C(x)$. We can write $E = \bigcup_{r=1}^n x_r E_r$ where $E_r \subset C(x)$ for all r. Then

$$M(Ex) = \sum_1^n M(x_r E_r x) = \sum_1^n M(x_r x E_r) = \sum_1^n M(x_r E_r) = M(E) ,$$

and $M \in \mathfrak{R}(G)$. It now follows that $\mathfrak{L}(G) = \mathfrak{R}(G)$.

NOTE 3.3. Contrary to the assertion of [1, Theorem 7], there are exponentially bounded groups G for which $\mathfrak{L}(G) \neq \mathfrak{R}(G)$. An example of such a group is the (nilpotent) discrete group of upper triangular, real, 3×3 matrices with diagonal entries equal to 1. (The latter group does not belong to $[FC]$.)

4. The nondiscrete case. We require three preliminary results.

LEMMA 4.1. *Let $G \in [IN]$ be such that for each $C \in \mathscr{C}(G)$, we have*

$$\sup_{D \in \mathscr{C}(G)} \left[\inf_{x \in G} \lambda(xCx^{-1} \cap D) \right] = \lambda(C) . \tag{1}$$

Then the set $\cup \{xCx^{-1} : x \in G\}$ is relatively compact for each $C \in \mathscr{C}(G)$.

Proof. Let U be an open, relatively compact subset of G. Approximating U by compact subsets and using the equation (1), the fact that G is unimodular, and the inner regularity of λ, we see that (1) is valid when C is replaced by U.

The desired result will follow once it has been shown that there exists $D_0 \in \mathscr{C}(G)$ such that $xUx^{-1} \subset D_0$ for all $x \in G$. Let N be a compact, invariant neighborhood of e. Since $\bar{U}$ is compact, we can find $x_1, \cdots, x_r$ in U such that

$$U \subset \bigcup_{i=1}^{r} x_i N . \tag{2}$$

Then $k = \min_i \lambda(U \cap x_i N)$ is positive. Find $E \in \mathscr{C}(G)$ such that for all $x \in G$,

$$\lambda(U \cap x^{-1}Ex) = \lambda(xUx^{-1} \cap E) > \lambda(U) - k . \tag{3}$$

Let $x_0 \in G$. By (2) and (3), we can find, for each i, an element $n_i \in N$ such that $x_i n_i \in x_0^{-1}Ex_0$. So

$$x_i N \subset x_i n_i N^{-1} N \subset x_0^{-1} E x_0 N^{-1} N = x_0^{-1}(EN^{-1}N)x_0 ,$$

and it follows that $x_0 U x_0^{-1} \subset EN^{-1}N$. Now take $D_0 = EN^{-1}N$.

LEMMA 4.2. *Let G be an amenable, compactly generated, locally compact group for which $\mathfrak{L}_t(G) = \mathfrak{R}_t(G)$. Then $G \in [IN]$.*

Proof. Assume that $\mathfrak{L}_t(G) = \mathfrak{R}_t(G)$, and that G is not an $[IN]$ group. By [11, Theorem 1.8], we have

$$\inf_{x \in G} \lambda(N \cap x^{-1}Nx) = 0$$

for all $N \in \mathscr{C}_e(G)$. It easily follows that

$$\inf_{x \in G} \lambda(N \cap x^{-1}Mx) = 0 \tag{1}$$

for all $N, M \in \mathscr{C}(G)$.

Let $C \in \mathscr{C}_e(G)$ be such that $G = \bigcup_{n=1}^{\infty} C^n$, and let $\varepsilon = (1/2)\lambda(C)$. Using (1), we can find, for each n, an element $x_n \in G$ such that

$$\lambda(C^{-1}C \cap x_n^{-1}C^{-n}C^n x_n) < \varepsilon 2^{-n} . \tag{2}$$

Let $T = \bigcup_{n=1}^{\infty} C^n x_n$. It is obvious that T is topologically left thick in G. The lemma will be established (by contradiction) once we have shown that T is not topologically right thick.

Let $x \in G$, and, for each n, let $C_n = xC \cap C^n x_n$. Let $c_n \in C_n$. Then

$$\lambda(C_n) = \lambda(c_n^{-1}C_n) \leqq \lambda(C^{-1}C \cap x_n^{-1}C^{-n}C^n x_n) < \varepsilon 2^{-n} ,$$

using (2). It follows that $\lambda(xC \cap T) < \varepsilon \sum_1^{\infty} 2^{-n} = \varepsilon$, and so

$$\lambda(xC \cap T) \leqq \frac{1}{2}\lambda(C) .$$

So T is not topologically right thick.

LEMMA 4.3. *Let G be an amenable, compactly generated, locally compact group for which $\mathfrak{L}_t(G) = \mathfrak{R}_t(G)$. Then*

$$\sup_{D \in \mathscr{C}(G)} \left[\inf_{x \in G} \lambda(xCx^{-1} \cap D)\right] = \lambda(C)$$

for all $C \in \mathscr{C}(G)$.

Proof. Suppose that $C_0 \in \mathscr{C}(G)$ is such that for some $\varepsilon > 0$,

$$\sup_{D \in \mathscr{C}(G)} \left[\inf_{x \in G} \lambda(xC_0x^{-1} \cap D)\right] \leqq \lambda(C_0) - \varepsilon . \tag{1}$$

By (4.2), $G \in [IN]$, and hence is unimodular. It follows that (1) remains valid when C_0 is replaced by any larger compact subset of G. This fact will be used in the remainder of the proof.

Let N be a compact, invariant neighborhood of e and let $C \in \mathscr{C}(G)$ be such that $G = \bigcup_{n=1}^{\infty} C^n$ and $C_0 \cup N \subset C$. We can suppose that $\lambda(N) \geqq \varepsilon$.

We now claim that if $D \in \mathscr{C}(G)$, and $\eta < \varepsilon$, then the set A, where

$$A = \{x \in G \colon \lambda(xCx^{-1} \cap D) \leqq \lambda(C) - \eta\} ,$$

is not relatively compact. For if $\bar{A} \in \mathscr{C}(G)$, and if $E = \bar{A}C(\bar{A})^{-1} \cup D$, then for all $x \in G$, we have $\lambda(xCx^{-1} \cap E) \geqq \lambda(C) - \eta > \lambda(C) - \varepsilon$, and the fact that (1) is valid, with C_0 replaced by C, is contradicted.

We now construct by induction a sequence $\{x_n\}$ in G such that for each $x \in G$ and each positive integer n, we have

$$\lambda\Big(xC \cap \Big(\bigcup_{r=1}^{n} C^r x_r\Big)\Big) \leqq \Big(\lambda(C) - \frac{1}{2}\varepsilon\Big) . \tag{2}$$

Let m be a positive integer and assume that $x_1, \cdots, x_{m-1}$ have been constructed such that (2) is valid for $1 \leqq n \leqq m-1$. Let $D = \bigcup_{r=1}^{m-1} C^r x_r$. Choose x_m such that:

(i) $x_m \notin C^{-m}DC^{-1}C$;

(ii) $\lambda(x_m C x_m^{-1} \cap NC^{-m}C^m) \leqq (\lambda(C) - (1/2)\varepsilon)$.

Let $x \in G$. We cannot have both of the sets $xC \cap D$ and $xC \cap C^m x_m$ not empty: for if this were so, then $DC^{-1} \cap C^m x_m C^{-1} \neq \emptyset$, and (i) is contradicted. So if $xC \cap D \neq \emptyset$, then (2) is trivially true with $n = m$.

Suppose then that $xC \cap D = \emptyset$, and set $E = xC \cap C^m x_m$. To complete the induction step, we show that

$$\lambda(E) \leqq \left(\lambda(C) - \frac{1}{2}\varepsilon\right) . \tag{3}$$

Two cases have to be considered. Suppose firstly that $xN \cap E = \emptyset$. Then

$$\lambda(E) \leqq \lambda(xC \sim xN) \leqq \lambda(C) - \varepsilon < \left(\lambda(C) - \frac{1}{2}\varepsilon\right)$$

and (3) is established. Now suppose that $xN \cap E \neq \emptyset$, and let $u \in N$ be such that $xu \in E$. Then

$$(xu)^{-1}E \subset u^{-1}C \cap x_m^{-1}C^{-m}C^m x_m ,$$

and since $Nx_m^{-1} = x_m^{-1}N$, it follows that

$$\lambda(E) \leqq \lambda(C \cap ux_m^{-1}C^{-m}C^m x_m) \leqq \lambda(x_m C x_m^{-1} \cap NC^{-m}C^m) .$$

The inequality (3) now follows using (ii).

Now let $T = \bigcup_{n=1}^{\infty} C^n x_n$. The set T is obviously topologically left thick in G. However, by (2), $\lambda(xC \cap T) \leqq \lambda(C) - 1/2\varepsilon$ for all $x \in G$, and so T is not topologically right thick. It follows that $\mathfrak{L}_t(G) \neq \mathfrak{R}_t(G)$, and the resultant contradiction establishes the lemma.

THEOREM 4.4. *Let G be an amenable, compactly generated, locally compact group. Then $\mathfrak{L}_t(G) = \mathfrak{R}_t(G)$ if and only if $G \in [FC]^-$.*

Proof. Assume that $\mathfrak{L}_t(G) = \mathfrak{R}_t(G)$. By (4.3) and (4.1), we have $G \in [FC]^-$. Conversely, assume that $G \in [FC]^-$. Let H be the closure of the commutator subgroup of G. By [4, Theorem 3.20], the group H is compact. Let μ be the normalized Haar measure of H. In the obvious way, μ will be regarded as a probability measure on G. Note that if $M \in \mathfrak{L}_t(G)(\mathfrak{R}_t(G))$ then $M(\phi\mu) = M(\phi)(M(\mu\phi) = M(\phi))$ for all $\phi \in L_\infty(G)$. Note also that $\delta_h * \mu = \mu = \mu * \delta_h$ for all $h \in H$.

Define

$$A = \{\phi \in C(G): \phi(xh) = \phi(x) \text{ for all } x \in G \text{ and all } h \in H\} .$$

If $\phi \in A$ and $x, y \in G$, then, since G/H is abelian, we have $xy = yxh_0$ for some $h_0 \in H$, and it follows that $\phi(xy) = \phi(yx)$, and hence that $\nu\phi = \phi\nu$ for all $\nu \in P(G)$.

Now let $M \in \mathfrak{R}_t(G)$, ν_0, $\nu \in P(G)$ and $\psi \in L_\infty(G)$. Then if $x \in G$ and $h \in H$, we have

$$(\mu\nu_0)\psi(xh) = \mu([(\nu_0\psi)x]|_H h) = (\mu\nu_0)\psi(x) ,$$

and so $(\mu\nu_0)\psi \in A$. Now if $\nu \in P(G)$, we obtain

$$M(\psi) = M(\nu(\mu\nu_0)\psi) = M([(\mu\nu_0)\psi]\nu) = M(\psi\nu) ,$$

and $M \in \mathfrak{L}_t(G)$. It easily follows that $\mathfrak{L}_t(G) = \mathfrak{R}_t(G)$.

NOTE 4.5. The two theorems of this paper suggest the following conjecture: if G is an amenable locally compact group, then $\mathfrak{L}_t(G) = \mathfrak{R}_t(G)$ if and only if $G \in [FC]^-$. More evidence in support of this conjecture is found in the following result ([3], [8], [9]): if $G \in [SIN] \cap [FC]^-$, then $\mathfrak{L}_t(G) = \mathfrak{R}_t(G)$.

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