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COHOMOLOGY OVER BANACH CROSSED PRODUCTS. APPLICATION TO BOUNDED DERIVATIONS AND CROSSED HOMOMORPHISMS

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The purpose of this work is to study the structure of bounded derivations and crossed homomorphisms of the Banach crossed product $\mathfrak{A}=L^1(G, A)$ of a Banach-*-algebra A acted upon by a locally compact group G. As bounded derivations and crossed homomorphisms are related to 1-cocycles, we first define and study cohomology over $\mathfrak{A}$, generalizing cohomology over group algebras. Then, if G is amenable and A is a C^*-algebra, or the dual of a Banach space, we show that a bounded derivation (resp. a crossed homomorphism) on $\mathfrak{A}$ is equivalent to some couple of a bounded derivation (resp. a crossed homomorphism) from A to $M_1(G, A)$ and a bounded measure on A with value in the centralizers of A (resp. an element of $\mathfrak{A}$).

1. Introduction. Crossed products of Banach algebras and locally compact groups are interesting objects from a mathematical point of view because they are generalizations of group algebras, from a physical point of view because they are useful tools in describing quantum dynamical systems. Hence it would be interesting to know the structure of their automorphisms and derivations. For a large class of automorphisms, the answer is given in [2]. In this paper, our aim is to begin the study of bounded derivations and crossed homomorphisms of Banach crossed products. For that purpose, cohomology techniques seem to be useful and this is the reason why we will begin with cohomology over Banach crossed products, a generalization of cohomology over group algebras worked out in [15].

Given a locally compact group G acting on a Banach $*$-algebra A, $\mathfrak{A} = L^1(G, A)$ will be the Banach crossed product of these two objects. In paragraph 2, we collect known results about centralizers on A and vector measures, and define several module structures on them in paragraph 3. Paragraph 4 is devoted to the definition of cohomology over $\mathfrak{A}$, while paragraph 5 contains a Riesz representation theorem for the elements of the spaces introduced in the preceding paragraph. In paragraph 6 we extend the cohomology over $\mathfrak{A}$ to its centralizers. Finally paragraph 8 characterizes the structure of derivations and crossed homomorphisms, using the notion of vector means developed in paragraph 7.

2. Notations and preliminary results. In the sequel, (A, G, σ) will denote a Banach dynamical system, that is to say the triplet of a (separable) Banach-$*$-algebra with norm $|\cdot|$ and (countable) approximate unit $\{e_\alpha\}_{\alpha\in I}$ contained in the unit ball, a second countable locally compact Hausdorff (hence Polish, i.e., second countable, metrizable and complete) group G with Haar measure dg, and a representation σ of G into Aut G (the group of continuous and isometric $*$-automorphisms of A), representation continuous in the sense that

$$(1) \qquad (a, g) \in A \times G \longrightarrow \sigma(g)a \in A$$

is continuous.

Once A is given, we call $M_L(A)$ (respectively $M_R(A)$, $M(A)$) the space of left centralizers (resp. right centralizers, centralizers) on A. Let us recall [2], [16], [3] that $M_L(A)$ (resp. $M_R(A)$) is the algebra with unit of continuous linear maps L (resp. R) on A such that $L(ab) = L(a)b$ (resp. $R(ab) = aR(b)$) for any a and b in A, the product being defined by $L_1 \cdot L_2 = L_1 \circ L_2$ (the composition of maps) (resp. $R_1 \cdot R_2 = R_2 \circ R_1$). $M(A)$ is the $*$-algebra with unit of couples (L, R) of (automatically linear and continuous) maps on A such that $aL(b) = R(a)b$ for any a and b in A, the product and $*$-operation being defined according to $(L_1, R_1)\cdot(L_2, R_2) = (L_1\cdot L_2, R_1\cdot R_2) = (L_1\circ L_2, R_2\circ R_1)$ and $(L, R)^* = (R', L')$ where $R'(a) = R(a^*)^*$ and $L'(a) = L(a^*)^*$. If $(L, R) \in M(A)$, then $L \in M_L(A)$ and $R \in M_R(A)$, and these algebras become Banach algebras under the operator norms:

$$||(L, R)|| = ||L|| = ||R|| = \lim_\alpha |L(e_\alpha)| = \lim_\alpha |R(e_\alpha)| \, .$$

Through the correspondence $a \in A \to L_a \in M_L(A)$: $L_a b = ab$, $b \in A$ (resp. $a \in A \to R_a \in M_R(A)$: $R_a b = ba$, $b \in A$), A becomes a closed left ideal (resp. right ideal, $*$-ideal) of $M_L(A)$ (resp. $M_R(A)$, $M(A)$) and $M(A)$ is the idealizer of A in $M_L(A)$ (or $M_R(A)$). Moreover, A is dense in $M_L(A)$ (resp. $M_R(A)$, $M(A)$) for the strong (resp. strong, strict) topology, i.e., the topology defined by the set of semi-norms $||L||_a = |L(a)| = |L\cdot a|$, $a \in A$ (resp. $||R||^a = |R(a)| = |a\cdot R|$; $||(L, R)||_a = ||L||_a$ and $||(L, R)||^a = ||R||^a$). The formulas

$$(2) \qquad \begin{cases} \sigma(g)(L\cdot a) = \sigma(g)L\cdot\sigma(g)a \\ \sigma(g)(a\cdot R) = \sigma(g)a\cdot\sigma(g)R \\ \sigma(g)(L, R) = (\sigma(g)L, \sigma(g)R) \end{cases}$$

allow to extend $\sigma(g)$ as a continuous automorphism of $M_L(A)$, $M_R(A)$ or $M(A)$.

If A is a C^*-algebra, $M_L(A)$ (resp. $M_R(A)$) is isomorphic to the algebra $LM(A)$ (resp. $RM(A)$) of left (resp. right) multipliers on A, i.e., the subalgebra of the enveloping Von Neumann algebra A'' of elements $L \in A''$ (resp. $R \in A''$) such that $La \in A$ (resp. $aR \in A$) and $M(A) = M_L(A) \cap M_R(A)$ is the idealizer of A in A''. Moreover, if we now call $M_L(A, A'')$ (resp. $M_R(A, A'')$) the algebra of continuous linear maps L (resp. R) from A to A'' such that $L(ab) = L(a)b$ (resp. $R(ab) = aR(b)$), then it is possible to prove in the same way that $M_L(A, A'') = M_R(A, A'') = A''$. And here too, $\sigma(g)$ extends to A'' by bitransposition as a normal automorphism.

We will now denote by $\mathfrak{X} = \mathscr{C}_0(G, A)$ the Banach space of continuous functions from G to A "vanishing at infinity" with the uniform norm $\|h\|_\infty = \sup_{g \in G} |h(g)|$, $h \in \mathfrak{X}$: it contains, as a dense set, the subspace $K(G, A)$ of continuous functions from G to A with compact support.

If $X = \mathscr{L}(E, F)$, the continuous operators from a Banach space E to a Banach space F, with norms $\|\cdot\|_E$ and $\|\cdot\|_F$ respectively, $M_1(G, X)$ will be the Banach space of regular Borel measures μ on G with bounded variation $|\mu|$ and norm $\|\mu\|_1 = |\mu|(G) < \infty$. Let us recall [7] that if $B \in B(G)$, the ring of Borel sets in G, the variation $|\mu|$ of μ is the positive scalar measure on G defined by

$$|\mu|(B) = \sup \sum_i \|\mu(B_i)\| \tag{3}$$

where the sup is over all (finite) families of mutually disjoint Borel sets B_i contained in B. Then μ is said with finite variation if $|\mu|(B) < \infty$ for any relatively compact B in $B(G)$ and with bounded variation if $|\mu|(G) < \infty$.

Let now U be a linear mapping from $K(G, E)$ to F. In the usual way, we define

$$\|U\| = \sup_{\|h\|_\infty \leqq 1} \|U(h)\|_F, \quad h \in K(G, E) . \tag{4}$$

It is a norm and

$$\|U(h)\|_F \leqq \|U\| \, \|h\|_\infty \tag{5}$$

so that, if $\|U\| < \infty$, U extends to $\mathscr{C}_0(G, E)$ by continuity. One can notice that $\|U\|$ can also be defined according to

$$\|U\| = \sup \|\sum_i U(h_i)\|_F \tag{4$^{\text{bis}}$}$$

where the sup is over all finite families of functions $h_i \in K(G, E)$ such that support $h_i \cap$ support $h_j = \varnothing$ for any $i \neq j$ and $\|h_i\|_\infty \leqq 1$ (or equivalently $\|\Sigma_i h_i\|_\infty \leqq 1$).

In an analogous way, we can now define [2], [7]

$$|||U||| = \sup \sum_i ||U(h_i)||_F \tag{6}$$

where the sup is taken over the some families as in (4^{bis}). It is a norm and

$$\sum_i ||U(h_i)||_F \leqq |||U|||\,||\sum_i h_i||_\infty \tag{7}$$

for any finite family of functions $h_i \in K(G, E)$ with support $h_i \cap$ support $h_j = \varnothing$ for $i \neq j$, while

$$||U|| \leqq |||U|||;\ ||U(h)||_F \leqq |||U|||\,||h||_\infty \tag{8}$$

so that, if $|||U||| < \infty$, U extends to $\mathscr{C}_0(G, E)$ by continuity.

It is now possible to prove the following theorem:

THEOREM 1. *Let A be a Banach-$*$-algebra and $X = \mathscr{L}(E, F)$, where E and F are two Banach spaces.*

(a) *There exists a one-to-one linear correspondence between $M_1(G, X)$ and the Banach space of linear mappings U from $\mathscr{C}_0(G, E)$ to F such that $|||U||| < \infty$, given by*

$$\mu \longleftrightarrow U_\mu:\ U_\mu(h) = \langle \mu, h\rangle = \int d\mu(g)h(g),\ h \in \mathscr{C}_0(G, E),\ \mu \in M_1(G, X) \tag{9}$$

with

$$|||U_\mu||| = ||\mu||_1\ . \tag{10}$$

(b) *This correspondence induces a one-to-one isometric correspondence between $M_1(G, M_L(A))$* (resp. *$M_1(G, M_R(A))$*) *and the Banach space of A-right linear* (resp. *A-left linear*) *mappings U from $\mathfrak{X}$ to A such that $|||U||| < \infty$.*

(c) *If A is a C^*-algebra, this correspondence induces a one-to-one isomorphic correspondence between $M_1(G, A'')$ and the Banach space of A-right linear* (*or A-left linear*) *mappings U from $\mathfrak{X}$ to A'' such that $|||U||| < \infty$.*

Proof. (a) and (b) come from ([7], § 19 no. 3, Theorem 2) and ([2], Theorem 3.9), while (c) can be proved in the same way as (b) thanks to

$$M_L(A, A'') = M_R(A, A'') = A''\ .$$

If we adopt the notations

$$M_1(G, M_L(A)) = \mathfrak{X}_L^{*,A},\ M_1(G, M_R(A)) = \mathfrak{X}_R^{*,A};\ M_1(G, A'') = \mathfrak{X}^{*,A''} \tag{11}$$

(any time we use A'' without comment, we mean implicitely that A is a C^*-algebra) we can write, for $\mu_L \in \mathfrak{X}_L^{*,A}$, $\mu_R \in \mathfrak{X}_R^{*,A}$, $\mu \in \mathfrak{X}^{*,A''}$, $h \in \mathfrak{X}$

$$(12)\qquad \begin{cases}\langle \mu_L, h\rangle \in A,\ \langle \mu_L, ha\rangle = \langle \mu_L, h\rangle a,\ a \in A\\ \langle \mu, h\rangle \in A'',\ \langle \mu, ha\rangle = \langle \mu, h\rangle a\end{cases}$$

$$(12^{\text{bis}})\qquad \begin{cases}\langle h, \mu_R\rangle \in A,\ \langle ah, \mu_R\rangle = a\langle h, \mu_R\rangle,\ a \in A\\ \langle h, \mu\rangle \in A'',\ \langle ah, \mu\rangle = a\langle h, \mu\rangle\ .\end{cases}$$

Given $g \in G$ and μ_L (resp. μ_R, μ) we define

$$(13)\qquad {}^g\mu_L(B) = \sigma(g)\mu_L(B)\ (\text{resp. } {}^g\mu_R(B) = \sigma(g)\mu_R(B),\ {}^g\mu(B) = \sigma(g)\mu(B)),\ B \in B(G)$$

or, in an equivalent way

$$(13^{\text{bis}})\qquad \begin{cases}\langle {}^g\mu_L, h\rangle = \sigma(g)\langle \mu_L, \sigma(g^{-1})h\rangle\ (\text{resp. } \langle h, {}^g\mu_R\rangle = \sigma(g)\langle \sigma(g^{-1})h, \mu_R\rangle,\\ \langle {}^g\mu, h\rangle = \sigma(g)\langle \mu, \sigma(g^{-1})h\rangle,\ \langle h, {}^g\mu\rangle = \sigma(g)\langle \sigma(g^{-1})h, \mu\rangle)\ .\end{cases}$$

Of course,

$$(14)\qquad \begin{cases}|{}_g\mu_L| = |\mu_L|,\ |{}^g\mu_R| = |\mu_R|,\ |{}^g\mu| = |\mu|\\ \|{}^g\mu_L\|_1 = \|\mu_L\|_1,\ \|{}^g\mu_R\|_1 = \|\mu_R\|_1,\ \|{}^g\mu\|_1 = \|\mu\|_1\ .\end{cases}$$

Then $\mathfrak{X}_L^{*,A}$, $\mathfrak{X}_R^{*,A}$, $\mathfrak{X}^{*,A''}$ become Banach algebras with unit (the unit being δ_e, the Dirac measure at the neutral element e of G) if we define the σ-convolution of measures according to [2], [8]:

$$(15)\qquad \begin{cases}\langle \mu_R *^\sigma \nu_L, h\rangle = \langle \mu_L(u), \langle {}^u\nu_L(v), h(uv)\rangle\rangle,\ \mu_L, \nu_L \in \mathfrak{X}_L^{*,A}\\ \langle \mu *^\sigma \nu, h\rangle = \langle \mu(u), \langle {}^u\nu(v), h(uv)\rangle\rangle,\ \mu, \nu \in \mathfrak{X}^{*,A''}\end{cases}$$

$$(15^{\text{bis}})\qquad \begin{cases}\langle h, \mu_R *^\sigma \nu_R\rangle = \langle\langle h(uv), {}^u\nu_R(v)\rangle, \mu_R(u)\rangle,\ \mu_R, \nu_R \in \mathfrak{X}_R^{*,A}\\ \langle h, \mu *^\sigma \nu\rangle = \langle\langle h(uv), {}^u\nu(v)\rangle, \mu(u)\rangle\ .\end{cases}$$

Through the correspondence

$$(16)\qquad f \in L^1(G, A) \longrightarrow \mu_f\colon \langle \mu_f, \varphi\rangle = \int f(g)\varphi(g)dg,\ \varphi \in K(G)$$

the Banach space $L^1(G, A)$ (for the norm $\|f\|_1 = \int |f(g)|dg$) of functions from G to A, Bochner-integrable with respect to the Haar measure, can be identified with a left ideal (resp. right ideal, subalgebra) of $\mathfrak{X}_L^{*,A}$ (resp. $\mathfrak{X}_R^{*,A}$, $\mathfrak{X}^{*,A''}$) and we have the following formulas, where δ is the modular function of G:

$$(17)\qquad \begin{aligned} f_1 *^\sigma f_2 &= \int f_1(u)\sigma(u)f_2(u^{-1}\cdot)du\\ &= \int f_1(\cdot u)\sigma(\cdot u)f_2(u^{-1})du\\ &= \int \delta(u^{-1})f_1(\cdot u^{-1})\sigma(\cdot u^{-2})f_2(u)du\\ &= \int \delta(u^{-1})f_2(u^{-1})\sigma(u^{-1})f_2(u^{-1}\cdot)du\ .\end{aligned}$$

In the sequel, we will usually omit the subscript σ to denote the σ-convolution. $L^1(G, A)$ may be called the Banach crossed product of G by A [4], [8] and the following theorem can be proved:

THEOREM 2. ([2], *Theorems* 4.10, 4.15, 4.19.)

Let A be a Banach-$*$-algebra, A'' its enveloping Von Neumann algebra when A is a C^*-algebra.

$$\begin{cases} M_L(L^1(G, A)) = \mathfrak{X}_R^{*,A} \\ M_R(L^1(G, A)) = \mathfrak{X}_R^{*,A} \\ M(L^1(G, A)) = \mathfrak{X}^{*,A} \end{cases} \tag{18}$$

where $\mathfrak{X}^{*,A}$ (in general different from $M_1(G, M(A))$ is the idealizer of $L^1(G, A)$ in $\mathfrak{X}_L^{*\ A}$ or $\mathfrak{X}_R^{*,A}$. $L^1(G, A)$, $\mathfrak{X}^{*,A}$ and $\mathfrak{X}^{*,A''}$ are Banach-$*$-algebras if we define

$$\begin{cases} f^*(g) = \delta(g^{-1})\sigma(g)f(g^{-1})^* \\ (\mu_L * f)^* = f^* * \mu_L^*;\ (f * \mu_R)^* = \mu_R^* * f^*,\ (\mu * f)^* = f^* * \mu^* \end{cases} \tag{19}$$

and $L^1(G, A)$ is a $*$-ideal (resp. $*$-subalgebra) of $\mathfrak{X}^{*,A}$ (resp. $\mathfrak{X}^{*,A''}$).

3. Module structures on $\mathfrak{X}$, $\mathfrak{X}_L^{*,A}$, $\mathfrak{X}_R^{*,A}$, $\mathfrak{X}^{*,A}$, $\mathfrak{X}^{*,A''}$. In this paragraph, we are going to define several natural module structures on the various objects we introduced in the preceding one. We first begin with G-module structures.

PROPOSITION 1. *$\mathfrak{X}$ is a Banach-G-module in two different ways corresponding to the two following different actions of G, denoted successively by a $\cdot$ and by a $\circ$:*

$$g \cdot h = h * \delta_{g^{-1}} = h(\cdot g);\ h \cdot g = \sigma(g)\{\delta_{g^{-1}}\} * h = h(g \cdot),\ h \in \mathfrak{X},\ g \in G \tag{20}$$

$$g \circ h = g \cdot h;\ h \circ g = h\ . \tag{20$^{\text{bis}}$}$$

Proof. First of all $g' \cdot (g \cdot h) = (g'g) \cdot h$ and $\|g \cdot h\|_\infty = \|h\|_\infty$. Then, given ε, let $k \in K(G, A)$ such that $\|h - k\|_\infty \leqq \varepsilon/3$ and $V(e)$ a neighborhood of e in G such that $\|g \cdot k - k\| < \varepsilon/3$ when $g \in V(e)$. Then $\|g \cdot h - h|_\infty \leqq \|g \cdot h - g \cdot k\|_\infty + \|g \cdot k - k\|_\infty + \|k - h\|_\infty \leqq \varepsilon$ when $g \in V(e)$. Same proof for the right action.

PROPOSITION 2. *$\mathfrak{X}_L^{*,A}$, $\mathfrak{X}_R^{*,A}$, $\mathfrak{X}^{*,A}$ and $\mathfrak{X}^{*,A''}$ are G-modules in two different ways corresponding to the two following different actions of G, denoted successively by a $\cdot$ and by $\circ$:*

$$
(21)\quad
\begin{cases}
g\cdot\mu_L = \delta_g*\mu_L;\ \mu_L\cdot g = \mu_L*\delta_g,\ \mu_L\in\mathfrak{X}_L^{*,A},\ g\in G\\
g\cdot\mu_R = \delta_g^*\mu_R;\ \mu_R\cdot g = \mu_R*\delta_g;\ \mu_R\in\mathfrak{X}_R^{*,A},\ g\in G\\
g\cdot(\mu_L,\mu_R) = (g\cdot\mu_L,\ g\cdot\mu_R)\ ;\\
(\mu_L,\mu_R)\cdot g = (\mu_L\cdot g,\ \mu_R\cdot g),\ (\mu_L,\mu_R)\in\mathfrak{X}^{*,A},\ g\in G\\
g\cdot\mu = \delta_g*\mu;\ \mu\cdot g = \mu*\delta_g,\ \mu\in\mathfrak{X}^{*,A},\ g\in G\ .
\end{cases}
$$

$$
(21^{\text{bis}})\quad
\begin{cases}
g\circ\mu_L = \mu_L;\ \mu_L\circ g = \mu_L\cdot g\\
g\circ\mu_R = \mu_R;\ \mu_R\circ g = \mu_R\cdot g\\
g\circ(\mu_L,\mu_R) = (\mu_L,\mu_R);\ (\mu_L,\mu_R)\circ g = (\mu_L\circ g,\ \mu_R\circ g)\\
g\circ\mu = \mu;\ \mu\circ g = \mu\cdot g\ .
\end{cases}
$$

Proof. The proof is straightforward, and left to the reader.

With the same notations, we then have the following formulas, relating these G-module structures:

PROPOSITION 3.

$$
(22)\quad
\begin{cases}
\langle g\cdot\mu_L, h\rangle = \langle {}^g\mu_L, h\cdot g\rangle;\ \langle h, g\cdot\mu_R\rangle = \langle h\cdot g, {}^g\mu_R\rangle\\
\langle g\cdot\mu, h\rangle = \langle {}^g\mu, h\cdot g\rangle;\ \langle h, g\cdot\mu\rangle = \langle h\cdot g, \mu\rangle
\end{cases}
$$

$$
(23)\quad
\begin{cases}
\langle \mu_L\cdot g, h\rangle = \langle \mu_L, g\cdot h\rangle;\ \langle h, \mu_R\cdot g\rangle = \langle g\cdot h, \mu_R\rangle\\
\langle \mu\cdot g, h\rangle = \langle \mu, g\cdot h\rangle;\ \langle h, \mu\cdot g\rangle = \langle g\cdot h, \mu\rangle
\end{cases}
$$

$$
(22^{\text{bis}})\quad
\begin{cases}
\langle g\circ\mu_L, h\rangle = \langle \mu_L, h\circ g\rangle = \langle \mu_L, h\rangle\ ;\\
\langle h, g\circ\mu_R\rangle = \langle h\circ g, \mu_R\rangle = \langle h, \mu_R\rangle\\
\langle g\circ\mu, h\rangle = \langle \mu, h\circ g\rangle = \langle \mu, h\rangle;\ \langle h, g\circ\mu\rangle = \langle h\circ g, \mu\rangle = \langle h, \mu\rangle
\end{cases}
$$

$$
(23^{\text{bis}})\quad
\begin{cases}
\langle \mu_L\circ g, h\rangle = \langle \mu_L, g\circ h\rangle;\ \langle h, \mu_R\circ g\rangle = \langle g\circ h, \mu_R\rangle\\
\langle \mu\circ g, h\rangle = \langle \mu, g\circ h\rangle;\ \langle h, \mu\circ g\rangle = \langle g\circ h, \mu\rangle
\end{cases}
$$

$$
(24)\quad
\begin{aligned}
&\|\mu_L\cdot g\|_1 = \|g\cdot\mu_L\|_1 = \|\mu_L\|_1;\ \|\mu_R\cdot g\| = \|g\cdot\mu_R\|_1 = \|\mu_R\|_1\ ;\\
&\qquad\|g\cdot\mu\|_1 = \|\mu\cdot g\|_1 = \|\mu\|_1\ .
\end{aligned}
$$

Proof. Formulas (22), (23), (22^{bis}), (23^{bis}) are just a matter of computation. Let us prove (24): with notations of (6),

$$
\begin{aligned}
\|\mu_L\cdot g\|_1 &= |||U_{\mu_L}\cdot g||| = \sup\sum_i |U_{\mu_L\cdot g}(h_i)| = \sup\sum_i |U_{\mu_L}(g\cdot h_i)|\\
&= \sup\sum_i |U_{\mu_L}(h_i)| = |||U_{\mu_L}||| = \|\mu_L\|_1
\end{aligned}
$$

$$
\begin{aligned}
\|g\cdot\mu_L\|_1 &= |||U_{g\cdot\mu_L}||| = \sup\sum_i |U_{g\cdot\mu_L}(h_i)| = \sup\sum_i |U_{{}^g\mu_L}(h_i\cdot g)|\\
&= \sup\sum_i |U_{{}^g\mu_L}(h_i)| = |||U_{{}^g\mu_L}||| = \|{}^g\mu_L\|_1 = \|\mu_L\|_1
\end{aligned}
$$

and the same for μ_R and μ.

PROPOSITION 4. *The functions* $g \to g\cdot\mu_L$, $g \to \mu_L\cdot g$ (resp. $g \to g\cdot\mu_R$, $g \to \mu_R\cdot g$; $g \to g\cdot\mu$, $g \to \mu\cdot g$) *are continuous in the* $\mathfrak{X}$*-weak-topology*, i.e., *the topology defined by the semi-norms*: $|\mu_L|_h = |\langle\mu_L, h\rangle|$ (resp. $|\mu_R|^h = |\langle h, \mu_R\rangle|$; $|\mu|_h$ *and* $|\mu|^h$).

Proof. We give the proof for the first function only. Then, given μ_L, h and $\varepsilon > 0$, there exists $V(e)$, neighborhood of e in G, such that, if $g \in V(e)$, $\|h\cdot g - h\|_\infty < \varepsilon/3\|\mu_L\|_1$, $|\langle\mu_L, h\rangle = \sigma(g^{-1})\langle\mu_L, h\rangle| < \varepsilon/3$ by continuity of σ, $|\langle\mu_L, \sigma(g^{-1})h - h\rangle| < \varepsilon/3$ by second countability of G and Lebesgue's dominated convergence theorem. Hence,

$$\begin{aligned}|\langle g\cdot\mu_L - \mu_L, h\rangle| &= |\langle{}^g\mu_L, h\cdot g\rangle - \langle\mu_L, h\rangle| = |\sigma(g)\langle\mu_L, \sigma(g^{-1})h(g\cdot)\rangle \\ &\quad - \langle\mu_L, h\rangle| = |\langle\mu_L, \sigma(g^{-1})h(g\cdot)\rangle - \sigma(g^{-1})\langle\mu_L, h\rangle| \\ &= |\langle\mu_L, \sigma(g^{-1})h(g\cdot) - \sigma(g^{-1})h + \sigma(g^{-1})h - h + h\rangle - \sigma(g^{-1})\langle\mu_L, h\rangle| \\ &\leqq |\langle\mu_L, \sigma(g^{-1})h(g\cdot) - \sigma(g^{-1})h\rangle| + |\langle\mu_L, \sigma(g^{-1})h - h\rangle| \\ &\quad + |\langle\mu_L, h\rangle - \sigma(g^{-1})\langle\mu_L, h\rangle| \leqq \|\mu_L\|_1\|h\cdot g - h\|_\infty + |\langle\mu_L, \sigma(g^{-1})h - h\rangle| \\ &\quad + |\langle\mu_L, h\rangle - \sigma(g^{-1})\langle\mu_L, h\rangle| < \varepsilon\,.\end{aligned}$$

In a second step, we now introduce more general module structures.

PROPOSITION 5. $\mathfrak{X}$ *becomes a unital Banach-*$\mathfrak{X}_L^{*,A}$ *(or* $\mathfrak{X}_R^{*,A}$*, or* $\mathfrak{X}^{*,A}$*)-module (and a neo-unital Banach-*$L^1(G, A)$*-module by restriction) according to*

$$\begin{cases}\mu_L\circ h = \mu_L\cdot h, & h\circ\mu_L = \mu_L \\ \mu_R\circ h = \mu_R\cdot h, & h\circ\mu_R = \mu_R\end{cases} \tag{25}$$

where

$$\begin{cases}(\mu_L\cdot h)(g) = \langle{}^g\mu_L(u), u\cdot h(g)\rangle = \langle{}^g\mu_L, h\cdot g\rangle \\ \qquad = \langle g\cdot\mu_L, h\rangle = \langle\delta_g*\mu_L, h\rangle \\ (\mu_R\cdot h)(g) = \langle u\cdot h(g), {}^g\mu_R(u)\rangle = \langle h\cdot g, {}^g\mu_R\rangle \\ \qquad = \langle h, g\cdot\mu_R\rangle = \langle h, \delta_g*\mu_R\rangle\,.\end{cases} \tag{26}$$

Then

$$\|\mu_L\cdot h\|_\infty \leqq \|\mu_L\|_1\|h\|_\infty;\ \|\mu_R\cdot h\|_\infty \leqq \|\mu_R\|_1\|h\|_\infty \tag{27}$$

$$\mu_L\cdot(\nu_L\cdot h) = (\mu_L*\nu_L)\cdot h;\ \mu_R\cdot(\nu_R\cdot h) = (\mu_R*\nu_R)\cdot h \tag{28}$$

and, in particular

$$\delta_g\cdot h = g\cdot h,\ \delta_0\cdot h = h\,. \tag{29}$$

If $\{\lambda_\beta\}_{\beta\in J}$ is a countable approximate unit in $L^1(G, A)$, then

$$(30)\qquad \mu_L\cdot h = \lim_{\alpha,\beta}(\mu_L * e_\alpha\lambda_\beta)\cdot h;\ \mu_R\cdot h = \lim_{\alpha,\beta}(\mu_R * e_\alpha\lambda_\beta)\cdot h\ .$$

Proof. $(\mu_L\cdot h)(g)$ is continuous by Proposition 4. Moreover, as μ_L is a regular Borel measure, given $\varepsilon > 0$, there exists a compact $K\subset G$ such that $|\mu_L|(G/K) < \varepsilon/2\|h\|_\infty$ for $h\in\mathfrak{X}$. Given $\varepsilon > 0$ and K, there exists a compact K' such that, if $g\notin K'$, $\sup_{u\in K}|h(gu)| < \varepsilon/2\|\mu_L\|_1$. So

$$\begin{aligned}|(\mu_L\cdot h)(g)| &\leqq \langle|\mu_L|, |h(g\cdot)|\rangle\\ &= \int_K d|\mu_L|(u)|h(gu)| + \int_{G/K} d|\mu_L|(u)|h(gu)|\\ &\leqq \|\mu_L\|_1\sup_{u\in K}|h(gu)| + |\mu_L|(G/K)\cdot\|h\|_\infty \leqq \varepsilon \text{ if } g\notin K'\ ,\end{aligned}$$

and then $\mu_L\cdot h\in\mathfrak{X}$, and also $\mu_R\cdot h\in\mathfrak{X}$. Formulas (26) and (29) are just a matter of computation. Moreover (29) proves the unital character of $\mathfrak{X}$, while its neo-unital character on $L^1(G, A)$ comes from (26): more precisely (26) shows the set $\{e_\alpha\lambda_\beta\cdot h\}_{\alpha\in I,\beta\in J}$ is dense in $\mathfrak{X}$, while the Curtis-Figa-Talamanca factorization theorem proves this set generates a closed subspace of $\mathfrak{X}$([6], p. 169–185). Hence any $h\in\mathfrak{X}$ can be written $h = f\cdot h'$ with $h'\in\mathfrak{X}$ and $f\in L^1(G, A)$. This allows to prove (30) because

$$\begin{aligned}\lim_{\alpha,\beta}(\mu_L * e_\alpha\lambda_\beta)\cdot h &= \lim_{\alpha,\beta}(\mu_L * e_\alpha\lambda_\beta)\cdot f\cdot h' = \lim_{\alpha,\beta}(\mu_L * e_\alpha\lambda_\beta * f)\cdot h'\\ &= (\mu_L * f)\cdot h' = \mu_L\cdot f\cdot h' = \mu_L\cdot h\ .\end{aligned}$$

It does not seem possible to define a nontrivial action of $\mathfrak{X}_L^{*,A}$ or $\mathfrak{X}_R^{*\ A}$ on the right of $\mathfrak{X}$ which turns it into a Banach module. Formula (30) of the preceding theorem means the action of $\mathfrak{X}_R^{*,A}$ or $\mathfrak{X}_R^{*,A}$ can be deduced from the one of $L_1(G, A)$ by extension to its left and right centralizers.

PROPOSITION 6. *$\mathfrak{X}_L^{*,A}$, $\mathfrak{X}_R^{*\ A}$, $\mathfrak{X}^{*,A}$ and $\mathfrak{X}^{*,A''}$ are unital Banach modules onto themselves in two different ways, corresponding to the two following different actions, denoted successively by a $\cdot$ and by a $\circ$:*

$$(31)\qquad \begin{cases}\nu_L\cdot\mu_L = \nu_L * \mu_L;\ \mu_L\cdot\nu_L = \mu_L * \nu_L\\ \nu_R\cdot\mu_R = \nu_R * \mu_R;\ \mu_R\cdot\nu_R = \mu_R * \nu_R\\ (\nu_L, \nu_R)\cdot(\mu_L, \mu_R) = (\nu_L * \mu_L, \nu_R * \mu_R);\ (\mu_L, \mu_R)\cdot(\nu_L, \nu_R) = (\mu_L * \nu_L, \mu_R * \nu_R)\\ \nu\cdot\mu = \nu * \mu;\ \mu\cdot\nu = \mu * \nu\end{cases}$$

$$\begin{cases}\nu_L\circ\mu_L = \mu_L;\ \mu_L\circ\nu_L = \mu_L * \mu_L\\ \nu_R\circ\mu_R = \mu_R;\ \mu_R\circ\nu_R = \mu_R * \nu_R\end{cases}$$

$$\begin{cases}(\nu_L, \nu_R)\circ(\mu_L, \mu_R) = (\mu_L, \mu_R); (\mu_L, \mu_R)\circ(\nu_L, \nu_R) = (\mu_L*\nu_L, \mu_R*\nu_R)\\ \nu\circ\mu = \mu; \mu\circ\nu = \mu*\nu .\end{cases}$$

Proof. The proof is straightforward and left to the reader.

With the same notations, we then have the following formulas, relating these module structures:

PROPOSITION 7.

$$\langle\mu_L\cdot\nu_L, h\rangle = \langle\mu_L, \nu_L\cdot h\rangle; \langle h, \mu_R\cdot\nu_R\rangle = \langle\nu_R\cdot h, \mu_R\rangle \tag{32}$$

$$\begin{cases}\langle\nu_L\circ\mu_L, h\rangle = \langle\mu_L, h\circ\nu_L\rangle = \langle\mu_L, h\rangle; \langle h, \nu_R\circ\mu_R\rangle = \langle h\circ\nu_R, \mu_R\rangle = \langle h, \mu_R\rangle\\ \langle\mu_L\circ\nu_L, h\rangle = \langle\mu_L, \nu_L\circ h\rangle; \langle h, \mu_R\circ\nu_R\rangle = \langle\nu_R\circ h, \mu_R\rangle .\end{cases} \tag{32$^{\text{bis}}$}$$

According to formulas (22), (23) and (32) one side, and (22^{bis}), (23^{bis}) and (32^{bir}) on the other side, we can refer to the structure denoted by $a\ \cdot$ as the "nondual" structure, and the structure denoted by a $\circ$ as the "dual" structure.

4. Homology and cohomology over $L_1(G, A)$. In the sequel, we will denote, for convenience,

$$\mathfrak{A} = L^1(G, A) \tag{33}$$

and

$$L_n(\mathfrak{A}, \mathfrak{X}) = \mathfrak{A}\hat{\otimes}\mathfrak{A}\hat{\otimes}\cdots\hat{\otimes}\mathfrak{A}\hat{\otimes}\mathfrak{X} = \mathfrak{A}^{\hat{\otimes} n}\hat{\otimes}\mathfrak{X} \tag{34}$$

where there are n copies of $\mathfrak{A}(n \geqq 0)$ and where $\hat{\otimes}$ denotes the projective tensor product [11].

Let us define the application D_n from $L_n(\mathfrak{A}, \mathfrak{X})$ into $L_{n-1}(\mathfrak{A}, \mathfrak{X})$ by the continuous linear extension of

$$\begin{aligned} D_n(f_1\otimes f_1\otimes f_2\otimes\cdots\otimes f_n\otimes h) &= f_2\otimes f_3\otimes\cdots\otimes f_n\otimes h\cdot\\ &+ \sum_{i=1}^{n-1}(-1)^i f_1\otimes\cdots\otimes f_i^* f_{i+1}\otimes\cdots f_n\otimes h\\ &+ (-1)^n f_1\otimes f_2\otimes\cdots\otimes f_{n-1}\otimes(f_n\circ h)\end{aligned} \tag{35}$$

where $f_i\in\mathfrak{A}$, $i = 1, \cdots, n$ and $h\in\mathfrak{X}$. (D_0 is defined as the null application.) Then $D_nD_{n+1} = 0$ for all $n \geqq 0$ and it is possible to introduce the quotient space

$$H_n(\mathfrak{A}, \mathfrak{X}) = \frac{\text{Ker } D_n}{\text{Im } D_{n+1}} . \tag{36}$$

One can notice that it is possible to write

(37) $$L_n(\mathfrak{A}, \mathfrak{X}) = L^1(G, A)^{\hat{\otimes} n} \hat{\otimes} \mathfrak{X} = L^1(G^n) \otimes A^{\hat{\otimes} n} \otimes \mathfrak{X}$$

or

(38) $$L_n(\mathfrak{A}, \mathfrak{X}) = L^1(G^n, A^{\hat{\otimes} n}) \hat{\otimes} \mathfrak{X}$$

or

(39) $$L_n(\mathfrak{A}, \mathfrak{X}) = L^1(G^n, A^{\hat{\otimes} n} \otimes \mathfrak{A}) .$$

Let now $L_n(\mathfrak{A}, \mathfrak{X})_L^{*,A}$ be the space of functionals T on $L_n(\mathfrak{A}, \mathfrak{X})$ with values in A which are n-linear on $\mathfrak{A}$, A-right linear on $\mathfrak{X}$, and bounded in the sense that, for any finite family (h_i) of functions $h_i \in K(G, A)$ with support $h_i \cap$ support $h_j = \emptyset$ for any $i \neq j$,

(40) $$\sum_i |T(f_1 \otimes \cdots \otimes f_n \otimes h_i)| \leqq K||f_1||_1 \cdots ||f_n||_1 ||\Sigma h_i||_\infty$$

where K is some constant.

Hence it is possible to identify $L_n(\mathfrak{A}, \mathfrak{X})_L^{*,A}$ with $L^n(\mathfrak{A} \cdot \mathfrak{X}_L^{*,A})$, the space of continuous n-linear functionals on $\mathfrak{A}$ with values in $\mathfrak{X}_L^{*,A}$ according to (we use the same letter T to denote the two corresponding objects):

(41) $$T(f_1 \otimes \cdots \otimes f_n \otimes h) = \langle T(f_1 \otimes \cdots \otimes f_n), h \rangle .$$

In the same way, we could introduce $L^n(\mathfrak{A}, \mathfrak{X}_R^{*,A})$, $L^n(\mathfrak{A}, \mathfrak{X}^{*,A})$ and $L^n(\mathfrak{A}, \mathfrak{X}^{*,A''})$.

Let us now define the application Δ^n from

$$L^{n-1}(\mathfrak{A}, \mathfrak{X}_L^{*A})(\text{resp. } L^{n-1}(\mathfrak{A}, \mathfrak{X}_R^{*,A}), L^{n-1}(\mathfrak{A}, \mathfrak{X}^{*,A}), L^{n-1}(\mathfrak{A}, \mathfrak{X}^{*,A''}))$$

into

$$L^n(\mathfrak{A}, \mathfrak{X}_L^{*,A}) \text{ (resp. } L^n(\mathfrak{A}, \mathfrak{X}_R^{*,A}), L^n(\mathfrak{A}, \mathfrak{X}^{*,A}), L^n(\mathfrak{A}, \mathfrak{X}^{*,A''}))$$

by the following formula, corresponding to the "nondual" structure:

(42) $$\begin{aligned} \Delta^n T(f_1 \otimes \cdots \otimes f_n) &= f_1 \cdot T(f_2 \otimes \cdots \otimes f_n) \\ &+ \sum_{i=1}^{n-1} (-1)^i T(f_1 \otimes \cdots \otimes f_i * f_{i+1} \otimes \cdots \otimes f_n) \\ &+ (-)^n T(f_1 \otimes \cdots \otimes f_{n-1}) \cdot f_n . \end{aligned}$$

Then $\Delta^{n+1}\Delta^n = 0$ and it is possible to introduce the quotient space

(43) $$H^n(\mathfrak{A}, \mathfrak{X}_L^{*,A}) = \frac{\text{Ker } \Delta^{n+1}}{\text{Im } \Delta^n} = \frac{Z^n(\mathfrak{A}, \mathfrak{X}_L^{*,A})}{N^n(\mathfrak{A}, \mathfrak{X}_L^{*,A})}$$

and, in the same way, $H^n(\mathfrak{A}, \mathfrak{X}_R^{*,A})$, $H^n(\mathfrak{A}, \mathfrak{X}^{*,A})$, $H^n(\mathfrak{A}, \mathfrak{X}^{*,A''})$.

In the "dual" structure case, we have to modify slightly our definitions, asking for T to be n-affine and continuous on $\mathfrak{A}$, and replacing formula (42) by

$$\begin{aligned}\Delta^n T(f_1 \otimes \cdots \otimes f_n) &= T(f_2 \otimes \cdots \otimes f_n) \\ &+ \sum_{i=1}^{n-1} (-1)^i T(f_1 \otimes \cdots \otimes f_i * f_{i+1} \otimes \cdots \otimes f_n) \\ &+ (-1)^n T(f_1 \otimes \cdots \otimes f_{n-1}) \circ f_n \, . \end{aligned} \tag{42bis}$$

In that case, it is easy to prove the "duality" formula:

$$\langle T, D_n(f_1 \otimes \cdots \otimes f_n \otimes h)\rangle = \langle \Delta^n T, f_1 \otimes \cdots \otimes f_n \otimes h\rangle \tag{44}$$

and we have the following theorem, relating (36) and (43) in the "dual" case.

THEOREM 3. *Let us assume there exists an $F \in A'$ such that $F(a) = 0$ imply $a = 0$ (it is the case if A is separable). Then $H_n(\mathfrak{A}, \mathfrak{X}) = 0$ and $\operatorname{Im} D_n$ closed is, in the "dual" case, equivalent to $H^n(\mathfrak{A}, \mathfrak{X}_L^{*\,A}) = 0$, $H^n(\mathfrak{A}, \mathfrak{X}_R^{*\,A}) = 0$, $H^n(\mathfrak{A}, \mathfrak{X}^{*,A}) = 0$ or $H^n(\mathfrak{A}, \mathfrak{X}^{*,A''}) = 0$.*

Proof. Let $L^n(\mathfrak{A}, \mathfrak{X}')$ the space of continuous n-linear functionals on $\mathfrak{A}$ with value in $\mathfrak{X}'$ (the dual of $\mathfrak{X}$), i.e., the dual of $L_n(\mathfrak{A}, \mathfrak{X})$. Given $T \in L^n(\mathfrak{A}, \mathfrak{X}_L^{*\,A})$ and $F \in A'$ let T_F be the element of $L^n(\mathfrak{A}, \mathfrak{X}')$ defined by

$$\langle T_F, f_1 \otimes \cdots \otimes f_n \otimes h\rangle = F\{\langle T, f_1 \otimes \cdots \otimes f_n \otimes h\rangle\}$$

and let $L^n(\mathfrak{A}, \mathfrak{X}_L^{*\,A})_F$ be the closed subspace of $L^n(\mathfrak{A}, \mathfrak{X}')$ generated by the set of T_F with $T \in L^n(\mathfrak{A}, \mathfrak{X}_L^{*\,A})$. By faithfulness of F, the correspondence $T \to T_F$ is injective and the spaces $L_n(\mathfrak{A}, \mathfrak{X})$ and $L^n(\mathfrak{A}, \mathfrak{X}_L^{*,A})_F$ are in duality. Moreover, if Δ_1^n means the equivalent of Δ^n on $L^n(\mathfrak{A}, \mathfrak{X}')$, we have, if we define $f \circ T_F = T_F$ and $T_F \circ f = (T \circ f)_F$,

$$\begin{aligned}\langle T_F, D^{n+1}(f_1 \otimes \cdots \otimes f_{n+1} \otimes h)\rangle &= \langle \Delta_1^{n+1} T_F, f_1 \otimes \cdots \otimes f_{n+1} \otimes h\rangle \\ &= F\{\langle T, D^{n+1}(f_1 \otimes \cdots \otimes f_{n+1} \otimes h)\rangle\} = F\{\langle \Delta^{n+1} T, f_1 \otimes \cdots \otimes f_n \otimes h\rangle\} \\ &= \langle (\Delta^{n+1} T)_F, f_1 \otimes \cdots \otimes f_{n+1} \otimes h\rangle \, .\end{aligned}$$

So $\Delta_1^{n+1} T_F = (\Delta^{n+1} T)_F$: Δ_1^{n+1}, when restricted to $L^n(\mathfrak{A}, \mathfrak{X}_L^{*\,A})_F$, maps it into $L^{n+1}(\mathfrak{A}, \mathfrak{X}_L^{*,A})_F$, and is the transpose of D^{n+1} in the duality $\langle L_n(\mathfrak{A}, \mathfrak{X}), L^n(\mathfrak{A}, \mathfrak{X}_L^{*,A})_F\rangle$. Hence, in the same way as in ([15], Corollary 1.3), it is possible to prove the theorem is true if $H^n(\mathfrak{A}, \mathfrak{X}_L^{*,A})$ is replaced by $H^n(\mathfrak{A}, \mathfrak{X}_L^{*\,A})_F$.

But let us now assume that $H^n(\mathfrak{A}, \mathfrak{X}_L^{*\,A}) = 0$, that is to say that $\Delta^{n+1} T = 0$ imply $T = \Delta^n T'$ with $T' \in L^{n-1}(\mathfrak{A}, \mathfrak{X}_L^{*\,A})$. By faithfulness of F, $\Delta^{n+1} T = 0$ is equivalent to $\Delta_1^{n+1} T_F = 0$ and $T = \Delta^n T'$ is equivalent to $T_F = \Delta_1^n T'_F$ with $T'_F \in L^{n-1}(\mathfrak{A}, \mathfrak{X}_L^{*,A})_F$. Hence $H^n(\mathfrak{A}, \mathfrak{X}_L^{*,A}) = 0$ is equivalent to $H^n(\mathfrak{A}, \mathfrak{X}_L^{*,A})_F = 0$ which proves the theorem. The proof is similar for $H^n(\mathfrak{A}, \mathfrak{X}_R^{*,A})$, $H^n(\mathfrak{A}, \mathfrak{X}^{*,A})$ and $H^n(\mathfrak{A}, \mathfrak{X}^{*,A''})$.

THEOREM 6. *Let p be a positive integer. Then*

$$\begin{cases} H_{n+p}(\mathfrak{A}, \mathfrak{X}) \sim H_n(\mathfrak{A}, L_p(\mathfrak{A}, \mathfrak{X})) \\ H^{n+p}(\mathfrak{A}, \mathfrak{X}_L^{*,A}) \sim H^n(\mathfrak{A}, L^p(\mathfrak{A}, \mathfrak{X}_L^{*,A})) \end{cases} \tag{45}$$

and equivalent formulas for $\mathfrak{X}_R^{*,A}$, $\mathfrak{X}^{*,A}$, $\mathfrak{X}^{*,A''}$.

Proof. This is the Hochschild's method for the reduction of dimension [14] [15]. It consists in defining the natural isometry τ_n from $L_{n+p}(\mathfrak{A}, \mathfrak{X})$ onto $L_n(\mathfrak{A}, L_p(\mathfrak{A}, \mathfrak{X}))$ thanks to the associativity of the tensor product, the action of $\mathfrak{A}$ onto $L_p(\mathfrak{A}, \mathfrak{X})$ by

$$\begin{aligned} f\circ(f_1 \otimes \cdots \otimes f_p \otimes h) &= f*f_1 \otimes \cdots \otimes f_p \otimes h \\ &= \sum_{i=1}^{n-1}(-1)^i f \otimes f_1 \otimes \cdots \otimes f_i * f_{i+1} \otimes \cdots \otimes f_p \otimes h \\ &\quad + (-1)^n f \otimes f_1 \otimes \cdots \otimes f_{p-1} \otimes f\circ h \\ (f_1 \otimes \cdots \otimes f_p \otimes h)\circ f &= f_1 \otimes \cdots \otimes f_p \otimes h \end{aligned}$$

and to notice that $\tau_{n-1}D_{n+p} = D'_n\tau_n$ if D'_n denotes the equivalent of D_n on $L_n(\mathfrak{A}, L_p(\mathfrak{A}, \mathfrak{X}))$.

In the same way, one can define an isometry τ^n from $L^{n+p}(\mathfrak{A}, \mathfrak{X}_L^{*,A})$ onto $L^n(\mathfrak{A}, L^p(\mathfrak{A}, \mathfrak{X}_L^{*,A}))$ by

$$[(\tau^n T)(f_1 \otimes \cdots \otimes f_n)](f_{n+1} \otimes \cdots \otimes f_{n+p}) = T(f_1 \otimes \cdots \otimes f_{n+p})$$

and the "nondual" action of $\mathfrak{A}$ onto $L^p(\mathfrak{A}, \mathfrak{X}_L^{*,A})$ by

$$\begin{aligned} (f\cdot T)(f_1 \otimes \cdots \otimes f_p) &= f * T(f_1 \otimes \cdots \otimes f_p) \\ (T\cdot f)(f_1 \otimes \cdots \otimes f_p) &= T(f*f_1 \otimes \cdots \otimes f_p) \\ &\quad + \sum_{i=1}^{n-1}(-1)^i T(f \otimes f_1 \otimes \cdots \otimes f_i * f_{i+1} \otimes \cdots \otimes f_p) \\ &\quad + (-1)^n T(f \otimes f_1 \otimes \cdots \otimes f_p)\,. \end{aligned}$$

A "dual" action could be defined according to $f\circ T = T$ and $T\circ f = T.f$.

We close this paragraph by giving an example of $T \in L^n(\mathfrak{A}, \mathfrak{X}_L^{*,A})$ in the "nondual" case. Let (k_i), $i = 1, \cdots, n$ be a family of functions in $L^\infty(G)$, (F_i), $i = 1, \cdots, n$ a family of continuous linear forms in the dual A' of A, and $\mu \in \mathfrak{X}_L^{*,A}$. Let us define

$$T = F_1\circ k_1 \otimes \cdots \otimes F_n\circ k_n \tag{46}$$

by

$$(47)\qquad \langle T, f_1 \otimes \cdots \otimes f_n \rangle = F_1(\langle k_1, f_1\rangle)F_2(\langle k_2, f_2\rangle)\cdots F_n(\langle k_n, f_n\rangle)$$

where

$$(48)\qquad \langle k_i, f_i\rangle = \int k_i(g)f_i(g)dg \in A\ .$$

It is easy to check that T has the desired properties for being an element of $L^n(\mathfrak{A}, \mathfrak{X}_L^{*,A})$. Considering $f_1 \otimes \cdots \otimes f_n$ as an element of $L_1(G^n, A^{\hat{\otimes} n})$ (37), T is a function on G^n with values in $\mathscr{L}(A^{\hat{\otimes} n}, \mathfrak{X}_L^{*,A})$ such that $\|T(g_1, \cdots, g_n)\|$ is in $L^\infty(G^n)$. It is the purpose of the next paragraph to prove that, under some hypothesis on A, any T can be represented by a function having these properties and conversely.

5. A Riesz representation theorem for the elements of $L^n(\mathfrak{A}, \mathfrak{X}_R^{*,A})$, $L^n(\mathfrak{A}, \mathfrak{X}_L^{*,A})$, or $L^n(\mathfrak{A}, \mathfrak{X}^{*,A''})$. In this section, we will restrict ourself to the case when $A = Z'$, the dual of some Banach space Z, or when A is a C^*-algebra (if A is both, it is a Von Neumann algebra), and to the "nondual" structure. It could be possible to adapt this paragraph to the "dual" one. We will denote by $\bar{a}$ and $\bar{g}$ an element of $A^{\hat{\otimes} n}$ and G^n respectively.

We begin by recalling a theorem which asserts that any T can be represented by a measure:

THEOREM 5. *If A is a Banach-*-algebra, there exists an isomorphism $T \leftrightarrow \mu_T$ between $L^n(\mathfrak{A}, \mathfrak{X}_L^{*,A})$* (resp. $L^n(\mathfrak{A}, \mathfrak{X}_R^{*,A})$, $L^n(\mathfrak{A}, \mathfrak{X}^{*,A''})$) *and the space of vector measures on G^n with finite variation and with value in $\mathscr{L}(A^{\hat{\otimes} n}, \mathfrak{X}_L^{*,A})$* (resp. $\mathscr{L}(A^{\hat{\otimes} n}, \mathfrak{X}_R^{*,A})$, $\mathscr{L}(A^{\hat{\otimes} n}, \mathfrak{X}^{*,A''})$) *such that $|\mu_T| = k(\bar{g})d|\bar{g}|$ with $h \in L^\infty(G^n)$, $k \geqq 0$, where $d|\bar{g}|$ means the absolute value of the Haar measure on G^n, given by*

$$(49)\qquad T(f_1 \otimes \cdots \otimes f_n) = \int f_1(g_1)\cdots f_n(g_n)d\mu_T(g_1, \cdots, g_n)$$

with

$$(50)\qquad \|T\| = \|k\|_\infty\ .$$

Proof. See ([7], 13, no. 3, Theorem 1, Corollary 2).

We are now going to prove that μ_T can be represented by a function, with the help of the following generalization of Lebesgue-Nikodym's theorem.

PROPOSITION 8. *Let ν be a regular Borel scalar measure on G^n, μ a measure on G^n with value in the Banach-$*$-algebra $A = Z'$* (resp. *in the $C*$-algebra A), with finite variation $|\mu|$, absolutely continuous with respect to ν* (i.e., *$|\mu|$ is absolutely continuous with respect to $|\nu|$ in the usual sense). Then there exists a function V_μ on G^n with value in A* (resp. *in A''*) *such that:*

(i) *$|V_\mu(\bar{g})|$ is locally-ν-integrable and $|\mu| = |V_\mu(\bar{g})||\nu|$,*
i.e.: $\int \varphi(\bar{g})d|\mu|(\bar{g}) = \int \varphi(\bar{g})|V_\mu(\bar{g})|d|\nu|(\bar{g}),\ \varphi \in L^1(G^n, |\mu|)$,

(ii) $\left\langle \int F(\bar{g})d\mu(\bar{g}), z \right\rangle = \int \langle V_\mu(\bar{g})F(\bar{g}), z\rangle d\nu(\bar{g}),\ F \in L^1(G^n, A, \mu)$
for any $z \in Z$ (resp. $z \in A'$),

(iii) *If A* (resp. A'') *is separable, $V_\mu(g^n)$ is locally-ν-integrable and*

$$\int F(\bar{g})d\mu(\bar{g}) = \int V_\mu(\bar{g})F(\bar{g})d\nu(\bar{g}) ,$$

(iv) *If Z* (resp. A') *is separable, then V_μ is unique.*

Proof. See ([7], § 13, $n°4$, Theorem 5).

THEOREM 6. *Let $A = Z'$, $T \in L^n(\mathfrak{A}, \mathfrak{X}_L^{*,A})$* (resp. $L^n(\mathfrak{A}, \mathfrak{X}_R^{*,A})$) *and μ_T the corresponding measure. There exists a function V_T on G^n with values in $\mathscr{L}(A^{\hat{\otimes} n}, \mathfrak{X}_L^{*A})$* (resp. $\mathscr{L}^n(A^{\hat{\otimes} n}, \mathfrak{X}_R^{*,A})$ *such that*

(i) *$\|V_T(\bar{g})\| = k(\bar{g}) \in L^\infty(G^n)$, where k is defined in Theorem* 5 *and*

$$\int \varphi(\bar{g})d|\mu_T|(\bar{g}) = \int \varphi(\bar{g})\|V_T(\bar{g})\|\, d|\bar{g}|,\ \varphi \in L^1(G^n, |\mu_T|)$$

with $\|T\| = \|k\|_\infty = \|\,\|V_T(\bar{g})\|\,\|_\infty$.

(ii) *$\langle V_T(\bar{g})\Phi(\bar{g}), h\rangle$ is integrable, where $h \in \mathfrak{X}$ and $\Phi \in L^1(G^n, A^{\hat{\otimes} n})$ and*

$$\langle T(\Phi), h\rangle = \left\langle \int \Phi(\bar{g})d\mu_T(\bar{g}), h \right\rangle = \int \langle V_T(\bar{g})\Phi(\bar{g}), h\rangle d\bar{g} . \tag{51}$$

(iii) *If A (and $\mathfrak{X}$) is separable, then V_T is unique.*

Proof. Let $B \in B(G^n)$ a Borel subset of G^n and $\mu^T_{\bar{a}\,h}$ the measure on G^n with value in A defined by

$$\mu^T_{\bar{a}\,h}(B) = \langle \mu_T(B)\bar{a}, h\rangle .$$

It is a measure with finite variation and absolutely continuous with respect to $d\bar{g}$ because of

$$|\mu^T_{\bar{a},h}| \leqq |\bar{a}|\|h\|_\infty|\mu_T| = |\bar{a}|\,\|h\|_\infty k(\bar{g})d|\bar{g}| .$$

Hence, by Proposition 8, there exists a function $G^T_{\bar{a},h}$ on G^n with values in A such that

$$\mu^T_{\bar{a},h} = G^T_{\bar{a},h} d\bar{g} \text{ and } |\mu^T_{\bar{a},h}| = |G^T_{\bar{a},h}|\, d|\bar{g}| \leqq |\bar{a}|\, ||h||_\infty k(\bar{g}) d|\bar{g}| .$$

Let $G^T_{\bar{a}}(\bar{g})$ be the correspondence $h \in \mathfrak{X} \to G^T_{\bar{a},h}(\bar{g})$. It is A-right linear and if (h_i) is a finite family of functions in $K(G, A)$ with support $h_i \cap$ support $h_j = \varnothing$ and $||\Sigma h_i||_\infty \leqq 1$, then, for any $B \in B(G^n)$,

$$\begin{aligned}\int_B \sum_i |G^T_{\bar{a},h_i}(\bar{g})|\, d|\bar{g} &= \sum_i |\mu^T_{\bar{a},h_i}|(B) = \sup_j \sum_i |\mu^T_{\bar{a},h_i}(B_j)| \\ &= \sup_j \sum_i |\langle \mu_T(B_j)\bar{a}, h_i\rangle| \leqq \sup_j ||\mu_T(B_j)\bar{a}||_1 \leqq \sup_j ||\mu_T(B_j)||\, |\bar{a}| \\ &= |\mu_T|(B)\, |\bar{a}| = \left(\int_B k(\bar{g}) d|\bar{g}|\right) |\bar{a}|\end{aligned}$$

from which we deduce that

$$\sum_i |G^T_{\bar{a},h_i}(\bar{g})| \leqq k(\bar{g})\, |\bar{a}|$$

almost everywhere or, else, $|||G^T_{\bar{a}}(\bar{g})||| \leqq k(\bar{g})\,|\bar{a}|$ almost everywhere.

By modifying it on set of measure zero if necessary, we have $G^T_{\bar{a}}(\bar{g}) \in \mathfrak{X}^{*,A}_L$ for any $\bar{g} \in G^n$ and

$$||G^T_{\bar{a}}(\bar{g})||_1 \leqq k(\bar{g})\, |\bar{g}| .$$

If $V_T(\bar{g})$ is the correspondence $\bar{a} \to G^T_{\bar{a}}(\bar{a})$, then

$$V_T(\bar{g}) \in \mathscr{L}(A^{\hat{\otimes} n}, \mathfrak{X}^{*,A}_L),\ ||V_T(\bar{g})|| \leqq k(\bar{g}), \text{ and } \langle V_T(\bar{g})\bar{a}, h\rangle = G^T_{\bar{a},h}(\bar{g}) .$$

For any step function Φ' on G^n with values in $A^{\hat{\otimes} n}$, we have

$$\langle T(\Phi'), h\rangle = \left\langle \int \Phi'(\bar{g}) d\mu_T(\bar{g}), h \right\rangle = \int \langle V_T(\bar{g})\Phi'(\bar{g}), h\rangle d\bar{g}$$

so that, if Φ'_i is a sequence of such step functions converging to Φ in $L^1(G^n, A^{\hat{\otimes} n})$, the Cauchy sequence $\langle V_T(\bar{g})\Phi'_i(\bar{g}), h\rangle$ converges to $\langle V_T(\bar{g})\Phi(\bar{g}), h\rangle$ in $L^1(G^n, A)$ while $\int \Phi'_i(\bar{g}) d\mu(\bar{g})$ converges to $\int \Phi(\bar{g}) d\mu(\bar{g})$ and (ii) is proved.

The inequality $||V_T(\bar{g})|| \leqq k(\bar{g})$ shows that $||V_T(\bar{g})|| \in L^\infty(G^n)$ and so is locally integrable. Then the inequality

$$\langle \mu_T(B)\bar{a}, h\rangle \leqq \int_B |\langle V_T(\bar{g})\bar{a}, h\rangle|\, d|\bar{g}|$$

implies that

$$||\mu_T(B)|| \leqq \int_B ||V_T(\bar{g})||\, d|\bar{g}|$$

and

$$|\mu_T|(B) \leqq \int_B \| V_T(\bar{g})\| d|\bar{g}|$$

because $|\mu_T|$ is the least positive measure such that $\|\mu_T(B)\| \leqq |\mu_T(B)|$. Hence

$$\int_B k(\bar{g})d|\bar{g}| \leqq \int_B \| V_T(\bar{g})\| d|\bar{g}| \leqq \int k(\bar{g})d|\bar{g}|$$

which proves (i).

If A is separable, $\mathfrak{X}$ is separable too by countability of G. If V'_T is another function representing μ_T,

$$\int_B \langle (V_T(\bar{g}) - V'_T(\bar{g}))\bar{a}, h\rangle d\bar{g} = 0$$

for any Borel B in $B(G^n)$, any $\bar{a}$ in $A^{\hat{\otimes} n}$ and any h in $\mathfrak{X}$. So $\langle V_T(\bar{g}) - V'_T(\bar{g}))\bar{a}, h\rangle = 0$ almost everywhere. Taking $\bar{a}$ and h into countable dense subsets we conclude that $V_T(\bar{g}) = V'_T(\bar{g})$ almost everywhere.

Next theorem is a converse of the preceding one.

THEOREM 7. *Let A be a Banach-$*$-algebra and V a function on G^n with values in $\mathscr{L}(A^{\hat{\otimes} n}, \mathfrak{X}_L^{*,A})$* (resp. $\mathscr{L}(A^{\hat{\otimes} n}, \mathfrak{X}_R^{*\,A})$) *such that $\| V(\bar{g})\| \in L^\infty(G^n)$ and $\langle V(\bar{g})\bar{a}, h\rangle$ is measurable for any $\bar{a} \in A^{\hat{\otimes} n}$ and $h \in \mathfrak{X}$. There exists a continuous linear map T from $L^1(G^n, A^{\hat{\otimes} n})$ to $\mathfrak{X}_L^{*,A}$* (resp. $\mathfrak{X}_R^{*,A}$) *such that*

$$\langle T(\Phi), h\rangle = \int \langle V(\bar{g})\Phi(\bar{g}), h\rangle d\bar{g},\ \Phi \in L^1(G^n, A^{\hat{\otimes} n}) . \tag{52}$$

If $A = Z'$, let V_T and k the corresponding functions (Theorems 6 and 5). Then

$$k(\bar{g}) \leqq \| V(\bar{g})\| \text{ a.e., and } \| T\| \leqq \| \| V(\bar{g})\| \|_\infty .$$

If A (and $\mathfrak{X}$) is separable, then $k(\bar{g}) = \| V(\bar{g})\|$ a.e., $V = V_T$ and $\| T\| = \| \| V(\bar{g})\| \|_\infty$.

Proof. As $|\langle V(\bar{g})\Phi(\bar{g}), h\rangle| \leqq \| V(\bar{g})\| |\Phi(\bar{g})| \|h\|_\infty$ and

$$\| V(\bar{g})\| |\Phi(\bar{g})| \in L^1(G^n),\ |\langle V(\bar{g})\Phi(\bar{g}), h\rangle|$$

is integrable. Let $T_h(\Phi) = \int \langle V(\bar{g})\Phi(\bar{g}), h\rangle d\bar{g}$: the correspondence $h \to T_h(\Phi)$, denoted $T(\Phi)$, is A-right linear and such that, if (h_i) is a finite family of functions in $K(G, A)$ with support $h_i \cap$ support

$h_j = \emptyset$ and $\|\sum_i h_i\|_\infty \leqq 1$, then

$$\sum_i |T_{h_i}(\Phi)| \leqq \int \sum_i |\langle V(\bar{g})\Phi(\bar{g}), h_i\rangle| d|\bar{g}| \leqq \int \| V(\bar{g})\Phi(\bar{g})\|_1 d|\bar{g}|$$
$$\leqq \int \| V(\bar{g})\| \, |\Phi(\bar{g})| d|\bar{g}| \leqq \| \| V(\bar{g})\| \|_\infty \|\Phi\|_1 .$$

Hence $T(\Phi) \in \mathfrak{X}_L^{*,A}$ and $\|T(\Phi)\|_1 \leqq \| \| V(\bar{g})\| \|_\infty \|\Phi\|_1$. So the correspondence $\Phi \to T(\Phi)$, denoted T, is linear and continuous with $\|T\| \leqq \| \| V(\bar{g})\| \| \|_\infty$: hence $T \in L^n(\mathfrak{A}, \mathfrak{X}_L^{*,A})$ and

$$\langle T(\Phi), h\rangle = \int \langle V(\bar{g})\Phi(\bar{g}), h\rangle d\bar{g} = \left\langle \int \Phi(\bar{g}) d\mu_T(\bar{g}), h \right\rangle$$

if μ_T is the corresponding measure according to Theorem 5 with $|\mu_T| = k(\bar{g})d|\bar{g}|$, $k \in L^\infty(G^n)$. So, in particular,

$$|\mu_T|(B) \leqq \int_B \| V(\bar{g})\| d|\bar{g}| \text{ and } |\mu_T| \leqq \| V(\bar{g})\| d|\bar{g}|, \text{ i.e., } k(\bar{g}) \leqq \| V(\bar{g})\|$$

almost everywhere. Moreover, if V_T is the function corresponding to T by Theorem 6,

$$\int \langle V(\bar{g})\Phi(\bar{g}), h\rangle d\bar{g} = \int \langle V_T(\bar{g})\Phi(\bar{g}), h\rangle d\bar{g}$$

and, in particular,

$$\langle V(\bar{g})\bar{a}, h\rangle = \langle V_T(\bar{g})\bar{a}, h\rangle \text{ a.e. },$$

which ends the proof of the theorem if A (and $\mathfrak{X}$) is separable.

REMARK. If the hypothesis $A = Z'$ is replaced by A is a C^*-algebra, then V_T is, in any case, a function on G^n with values in $\mathscr{L}(A^{\hat{\otimes} n}, \mathfrak{X}^{*,A''})$: the proof of Theorem 6 is similar but the difference comes from Radon-Nikodym's theorem (Proposition 8). But then Theorem 7 associates to V an element of $L^n(\mathfrak{A}, \mathfrak{X}^{*,A''})$ and so is no longer the converse of Theorem 6. The symmetry is restored if we start with $L^n(\mathfrak{A}, \mathfrak{X}^{*,A''})$: if A (and $\mathfrak{X}$) is separable, this space is isometrically isomorphic to the space of functions V on G^n with values in $\mathscr{L}(A^{\hat{\otimes} n}, \mathfrak{X}^{*,A''})$ such that $\| V(\bar{g})\| \in L^\infty(G^n)$ and $\langle V(\bar{g})\Phi(\bar{g}), h\rangle$ is measurable for any $\Phi \in L^1(G^n, A^{\hat{\otimes} n})$ and $h \in \mathfrak{X}$.

6. Extension from $\mathfrak{A}$ to $\mathfrak{X}_L^{*,A}$, $\mathfrak{X}_R^{*,A}$ or $\mathfrak{X}^{*,A}$. By Propositions 5 and 6, it is possible to define $L_n(\mathfrak{X}_L^{*,A}, \mathfrak{X})$ and $L^n(\mathfrak{X}_L^{*,A}, \mathfrak{X}_L^{*,A})$, or $L^n(\mathfrak{X}_L^{*,A}, \mathfrak{X}^{*,A''})$ if A is a C^*-algebra, (and the same for $\mathfrak{X}_R^{*,A}$ or $\mathfrak{X}^{*,A}$) in the same way as we did for $\mathfrak{A}$. In this paragraph, we shall state the relation between these spaces and the corresponding ones for $\mathfrak{A}$, under the hypothesis: $A = Z'$ or A is a C^*-algebra.

THEOREM 8. *Let $A = Z'$ be a Banach-*-algebra. It is possible to extend $T \in L^n(\mathfrak{A}, \mathfrak{X}_L^{*,A})$ (resp. $L^n(\mathfrak{A}, \mathfrak{X}_R^{*,A})$, $L^n(\mathfrak{A}, \mathfrak{X}^{*,A})$) as an element $\tilde{T}$ of $L^n(\mathfrak{X}_L^{*,A}, \mathfrak{X}_L^{*,A})$ (resp. $L^n(\mathfrak{X}_R^{*,A}, \mathfrak{X}_R^{*,A})$, $L^n(\mathfrak{X}^{*,A}, \mathfrak{X}^{*,A})$) with $\|T\| = \|\tilde{T}\|$.*

Proof. Let (μ_i), $i = 1, \cdots, n$ with $\mu_i \in \mathfrak{X}_L^{*,A}$, $h \in \mathfrak{X}$ and $\{e_\alpha \lambda_\beta\}_{\alpha \in I; \beta \in J}$ a (countable) approximate unit in $\mathfrak{A}$. The set of $\langle T(\mu_1 * e_{\alpha_1}\lambda_{\beta_1}, \cdots, \mu_n * e_{\alpha_n}\lambda_{\beta_n}), h\rangle$ is bounded in A, uniformly with respect to α_i and β_i, and also in the bidual A''. So there exists a (nonunique) sequence of such elements converging weakly in the bidual A'' towards an element $\tilde{T}(\mu_1, \cdots, \mu_n, h)$ which is evidently linear in $\mu_1, \cdots, \mu_n$. It is A-right linear in h because, for any $F \in A'$,

$$\begin{aligned}
\lim_{i_k, j_k} \langle\langle T(\mu_1 * e_{i_1}\lambda_{j_1}, \cdots, \mu_n * e_{i_n}\lambda_{j_n}, ha), F\rangle \\
&= \langle \tilde{T}(\mu_1, \cdots, \mu_n, ha), F\rangle \\
&= \lim_{i_k, j_k} \langle T(\mu_1 * e_{i_1}\lambda_{j_1}, \cdots, \mu_n * e_{i_n}\lambda_{j_n}, h)a, F\rangle \\
&= \lim_{i_k, j_k} \langle T(\mu_1 * e_{i_1}\lambda_{j_1}, \cdots, \mu_n * e_{i_n}\lambda_{j_n}), a^t F\rangle \\
&= \langle \tilde{T}(\mu_1, \cdots, \mu_n), a^t F\rangle = \langle \tilde{T}(\mu_1 \cdots \mu_n)a, F\rangle
\end{aligned}$$

where the multiplication on the right of an element in the bidual A'' by an element in A is defined through bitransposition [1], [5]. Moreover, as $A = Z'$ (i.e., $A' = Z'' \supset Z$), $\tilde{T}(\mu_1, \cdots, \mu_n, h)$ is in fact in $A \subset A''$, as we can see computing the limit for $F = z \in Z \subset Z''$, and the right multiplication by a coincide with the product in A.

Now, given $\varepsilon > 0$ and $z \in Z$, there exists $i_{0,k}$ and $j_{0,k}$ such that, if $i_k > i_{0,k}$ and $j_k > j_{0,k}$,

$$|\langle \tilde{T}(\mu_1, \cdots, \mu_n, h), z\rangle| \leqq |\langle\langle T(\mu_1 * e_{i_1}\lambda_{j_1}, \cdots, \mu_n * e_{i_n}\lambda_{j_n}), h\rangle, z\rangle| + \varepsilon .$$

So, if (h_l), $l = 1, \cdots, m$ is a finite family in $K(G, A)$ with support $h_l \cap$ support $h_{l'} = \emptyset$ for $l \neq l'$ and $\|\sum_l h_l\|_\infty \leqq 1$, then

$$\begin{aligned}
\sum_l |\langle \tilde{T}(\mu_1, \cdots, \mu_n, h_l), z\rangle| &\leqq \sum_l \langle\langle T(\mu_1 * e_{i_1}\lambda_{j_1}, \cdots, \mu_n * e_{i_n}\lambda_{j_n}), h_l\rangle, z\rangle| + m\varepsilon \\
&\leqq \|z\| \sum_l |\langle T(\mu_1 * e_{i_1}\lambda_{j_1}, \cdots, \mu_n * e_{i_n}\lambda_{j_n}), h_l\rangle| + m\varepsilon \\
&\leqq \|z\| \, \|T(\mu_1 * e_{i_1}\lambda_{j_1}, \cdots \mu_n * e_{i_n}\lambda_{j_n})\|_1 + m\varepsilon \\
&\leqq \|z\| \, \|T\| \, \|\mu_1\|_1 \cdots \|\mu_n\|_1 + m\varepsilon .
\end{aligned}$$

Consequently,

$$\sum_l |\tilde{T}(\mu_1, \cdots, \mu, h_l)| \leqq \|T\| \, \|\mu_1\| \cdots \|\mu_n\|$$

which proves that $\tilde{T} \in L^n(\mathfrak{X}_L^{*,A}, \mathfrak{X}_L^{*,A})$ with $\|\tilde{T}\| \leqq \|T\|$. But, if (f_i),

$i = 1, \cdots, n$ is a finite family in $\mathfrak{A}$,

$$\operatorname*{limit}_{i_k, j_k} \langle T(f_1 * e_{i_1}\lambda_{j_1}, \cdots, f_n * e_{i_n}\lambda_{j_n}), h\rangle = \tilde{T}(f_1, \cdots, f_n, h)$$
$$= \langle T(f_1, \cdots, f_n), h\rangle$$

where the convergence is even in norm. Hence $\tilde{T}$ restricted to $\mathfrak{A}$ is T and $\|\tilde{T}\| = \|T\|$ because

$$\|\tilde{T}\| = \sup_{\|\mu_i\|_1 \leqq 1} \|\tilde{T}(\mu_1, \cdots, \mu_n)\|_1 \geqq \sup_{\|f_i\|_1 \leqq 1} \|T(f_1, \cdots, f_n)\|_1 = \|T\| .$$

REMARK. If the hypothesis $A = Z'$ is replaced by A is a C^*-algebra, the preceding proof shows that $\tilde{T} \in L^n(\mathfrak{X}_L^{*,A}, \mathfrak{X}^{*,A''})$. So, once again (see remark at the end of the preceding paragraph) we see that, in the case of a C^*-algebra, $L^n(\mathfrak{A}, \mathfrak{X}^{*,A''})$ is a more natural object than $L^n(\mathfrak{A}, \mathfrak{X}_L^{*,A})$.

Thanks to Theorem 8, we can give a meaning to the function $\tilde{T}(\bar{g})$ on G^n by $\tilde{T}(\bar{g}) = \tilde{T}(\delta_{g_1}, \cdots, \delta_{g_n})$ if $\bar{g} = (g_1, \cdots, g_n) \in G^n$. We are going to see that this function is closely related to the function V_T defined in Theorem 6 (see also the remark at the end of the preceding paragraph).

THEOREM 9. *Let A (and $\mathfrak{X}$) be separable and let λ_j be a countable approximate unit in $L^1(G)$ such that $\lim_j \lambda_j * k = k$ almost everywhere for any $k \in L^\infty(G, A)$ or $L^\infty(G, A'')$. (This condition is discussed in* ([12, Theorem 44. 18]). *If $\bar{a} = (a_1, \cdots, a_n) \in A^{\hat{\otimes} n}$ and $\bar{g} = (g_1, \cdots, g_n) \in G^n$, then, with any of the two hypothesis on A,*

$$\tilde{T}(a_1\delta_{g_1}, \cdots, a_n\delta_{g_n}) = V_T(\bar{g})\bar{a} \tag{53}$$

almost everywhere.

Proof. Let (φ_i), $i = 1, \cdots, n$ *with* $\varphi_i \in L^1(G)$. For any $h \in \mathfrak{X}$ $\langle T(a_1\varphi_1, \cdots, a_n\varphi_n), h\rangle = \int \varphi_1(g_1) \cdots \varphi_n(g_n) \langle V_T(\bar{g})\bar{a}, h\rangle d\bar{g}$ and $\langle V_T(\bar{g})\bar{a}, h\rangle \in L^\infty(G^n, A)$ or $L^\infty(G^n, A'')$. In particular,

$$T(a_1\delta_{g_1} * \lambda_{j_1}, \cdots, a_n\delta_{g_n} * \lambda_{j_n}), h\rangle = \int \lambda_{j_1}(g_1^{-1}u_1) \cdots \lambda_{j_n}(g_n^{-1}u_n) \langle V_T(\bar{u})\bar{a}, h\rangle d\bar{u} .$$

By hypothesis on λ_j and by successive applications of Lebesgue's dominated convergence theorem, we obtain

$$\langle \tilde{T}(a_1\delta_{g_1}, \cdots, a_n\delta_{g_n}), h\rangle = \langle V_T(\bar{g})\bar{a}, h\rangle$$

almost everywhere and, by separability of $\mathfrak{X}$,

$$\tilde{T}(a_1\delta_{g_1}, \cdots, a_n\delta_{g_n}) = V_T(\bar{g})\bar{a}, \quad \text{a.e.}$$

We then have the following formulas

$$(54)\qquad \langle T(\bar{a}\Phi), h\rangle = \int \Phi(\bar{g})\langle \tilde{T}(a_1\delta_{g_1}, \cdots, a_n\delta_{g_n}), h\rangle d\bar{g},\ \Phi \in L^1(G^n)\ , \quad \bar{a} = (a_1, \cdots, a_n),\ \bar{g} = (g_1, \cdots, g_n)$$

$$(55)\qquad \begin{aligned}\langle T(a_1\varphi_1 * a_2\varphi_2), h\rangle &= \int \varphi_1(g_1)\varphi_2(g_2)\langle \tilde{T}(a_1\sigma(g_1)a_2\delta_{g_1g_2}), h\rangle dg_1dg_2\\ &= \int \varphi_1(g_1)\varphi_2(g_2)\langle V_T(g_1, g_2)a_1\sigma(g_1)a_2, h\rangle dg_1dg_2,\ \varphi_1, \varphi_2 \in L^1(G)\ .\end{aligned}$$

We can now determine the relation between $L^n(\mathfrak{A}, \mathfrak{X}_L^{*,A})$ (resp. $L^n(\mathfrak{A}, \mathfrak{X}^{*,A''})$) and $L^n(\mathfrak{X}_L^{*,A}, \mathfrak{X}_L^{*,A})$ (resp. $L^n(\mathfrak{X}_L^{*,A}, \mathfrak{X}^{*,A''})$).

THEOREM 10. *The restriction operation $\tilde{T} \rightarrow T$ from $\mathfrak{X}_L^{*,A}$ to $\mathfrak{A}$ and Δ^n commute. This restriction induces a map from $L^n(\mathfrak{X}_L^{*,A}, \mathfrak{X}_L^{*,A})$ (resp. $L^n(\mathfrak{X}_L^{*,A}, \mathfrak{X}^{,A''})$) to $L^n(\mathfrak{A}, \mathfrak{X}_L^{*,A})$ (resp. $L^n(\mathfrak{A}, \mathfrak{X}^{*,A''})$) such that $H^n(\mathfrak{X}_L^{*,A}, \mathfrak{X}_L^{*,A})$ (resp. $H^n(\mathfrak{X}_L^{*,A}, \mathfrak{X}^{*,A''})$) is isomorphic to $H^n(\mathfrak{A}, \mathfrak{X}_L^{*,A})$ (resp. $H^n(\mathfrak{A}, \mathfrak{X}^{*,A''})$).*

Proof. The proof is by induction, identical to the corresponding one in ([15], Prop. 1.9 and Lemma 1.10), the proof of Lemma 1.10 being now achieved by noticing that $f * \mu_L = 0$ for any $f \in \mathfrak{X}$ imply $\mu_L = 0$.

7. **Vector means.** This paragraph collects and proves results, some of which will be useful in the sequel, about vector means, a natural extension of usual means on locally compact groups [10]. This notion appeared for the first time in [9].

We shall consider the following functional spaces:

$X_1 = L^\infty(G, A)$;

$X_2 = CB(G, A)$, the space of bounded continuous functions on G with value in A;

$X_3 = UCB_r^\sigma(G, A)$, the space of bounded continuous functions k on G with value in A with the property that, for any $\varepsilon > 0$, there exists $\mathscr{V}(e)$, neighborhood of e in G, such that, for any $g \in \mathscr{V}(e)$, $|k(v) - \sigma(g)k(g^{-1}v)| \leqq \varepsilon$ for any $v \in G$;

$X_4 = UCB_r(G, A)$, the space of bounded continuous functions k on G with value in A with the property that, for any $\varepsilon > 0$, there exists $\mathscr{V}(e)$, neighborhood of e in G, such that, for any $g \in \mathscr{V}(e)$, $|k(v) - k(g^{-1}v)| \leqq \varepsilon$ for any $v \in G$;

$X_5 = UCB_l(G, A)$, the space of bounded continuous functions k on G with value in A with the property that, for any $\varepsilon > 0$, there exists $\mathscr{V}(e)$, neighborhood of e in G, such that, for any $g \in \mathscr{V}(e)$, $|k(v) - k(vg)| \leqq \varepsilon$ for any $v \in G$;

$X_6 = X_3 \cap X_5$;
$X_7 = X_4 \cap X_5$.

Each of these spaces is a C^*-algebra for the usual product of functions, the uniform norm and the involution $k \to \bar{k}$: $\bar{k}(g) = k(g)^*$, and we have the following inclusions:

$$\begin{array}{ccccccc} & & X_3 & & & & \\ & \subset & & \cap & & & \\ X_6 & & & & & & \\ & \cap & & & & & \\ & & X_5 & \subset & X_2 & \subset & X_1 \\ & \subset & & & & & \\ X_7 & & & \cup & & & \\ & \cap & & & & & \\ & & X_4 & & & & \end{array}$$

DEFINITIONS. An A-mean on X_i is an A-linear positive continuous map on X_i with value in A, with norm less than or equal to 1.

An A-mean on X_1, X_2, X_4 or X_7 is said left invariant if

$$\langle M, k(g^{-1}.)\rangle = \langle M, k\rangle \tag{56}$$

and topologically left invariant if (here $*$ exceptionally means the usual convolution)

$$\langle M, \varphi * k\rangle = \langle M, k\rangle \int \varphi(u)du, \quad \varphi \in L^1(G) . \tag{57}$$

An A-mean on X_1, X_2, X_3 or X_6 is said σ-left invariant if

$$\langle M, \sigma(g)k(g^{-1}.)\rangle = \sigma(g)\langle M, k\rangle \tag{58}$$

and topologically σ-left invariant if

$$\langle M, \varphi *^{\sigma} k\rangle = \int \varphi(u)\sigma(u)\langle M, k\rangle du, \quad \varphi \in L^1(G) . \tag{59}$$

An A-mean on X_1, X_2, X_5 or X_7 is said right invariant if

$$\langle M, k(\cdot g)\rangle = \langle M, k\rangle \tag{60}$$

and topologically right invariant if (here usual and σ-convolution coincide)

$$\langle M, k * \tilde{\varphi}\rangle = \langle M, k\rangle \int \varphi(u)du, \varphi \in L^1(G), \tilde{\varphi}(u) = \varphi(u^{-1}) . \tag{61}$$

These definitions make sense thanks to the following lemmas:

LEMMA 1. *If $\varphi \in L^1(G)$ and $k \in L^\infty(G, A)$, then $\varphi * k$ (the usual convolution) is in X_4 and $k * \tilde{\varphi}$ in X_5. If k is in X_5, $\varphi * k$ is in X_7*

and if k is in X_4, $k*\tilde{\varphi}$ is in X_7.

Proof.

$$|(\varphi*k)(v) - (\varphi*k)(g^{-1}v)| \leqq \int |\varphi(vu) - \varphi(g^{-1}vu)|\,|k(u^{-1})|\,du$$
$$\leqq ||\varphi - \varphi(g^{-1}.)||_1 ||k||_\infty$$

and

$$|(k*\tilde{\varphi})(v) - (k*\tilde{\varphi})(vg)| \leqq \int |k(u)|\,|\varphi(v^{-1}u) - \varphi(g^{-1}v^{-1}u)|\,du$$
$$\leqq ||\varphi - \varphi(g^{-1}.)||_1\, ||k||_\infty\,,$$

and the conclusion comes from ([13], Theorem (20.4)).

If $k \in X_5$, it is enough to prove that $\varphi*k$ is in X_5 but

$$|(\varphi*k)(v) - (\varphi*k)(vg)| \leqq \int |\varphi(u)|\,|k(u^{-1}v) - k(u^{-1}vg)|\,du$$
$$\leqq ||\varphi||_1\, ||k - k(.g)||_\infty\,.$$

If $k \in X_4$, it is enough to prove that $k*\tilde{\varphi}$ is in X_4 but

$$|(k*\tilde{\varphi})(v) - (k*\tilde{\varphi})(g^{-1}v)| \leqq \int |k(vu) - k(g^{-1}vu)|\,|\varphi(u)|\,du$$
$$\leqq ||\varphi||_1\, ||k - k(g^{-1}.)||_\infty\,.$$

LEMMA 1 bis. *If $\varphi \in L^1(G)$ and $k \in L^\infty(G, A)$, then $\varphi*^\sigma k \in X_3$ and $k*^\sigma\tilde{\varphi} \in X_5$. If $k \in X_5$ then $\varphi*^\sigma k \in X_6$ and if $k \in X_3$ then $k*^\sigma\tilde{\varphi} = k*\tilde{\varphi} \in X_6$.*

Proof.

$$|(\varphi*^\sigma k)(v) - \sigma(g)(\varphi*k)(g^{-1}v)| = |\int\varphi(vu)\sigma(vu)k(u^{-1})\,du - \sigma(g)$$
$$\times \int\varphi(g^{-1}vu)\sigma(g^{-1}vu)k(u^{-1})du| \leqq \int |\varphi(vu) - \varphi(g^{-1}vu)|\,|\sigma(vu)k(u^{-1})|\,du$$
$$\leqq ||\varphi - \varphi(g^{-1}.)||_1\, ||k||_\infty$$

and $k*^\sigma\tilde{\varphi} = k*\tilde{\varphi}$ (see above).

If $k \in X_5$, it is enough to prove that $\varphi*k \in X_5$ but

$$|(\varphi*k)(v) - (\varphi*k)(vg)| \leqq \int |\varphi(u)|\,|\sigma(u)k(u^{-1}v) - \sigma(u)k(u^{-1}vg)|\,du$$
$$\leqq ||\varphi||_1\, ||k - k(.g)|||_\infty$$

and if $k \in X_3$ it is enough to prove that $k*\tilde{\varphi}$ is in X_3, but

$$|(k*\tilde{\varphi})(v) - \sigma(g)(k*\tilde{\varphi})(g^{-1}v)| \leqq \int |k(vu) - \sigma(g)k(g^{-1}vu)|\,|\varphi(u)|\,du$$
$$\leqq ||\varphi||_1\, ||k - \sigma(g)k(g^{-1}.)||_\infty\,.$$

The relation between these different means and these different spaces comes from the following proposition.

PROPOSITION 9. *If there exists a left invariant A-mean, there exists a right invariant one on the corresponding space, and conversely.*

Proof. Let us consider the transformation $k \to \tilde{k}: k(u) = k(u^{-1})$. It changes X_1, X_2, X_4 or X_7 into X_1, X_2, X_5 or X_7 because if $k \in X_4$ then $\tilde{k}(v) - \tilde{k}(vg) = k(v^{-1}) - k(g^{-1}v^{-1})$. Let $\langle \bar{M}, \tilde{k}\rangle = \langle M, k\rangle$. Then, if M is left invariant, $\langle \bar{M}, \tilde{k}(.g)\rangle = \langle M, k(g^{-1}.)\rangle = \langle M, k\rangle = \langle \bar{M}, \tilde{k}\rangle$.

THEOREM 11. *Let M a σ-left, left or right invariant A-mean on X_1 or X_2. Then M restricted to $\mathfrak{X} = \mathscr{C}_0(G, A)$ is null if G is non- compact.*

Proof. Let V be a relatively compact neighborhood of e in G. By noncompacity of G, there is no finite sequence (g_i) in G such that $(g_i V)$ is a covering of G. So it is possible to choose an infinite sequence (g_i) in G such that $g_{n+1} \notin \bigcup_{i=1}^n g_i V$ for all integer n. Hence if U is a symmetric relatively compact neighborhood of e such that $U^2 \subset V$, the sequence $(g_i U)$ is made of pairwise disjoint sets. Let $\tilde{M}$ the canonical extension of M to $\tilde{X}_1$ or $\tilde{X}_2$, the functional spaces corresponding to $\tilde{A}$, the associative Banach-$*$-algebra with unit, defined according to $\langle M, a\varphi\rangle = a\langle \tilde{M}\cdot\varphi\rangle$ for any $a \in A$ and φ any function with scalar values belonging to $\tilde{X}_1$ or $\tilde{X}_2$. Then $a\langle \tilde{M}, \varphi(g^{-1}.)\rangle = \langle M, a\varphi(g^{-1}.)\rangle = \langle M, \sigma(g)\sigma(g^{-1})a\varphi(g^{-1}.)\rangle = \sigma(g)\langle M, \sigma(g^{-1})a\varphi\rangle = a\langle \tilde{M}, \varphi\rangle$, i.e., $\langle \tilde{M}, \varphi(g^{-1}.)\rangle = \langle \tilde{M}, \varphi\rangle$. Let $\xi_U \in \tilde{X}_1$ or $\tilde{X}_2$ be the characteristic function of U: as $\sum_{i=1}^n \xi_{g_i U} \leqq 1$, we have $\sum_{i=1}^n \langle \tilde{M}, \xi_U\rangle \leqq 1$, i.e., $\langle \tilde{M}, \xi_U\rangle = 0$. Let $h \in \mathfrak{X}$. Given ε, there exists a compact $K \subset G$ such that $|h(g)| < \varepsilon$ for $g \notin K$ and a finite sequence (g'_j) in G such that $K \subset \bigcup_j g'_j U$. So $|h(g)| \leqq \|h\| \sum_j \xi_{g'_j U} + \varepsilon$ and $|\langle M, h\rangle| \leqq \langle \tilde{M}, \varepsilon\rangle = \varepsilon\langle \tilde{M}, 1\rangle \leqq \varepsilon$. The proof is similar if M is left or right invariant.

The relation between topological left or right invariance and left or right invariance is given by the two following theorems.

THEOREM 12. *A σ-left invariant A-mean is topologically σ-left invariant. A left (or right) invariant A-mean is topologically left (or right) invariant.*

Proof. Let $\varphi \in L_1(G)$. Then, for $k \in X_i$, $i = 1, 2, 3$, or 6,

$$\int \varphi(u)\sigma(u)\langle M, \sigma(g)k(g^{-1}.)\rangle du = \langle M, \varphi*\sigma(g)k(g^{-1}.)\rangle$$

$$= \left\langle M, \int \varphi(u)\sigma(ug)k(g^{-1}u^{-1}.)du \right\rangle$$

$$= \left\langle M, \delta(g^{-1})\int \varphi(ug^{-1})\sigma(u)k(u^{-1}.)du \right\rangle$$

$$= \langle M, \delta(g^{-1})\varphi(.g^{-1})*k \rangle = \delta(g^{-1})\int \varphi(ug^{-1})\sigma(u)\langle M, k\rangle du$$

$$= \int \varphi(u)\sigma(u)\sigma(g)\langle M, k\rangle du \ .$$

This being true for any φ, $\langle M, \sigma(g)k(g^{-1}.)\rangle = \sigma(g)\langle M, k\rangle$. The proof is similar for left and right invariance.

THEOREM 13. *A σ-left invariant A-mean on X_3 (or X_6) is topologically σ-invariant. A left invariant A-mean on X_4 (or X_7) is topologically invariant, and the same for right invariance on X_5 (or X_7).*

Proof. It is sufficient to prove the theorem for $\varphi \in L^1(G)$ continuous with compact support. Let $k \in X_3$. The map $g \in \text{support } \varphi \to \sigma(g)k(g^{-1}.) \in X_3$ is continuous because, if $g'g = gg''$,

$$\|\sigma(g)k(g^{-1}.) - \sigma(g'g)k((g'g)^{-1}.)\|_\infty = \sup_v |\sigma(g)k(g^{-1}v) - \sigma(gg'')k((gg'')^{-1}v)| = \sup_v |k(g^{-1}v) - \sigma(g'')k(g''^{-1}g^{-1}v)| < \varepsilon$$

if g' is in some neighborhood of e in G. The set $\{\sigma(g)k(g^{-1}.),\ g \in \text{support } \varphi\}$ is compact in X_3 and if

$$\Phi \in X_3', \left\langle \Phi, \int \sigma(g)k(g^{-1}.)\varphi(g)dg \right\rangle = \int \langle \Phi, \sigma(g)k(g^{-1}.)\rangle \varphi(g)dg \ .$$

Let us choose $\Phi = F \circ M$ with $F \in A'$. Then

$$F\left\{\left\langle M, \int \sigma(g)k(g^{-1}.)\varphi(g)dg \right\rangle\right\} = \int F\{\langle M, \sigma(g)k(g^{-1}.)\rangle\}\varphi(g)dg \ .$$

In the same way, if we choose $\Phi = F \circ \delta_t$, $t \in G$:

$$F\left\{\left\langle \delta_t, \int \sigma(g)k(g^{-1}.)\varphi(g)dg \right\rangle\right\} = \int F\{\langle \delta_t, \sigma(g)k(g^{-1}.)\rangle\}\varphi(g)dg$$

$$= \int F\{\varphi(g)\sigma(g)k(g^{-1}t)\}dt = F\{\langle \delta_t, \varphi*k\rangle\} \ .$$

So $\int \sigma(g)k(g^{-1}.)\varphi(g)dg = \varphi*k$ and

$$F\{\langle M, \varphi*k\rangle\} = \int F\{\varphi(g)\langle M, \sigma(g)k(g^{-1}.)\rangle\}dg$$

$$= \int F\{\varphi(g)\sigma(g)\langle M, k\rangle\}dg = F\left\{\int \varphi(g)\sigma(g)\langle M, k\rangle \right\}dg \ .$$

This being true for any F,

$$\langle M, \varphi * k\rangle = \int \varphi(g)\sigma(g)\langle M, k\rangle dg .$$

The proof is similar for left and right invariance.

The next theorem asserts the existence of σ-left, left and right invariant A-means on G if G is amenable.

THEOREM 14. *Let A be a Banach-$*$-algebra such that $A = Z'$* (resp. *A is a C^*-algebra*). *If G is ameanable, there exist σ-left, left and right invariant means on G with value in A* (resp. *in A''*).

Proof. Let m be an invariant mean on G, $k \in L^\infty(G, A)$, $F \in A'$: then $F \circ k \in L^\infty(G)$, $||F \circ k||_\infty \leqq ||F|| \, ||k||_\infty$ and $|\langle m, F \circ k\rangle|| \leqq ||F|| \, ||k||_\infty$. The correspondence $F \to \langle m, F \circ k\rangle$ is linear, continuous, and define an element $\langle M, k\rangle$ in the bidual A'' such that $\langle m, F \circ k\rangle = \langle F \langle M, k\rangle\rangle$ and $|\langle M, k\rangle| \leqq ||k||_\infty$. So the correspondence $k \to \langle M, k\rangle$ is continuous with norm less than or equal to one. If k and F are positive, $\langle F, \langle M, k\rangle\rangle$ is positive and so $\langle M, k\rangle$ is positive. Moreover, $\langle F, \langle M, ka\rangle\rangle = \langle m, F \circ ka\rangle = \langle m, a^t F \circ k\rangle = \langle a^t F, \langle M, k\rangle\rangle = \langle F, \langle M, k\rangle a\rangle$ where the product $\langle M, k\rangle a$ is defined through bitransposition [1], [5]. In the same way, $\langle M, ak\rangle = a\langle M, k\rangle$ and $\langle F, \langle M, \sigma(g)k(g^{-1}.)\rangle\rangle = \langle m, F \circ \sigma(g)k(g^{-1}.)\rangle = \langle m, \sigma(g)^t F \circ k(g^{-1}.)\rangle = \langle m, \sigma(g)^t F \circ k\rangle = \langle \sigma(g)^t F, \langle M, k\rangle\rangle = \langle F, \sigma(g)^{tt}\langle M, k\rangle\rangle$. If A is a C^*-algebra, the left and right multiplications in A'' by element from A coincide with the product in the Von Neumann evelopping algebra, while $\sigma(g)^{tt}$ is the natural extension of $\sigma(g)$ to A''. If $A = Z'$, then $\langle M, k\rangle \in A$ as we can see by choosing $F = z \in Z \subset Z'' = A'$ and the left and right multiplication by $a \in A$ is the product in A. The same kind of proof shows that M is also left and right invariant.

REMARK. The converse of the preceding theorem is obvious: if, for any Banach-$*$-algebra A, G is such that there exist invariant A-means, then G is ameanable, because it is sufficient to take $A = C$, the complex numbers.

8. The case $n = 1$. Bounded derivations and crossed homomorphisms. 1st Part: the "nondual" structure and bounded derivations. Let T be an element of $Z^1(\mathfrak{A}, \mathfrak{X}_L^{*,A})$: it is a linear and continuous application from $\mathfrak{A}$ to $\mathfrak{X}_L^{*,A}$ such that

$$T(f_1 * f_2) = T(f_1) * f_2 + f_1 * T(f_2) . \tag{62}$$

By Theorems 8 and 10, its extension $\tilde{T}$ is also in $Z^1(\mathfrak{X}_L^{*,A}, \mathfrak{X}_L^{*,A})$:

$$\tilde{T}(\mu_L * \nu_L): \tilde{T}(\mu_L) * \nu_L + \mu_L * \tilde{T}(\nu_L) . \tag{63}$$

However, in the case $n = 1$, it is possible to obtain a more precise result than in Theorem 8:

THEOREM 15. *Let $T \in Z^1(\mathfrak{A}, \mathfrak{X}_L^{*,A})$ where $A = Z'$ or A is a C^*-algebra. Then T extends in a unique way as an element $\tilde{T}$ of $Z^1(\mathfrak{X}_L^{*,A}, \mathfrak{X}_L^{*,A})$ which is continuous from $\mathfrak{X}_L^{*,A}$ with the strong topology to $\mathfrak{X}_L^{*,A}$ with the $\mathfrak{X}$-weak-topology and can be defined by continuous extension*:

$$\langle \tilde{T}(\mu_L), h\rangle = \operatorname*{limit}_{\alpha,\beta} \langle T(\mu_L * e_\alpha \lambda_\beta), h\rangle, \quad h \in \mathfrak{X} . \tag{64}$$

Moreover, if $T = \Delta\mu_L$, then $\tilde{T} = \Delta\tilde{\mu}_L$.

Proof. Our hypothesis on A insure the existence of $\tilde{T}$ and its uniqueness comes from $\tilde{T}(\mu_L) * f = T(\mu_L * f) - \mu_L * T(f)$, and the last statement of the theorem is evident. Now

$$\begin{aligned}\langle T(\mu_L * e_\alpha\lambda_\beta), h\rangle &= \langle T(\mu_L * e_\alpha\lambda_\beta) * f, h'\rangle \\ &= \langle T(\mu_L * e_\alpha\lambda_\beta * f), h'\rangle - \langle \mu_K * e_\alpha\lambda_\beta * T(f), h'\rangle \\ &= \langle T(\mu_L * e_\alpha\lambda_\beta * f\rangle, h'\rangle - \langle \mu_L * e_\alpha\lambda_\beta * f', h''\rangle\end{aligned}$$

if we write $h = f.h'$ and $T(f).h' = f'.h''$ thanks to the neo-unital character of $\mathfrak{X}$. Taking the limit in the norm of A, we have

$$\begin{aligned}\operatorname*{limit}_{\alpha,\beta} \langle T(\mu_L * e_\alpha\lambda_\beta), h\rangle &= \langle T(\mu_L * f), h'\rangle - \langle \mu_L * f', h''\rangle \\ &= \langle T(\mu_L * f) - \mu_L * T(f), h'\rangle = \langle \tilde{T}(\mu_L) * f, h'\rangle = \langle \tilde{T}(\mu_L), h\rangle\end{aligned}$$

which proves the continuity property of $\tilde{T}$, formula (64) and the fact that $\tilde{T} \in Z^1(\mathfrak{X}_L^{*,A}, \mathfrak{X}_L^{*,A})$.

REMARK. In the case of $Z^1(\mathfrak{A}, \mathfrak{X}_L^{*,A})$, even if A is a C^*-algebra, $\tilde{T}$ takes its values in $\mathfrak{X}_L^{*,A}$ (and not $\mathfrak{X}^{*,A''}$) and $H^1(\mathfrak{A}, \mathfrak{X}_L^{*,A})$ is isomorphic to $H^1(\mathfrak{X}_L^{*,A}, \mathfrak{X}_L^{*,A})$ (see Theorem 10). The preceding proof works for $Z^1(\mathfrak{X}^{*,A}, \mathfrak{X}^{*,A})$ but not for $Z^1(\mathfrak{X}_R^{*,A}, \mathfrak{X}_R^{*,A})$. Of course if $T \in Z^1(\mathfrak{A}, \mathfrak{X}^{*,A''})$ then $T \in Z^1(\mathfrak{X}_L^{*,A}, \mathfrak{X}^{*,A''})$ and $H^1(\mathfrak{A}, \mathfrak{X}^{*,A''})$ is isomorphic to $H^1(\mathfrak{X}_L^{*,A}, \mathfrak{X}^{*,A''})$.

Let us now consider the following function, linear on A,

$$(g, a) \in G \times A \longrightarrow \langle \tilde{T}(a\delta_g), h\rangle \in A,\ T \in Z^1(\mathfrak{A}, \mathfrak{X}_L^{*,A}) . \tag{65}$$

Because $|\langle \tilde{T}(a\delta_g, h\rangle| < \|T\|\,|a|\,\|h\|_\infty$, we can write $\langle \tilde{T}(a\delta_g), h\rangle = \langle V_T(g)a, h\rangle$ where $V_T(g) \in \mathscr{L}(A, \mathfrak{X}_L^{*,A})$ and $\langle V_T(g)a, h\rangle \in L^\infty(G, A)$: we recover the function V_T of Theorems 6 and 9. Of course,

$$\begin{cases}\tilde{T}(a\delta_g * b\delta_{g'}) = \tilde{T}(a\delta_g) * b\delta_{g'} + a\delta_g * \tilde{T}(b\delta_{g'}) \\ V_T(gg', a\sigma(g)b) = V_T(g)a * b\delta_{g'} + a\delta_g * V_T(g')b .\end{cases} \tag{66}$$

Conversely, let

$$(g, a) \in G \times A \longrightarrow F(g, a) \in \mathfrak{X}_L^{*, A} \tag{67}$$

a mapping, linear on A, such that $\langle F(g, a), h\rangle \in L^\infty(G, A)$, $\|F\| = \sup_{|a|\leq 1, g\in G}\|F(g, a)\|_1 < \infty$ and $F(gg', a\sigma(g)b) = F(g, a)*b\delta_{g'} + a\delta_g * F(g', b)$. Then there exists $V(g) \in \mathscr{L}(A, \mathfrak{X}_L^{*, A})$ such that $\langle F(g, a), h\rangle = \langle V(g)a, h\rangle$ and F extends from $G \times A$ to $\mathfrak{A}$ by

$$\langle F(a\varphi), h\rangle = \int\varphi(g)\langle F(g, a), h\rangle dg, \quad \varphi \in L^1(G) \tag{68}$$

and continuous extension. Then $F \in Z^1(\mathfrak{A}, \mathfrak{X}_L^{*, A})$ because

$$\begin{aligned}
\langle F(a\varphi_1)*b\varphi_2, h\rangle &= \langle F(a\varphi_1)(u), \langle\sigma(u)b\varphi_2(v), h(uv)\rangle\rangle \\
&= \int\varphi_1(g)\langle F(g, a)(u), \langle\sigma(u)b\varphi_2(v), h(uv)\rangle\rangle dg \\
&= \int\varphi_1(g)\left\langle F(g, a)(u), \int\varphi_2(w)\langle\sigma(u)b\delta_w(v), h(uv)\rangle dw\right\rangle dg \\
&= \int\varphi_1(g)\int\varphi_2(w)\langle F(g, a)(u), \langle\sigma(u)b\delta_w(v), h(uv)\rangle\rangle dwdg \\
&= \int\varphi_1(g)\varphi_2(w)\langle F(g, a)*b\delta_w, h\rangle dwdg
\end{aligned}$$

where we used Fubini's theorem to interchange the measures defined by $F(g, a)$ and φ_2, while on the other hand,

$$\begin{aligned}
\langle a\varphi_1 * F(b\varphi_2), h\rangle &= \langle a\varphi_1(u), \langle {}^uF(b\varphi_2)(v), h(uv)\rangle\rangle \\
&= a\int\varphi_1(u)\sigma(u)\langle F(b\varphi_2)(v), \sigma(u^{-1})h(uv)\rangle du \\
&= a\int\varphi_1(u)\sigma(u)\int\varphi_2(w)\langle F(b, w)(v), \sigma(u^{-1})h(uv)\rangle dwdu \\
&= a\int\varphi_1(u)\int\varphi_2(w)\langle {}^uF(b, w)(v), h(uv)\rangle dwdu \\
&= \int\varphi_2(w)\int a\varphi_1(u)\langle {}^uF(b, w)(v), h(uv)\rangle dudw \\
&= \int\varphi_2(w)\varphi_1(g)\langle a\delta_g(u), \langle {}^uF(b, w)(v), h(uv)\rangle\rangle dgdw \\
&= \int\varphi_1(g)\varphi_2(w)\langle a\delta_g * F(b, w), h\rangle dgdw
\end{aligned}$$

thanks, once again, to Fubini's theorem which is valid if we take for instance φ_1 and φ_2 with compact support. So, if we add the two preceding results,

$$\begin{aligned}
&\langle F(a\varphi_1)*b\varphi_2 + a\varphi_1 * F(b\varphi_2), h\rangle \\
&\quad = \int\varphi_1(g)\varphi_2(w)\langle F(g, a)*b\delta_w + a\delta_g * F(b, w), h\rangle dwdg
\end{aligned}$$

$$= \int \varphi_1(g)\varphi_2(w)\langle F(gw,\, a\sigma(g)b),\, h\rangle dwdg$$
$$= \int \varphi_1(g)\varphi_2(g^{-1}w')\langle F(w',\, a\sigma(g)b,\, h\rangle dw'dg$$
$$= \left\langle F\left(\int \varphi_1(g)\varphi_2(g^{-1}w')a\sigma(g)bdg\right),\, h\right\rangle$$
$$= \langle F(a\varphi_1 * b\varphi_2),\, h\rangle\ .$$

Of course the preceding discussion can be adapted to $T \in Z^1(\mathfrak{A}, \mathfrak{X}^{*,A''})$.

Let us now consider the three functions on G with value in $\mathfrak{X}_L^{*,A}$:

$$\begin{cases} \chi(g) = \tilde{T}(\delta_g) * \delta_{g^{-1}} = -\ \delta_g * \tilde{T}(\delta_{g^{-1}}) \\ \chi^a(g) = \tilde{T}(a\delta_g) * \delta_{g^{-1}} = \tilde{T}(a\delta_0) - a\delta_g * \tilde{T}(\delta_{g^{-1}}),\ a \in A \\ {}^a\chi(g) = \delta_g * \tilde{T}(\sigma(g^{-1}(a\delta_0) * \delta_{g^{-1}},\ a \in A \end{cases} \tag{69}$$

for $T \in Z^1(\mathfrak{A}, \mathfrak{X}_L^{*,A})$.

LEMMA 2. *For any $h \in \mathfrak{A}$ and $a \in A$, the three functions $\langle \chi(g), h\rangle$, $\langle \chi^a(g), h\rangle$ and $\langle {}^a\chi(g), h\rangle$ are in the space X_2 (see paragraph 7).*

Proof. These three functions are bounded by $\|T\|\,\|h\|_\infty$ and $\|T\|\,\|h\|_\infty\,|a|$ respectively. Moreover,

$$\begin{aligned} \langle \chi^a(g) - \chi^a(g'),\, h\rangle &\leqq |\langle \tilde{T}(a\delta_g) - \tilde{T}(a\delta_{g'}),\, g^{-1}.h\rangle| \\ &\quad + |\langle \tilde{T}(a\delta_{g'}),\, g^{-1}.h - g'^{-1}.h\rangle| \\ &\leqq |\langle \tilde{T}(a\delta_g) - \tilde{T}(a\delta_{g'}),\, g^{-1}.h\rangle| + \|T\|\,|a|\,\|g^{-1}.h - g'^{-1}.h\|_\infty \\ &\leqq 2\varepsilon + |\langle T(a\delta_g * e_\alpha\lambda_\beta) - T(a\delta_{g'} * e_\alpha\lambda_\beta)\rangle| + \|T\|\,|a|\,\|g^{-1}.h - g'^{-1}.h\|_\infty \end{aligned}$$

for α and β large enough, and less than 4ε for g' in a suitable neighborhood of g. The proof for $\chi(g)$ and ${}^a\chi(g)$ would be similar.

PROPOSITION 10. *Let us assume there exists a σ-left invariant A (resp. A'')-mean on X_2 (for instance G ameanable). The formulas*

$$\begin{cases} \langle \mu_L, h\rangle = M\{\langle \chi(g), h\rangle\} \\ \langle \mu_L^a, h\rangle = M\{\langle \chi^a(g), h\rangle\} \\ \langle {}^a\mu_L, h\rangle = M\{\langle {}^a\chi(g), h\rangle\} \end{cases} \tag{70}$$

define μ_L, μ_L^a and ${}^a\mu_L$ as elements of $\mathfrak{X}_L^{*,A}$ (resp. $\mathfrak{X}^{*,A''}$) related by

$$\begin{cases} \mu_L^a = \tilde{T}(a\delta_0) + a\delta_0 * \mu_L = \mu_L * a\delta_0 + {}^a\mu_L \\ {}^a\mu_L = \tilde{T}(a\delta_0) + a\delta_0 * \mu_L - \mu_L * a\delta_0 = \tilde{T}(a\delta_0) + \Delta\tilde{\mu}_L(a\delta_0)\ . \end{cases} \tag{71}$$

Proof. The right hand sides of formulas (70) have a meaning by Lemma 2. They are A-right linear in h and if (h_i) is a finite family in $K(G, A)$ with support $h_i \cap$ support $h_j = \emptyset$ for $i \neq j$ and $\|\sum_i h_i\|_\infty \leqq 1$, then

$$\sum_i |M\{\langle \chi(g), h_i\rangle\}| \leqq \sum_i \sup_g |\langle \chi(g), h_i\rangle| \leqq \sup_g \|\chi(g)\|_1 \leqq \|T\|$$

which proves that $\mu_L \in \mathfrak{X}_L^{*,A}$ (resp. $\mathfrak{X}^{*,L''}$) with $\|\mu_L\|_1 \leqq \|T\|$. In the same way μ_L^a and ${}^a\mu_L \in \mathfrak{X}_L^{*,A}$ (resp. $\mathfrak{X}^{*,A''}$) with norm less than $\|T\|\,|a|$. Formulas (71) comes from the equality between

$$\tilde{T}(a\delta_g)*\delta_{g^{-1}} = \tilde{T}(a\delta_0*\delta_g)*\delta_{g^{-1}} = \tilde{T}(a\delta_0) + a\delta_0*\tilde{T}(\delta_g)*\delta_{g^{-1}}$$

and

$$\begin{aligned}\tilde{T}(a\delta_g)*\delta_{g^{-1}} &= \tilde{T}(\delta_g*\delta_{g^{-1}}*a\delta_0*\delta_g)*\delta_{g^{-1}} \\ &= \tilde{T}(\delta_g)*\delta_{g^{-1}}*a\delta_0 + \delta_g*\tilde{T}(\sigma(g^{-1})a\delta_0)*\delta_{g^{-1}} .\end{aligned}$$

THEOREM 16. *Let* $T \in Z^1(\mathfrak{A}, \mathfrak{X}_L^{*,A})$ (*or* $Z^1(\mathfrak{A}, \mathfrak{X}^{*,A''})$) *and* M *a* σ*-left invariant* A (*or* A'')*-mean or* X_2 (*this is the case if* G *is ameanable*). *Then*

$$(72)\quad \begin{cases}\tilde{T}(a\delta_g) = \tilde{T}(a\delta_0)*\delta_g - a\delta_0*\Delta\tilde{\mu}_L(\delta_g) = -\widetilde{\Delta\mu}_L(a\delta_g) + {}^a\mu_L*\delta_g \\ \tilde{T}(\delta_g) = -\widetilde{\Delta\mu}_L(\delta_g) \\ \tilde{T}(a\delta_0) = -\widetilde{\Delta\mu}_L(a\delta) + {}^a\mu_L\end{cases}$$

and

$$(73)\quad \begin{cases}T(a\varphi) = \tilde{T}(a\delta_0)*\varphi - a\delta_0*\widetilde{\Delta\mu}_L(\varphi) = -\Delta\mu_L(a\varphi) + {}^a\mu_L*\varphi \\ \tilde{T}(\varphi) = -\widetilde{\Delta\mu}_L(\varphi) .\end{cases}$$

Moreover,

$$(74)\quad \sigma(g)a\widetilde{\Delta\mu}_L(\delta_g) - \tilde{T}(\sigma(g)a\delta_0)*\delta_g = \widetilde{\Delta\mu}_L(\delta_g)*a\delta_0 - \delta_g*\tilde{T}(a\delta_0)$$

or, equivalently

$$(75)\quad \sigma(g)^a\mu_L*\delta_g = \delta_g*{}^a\mu_L .$$

If $\tilde{T}$ *restricted to* A *is inner,* (i.e., *is a coboundary*), *then* T *is inner.*

Conversely, if $t \in Z^1(A, \mathfrak{X}_L^{*,A})$ (*or* $Z^1(A, \mathfrak{X}^{*,A''})$) *and* $\mu_L \in \mathfrak{X}_L^{*,A}$ (*or* $\mathfrak{X}^{*,A}$) *are related according to* (74) *or* (75) *where* ${}^a\mu_L$ *is defined by* (71), *then formulas* (72) *and* (73) *define an element* T *of* $Z^1(\mathfrak{A}, \mathfrak{X}_L^{*,A})$ (*or* $Z^1(\mathfrak{A}, \mathfrak{X}^{*,A''})$). *If* t *is inner then* T *is inner.*

Finally, if $T(\mathfrak{A}) \subset \mathfrak{A}$, *i.e., if* T *is a derivation on* $\mathfrak{A}$, *then* $\tilde{T}(A) \subset M_1(G, A)$ *and* $\mu_L \in \mathfrak{X}^{*,A} \cap M_1(G, M(A))$, *and conversely.*

Proof.

$$\begin{aligned}\langle a\delta_g*\mu_L, h\rangle &= a\langle \delta_g*\mu_L, h\rangle = a\langle {}^g\mu_L, h.g\rangle \\ &= a\sigma(g)\langle \mu_L, \sigma(g^{-1})h.g\rangle = a\sigma(g)M\{\langle \chi(u), \sigma(g^{-1})h.g\rangle\} \\ &= aM\{\sigma(g)\langle \chi(g^{-1}u), \sigma(g^{-1})h.g\rangle\} \\ &= M\{a\langle {}^g\chi(g^{-1}u), h.g\rangle\} = M\{a\langle g.\chi(g^{-1}u), h\rangle\} \\ &= M\{\langle a\delta_g*\tilde{T}(\delta_{g^{-1}u})*\delta_{u^{-1}g}, h\rangle\} \\ &= M\{\langle \tilde{T}(a\delta_u)*\delta_{u^{-1}}, g.h\rangle\} - M\{\langle \tilde{T}(a\delta_g), h\rangle\} \\ &= \langle \mu_L^a, g.h\rangle - \langle \tilde{T}(a\delta_g), h\rangle .\end{aligned}$$

So $\tilde{T}(a\delta_g) = \mu_L^a*\delta_g - a\delta_g*\mu_L$ which gives (72) with the help of (71). Formulas (73) are then coming from $\langle T(a\varphi), h\rangle = \int\varphi(g)\langle \tilde{T}(a\delta_g),h\rangle dg$. We obtain formula (74) from the equality between

$$\tilde{T}(\sigma(g)a\delta_g) = \tilde{T}(\sigma(g)a\delta_0*\delta_g) = \tilde{T}(\sigma(g)a\delta_0)*\delta_g + \sigma(g)a\delta_0*\tilde{T}(\delta_g)$$

and

$$\tilde{T}(\sigma(g)a\delta_g) = \tilde{T}(\delta_g*a\delta_0) = \delta_g*\tilde{T}(a\delta_0) + \tilde{T}(\delta_0)*a\delta_0$$

equality which gives also (75) with the help of (71).

If $\tilde{T}(a\delta_0) = -\widetilde{\Delta\mu'}(a\delta_0)$, then $\tilde{T} + \widetilde{\Delta\mu'}$ is zero on A and the corresponding ${}^a\mu'_L$ is zero, so $\tilde{T} + \widetilde{\Delta\mu'} = -\widetilde{\Delta\mu'_L}$ or $T = -\Delta(\mu' + \mu'_L)$.

Conversely, given t and μ_L, let $\tilde{T}(a\delta_g) = t(a\delta_0)*\delta_g - a\delta_0*\widetilde{\Delta\mu_L}(\delta_g)$. Then

$$\begin{aligned}\tilde{T}(a\delta_g*b\delta_{g'}) &= \tilde{T}(a\sigma(g)b\delta_{gg'}) \\ &= t(a\sigma(g)b\delta_0)*\delta_{gg'} - a\sigma(g)b\delta_0*\widetilde{\Delta\mu_L}(\delta_{gg'}) \\ &= t(a\delta_0*\sigma(g)b\delta_0)*\delta_{gg'} - a\delta_g*b\delta_{g^{-1}}*\widetilde{\Delta\mu_L}(\delta_{gg'}) \\ &= t(a\delta_0)*\sigma(g)b\delta_{gg'} + a\delta_0*t(\sigma(g)b\delta_0)*\delta_{gg'} - a\delta_g*b\delta_{g^{-1}}*\widetilde{\Delta\mu_L}(\delta_{gg'}) \\ &= t(a\delta_0)*\delta_g*b\delta_{g'} + a\delta_g*\delta_{g^{-1}}*t(\sigma(g)b\delta_0)*\delta_{gg'} - a\delta_g*b\delta_{g^{-1}}*\widetilde{\Delta\mu_L}(\delta_g)*\delta_{g'} \\ &\qquad - a\delta_g*b\delta_{g^{-1}}*\delta_g*\widetilde{\Delta\mu_L}(\delta_{g'}) \\ &= t(a\delta_0)*\delta_g*b\delta_{g'} - a\delta_g*b\delta_{g^{-1}}*\widetilde{\Delta\mu_L}(\delta_g)*\delta_{g'} \\ &\qquad + a\delta_g*\{\delta_{g^{-1}}*t(\sigma(g)b\delta_0)*\delta_{gg'} - b\delta_0*\widetilde{\Delta\mu_L}(\delta_{g'})\} \\ &= \{t(a\delta_0)*\delta_g*b\delta_{g'} - a\sigma(g)b\delta_0*\widetilde{\Delta\mu_L}(\delta_g)*\delta_{g'} \\ &\qquad + a\delta_g*\{\delta_{g^{-1}}*t(\sigma(g)b\delta_0)*\delta_{gg'} - b\delta_0*\widetilde{\Delta\mu_L}(\delta_{g'})\}\end{aligned}$$

which, thanks to (74), gives

$$\begin{aligned}t(a\delta_0)*\delta_g*b\delta_{g'} - a\delta_0*\widetilde{\Delta\mu_L}(\delta_g)*b\delta_{g'} + a\delta_g*\{t(b\delta_0)*\delta_{g'} - b\delta_0*\widetilde{\Delta\mu_L}(g')\} \\ = \tilde{T}(a\delta_g)*b\delta_{g'} + a\delta_g*\tilde{T}(b\delta_{g'}) .\end{aligned}$$

Finally, if $T(a\varphi) \in \mathfrak{A}$ for any a, φ, this is equivalent to $\tilde{T}(a\delta_0)*\varphi$, $a\delta_0*\mu_L*\varphi$ and $a\varphi*\mu_L$ be in $\mathfrak{A}$ for any a and φ. The last condition means that $\mu_L \in \mathfrak{X}^{*,A}$ instead of $\mathfrak{X}_L^{*,A}$. If $\mu_L \in \mathfrak{X}_L^{*,A}$ and $\varphi \in L^1(G)$, it is easy to see that $\mu_L * \varphi \in L^1(G, M_L(A))$. If $\varphi = \xi_B$, the characteristic function of some Borel $B \in B(G)$, the first condition reads $\{\tilde{T}(a\delta_0)*\varphi\}(g) = \tilde{T}(a\delta_0)(gB^{-1}) \in A$ or else $\tilde{T}(a\delta_0) \in M_1(G, A)$. The second condition reads $a\mu_L(g^{-1}B) \in A$, which means that $\mu_L \in M_1(G, M(A))$.

If G is discrete, this result is very close to ([17, Theorem 1]).

DEFINITION. $T \in Z^1(\mathfrak{A}, \mathfrak{X}_L^{*,A})$ will be called special if $\tilde{T}(A) \subset A$, i.e., $\tilde{T}(a\delta_0) = t(a)\delta_0$ where t is a derivation on A.

THEOREM 17. *Let $T \in Z^1(\mathfrak{A}, \mathfrak{X}_L^{*,A})$ be special. Then T is equivalent to the couple (t, μ_L) where t is a derivation on A and μ_L an element of $\mathfrak{X}_L^{*,A}$ related by* (74) *or* (75) *(where $\tilde{T}(a\delta_0) = t(a)\delta_0$). If t is inner, then T is inner. Finally $T(\mathfrak{A}) \subset \mathfrak{A}$, i.e., T is a special derivation on $\mathfrak{A}$, if $\mu_L \in \mathfrak{X}^{*,A} \cap M_1(G, M(A))$ and conversely. If $T \in Z^1(\mathfrak{A}, \mathfrak{X}^{*,A''})$ is special, $\mu_L \in \mathfrak{X}^{*,A''}$ and conversely.*

Proof. It is an adaptation of Theorem 16, noticing that condition $\tilde{T}(a\delta_0) \in M_1(G, A)$ is now automatic.

THEOREM 18. *If T is such that $\delta_g * \tilde{T}(\sigma(g^{-1})a\delta_0)*\delta_{g^{-1}} = \tilde{T}(a\delta_0)$ (which, in the case of a special T means that $t(\sigma(g)a) = (\sigma(g)t(a))$, then ${}^a\mu_L = \tilde{T}(a\delta_0)$ or, equivalently, $a\delta_0 * \mu_L = \mu_L * a\delta_0$* (i.e., $\widetilde{\Delta\mu_L}(a\delta_0) = 0$), *and conversely.*

Proof. It is a straightforward application of Theorem 16.

2nd part: The "dual" structure and crossed homomorphisms. All what has been done in the first part can be adapted to the "dual" structure. An element T of $Z^1(\mathfrak{A}, \mathfrak{X}_L^{*,A})$ is now an affine and continuous application from $\mathfrak{A}$ to $\mathfrak{X}_L^{*,A}$ such that

$$T(f_1 * f_2) = T(f_1)*f_2 + T(f_2) . \tag{76}$$

Theorem 15 works in exactly the same way, proving the uniqueness of $\tilde{T} \in Z^1(\mathfrak{X}_L^{*,A}, \mathfrak{X}_L^{*,A})$ with

$$\tilde{T}(\mu_L * \nu_L) = \tilde{T}(\mu_L)*\nu_L + \tilde{T}(\nu_L) . \tag{77}$$

Substituting to the affine application T the linear one $T_1 = T - T(0)$, we can repeat all what has been done in the preceding part, the only change being a new definition of ${}^a\chi(g)$:

$${}^a\chi(g) = \tilde{T}(\sigma(g^{-1})a\delta_0)*\delta_{g^{-1}} \tag{78}$$

and the use of a left invariant mean instead of σ-left invariant one, so that now

$$(79)\qquad \begin{cases}\mu_L^a = \widetilde{T}(a\delta_0) + \mu_L = \mu_L * a\delta_0 + {}^a\mu_L \\ {}^a\mu_L = \widetilde{T}(a\delta_0) + \mu_L - \mu_L * a\delta_0 = \widetilde{T}(a\delta_0) + \widetilde{\Delta\mu}_L(a\delta_0)\end{cases}$$

and we obtain the equivalent of Theorem 16:

THEOREM 16^{bis}. *Let $T \in Z^1(\mathfrak{A}, \mathfrak{X}_L^{*,A})$ (or $Z^1(\mathfrak{A}, \mathfrak{X}^{*,A''})$) and M a left invariant A (or A'')-mean on X_2 (this is the case if G is ameanable). Then*

$$(80)\qquad \begin{cases}\widetilde{T}(a\delta_g) = \widetilde{T}(a\delta_0) * \delta_g - \widetilde{\Delta\mu}_L(\delta_g) = -\widetilde{\Delta\mu}_L(a\delta_g) + {}^a\mu_L * \delta_g \\ \widetilde{T}(\delta_g) = -\widetilde{\Delta\mu}_L(\delta_g) \\ \widetilde{T}(a\delta_0) = -\widetilde{\Delta\mu}_L(a\delta_0) + {}^a\mu_L\end{cases}$$

and

$$(81)\qquad \begin{cases}T(a\varphi) = \widetilde{T}(a\delta_0) * \varphi - \widetilde{\Delta\mu}_L(\varphi) = -\Delta\mu_L(a\varphi) + {}^a\mu_L * \varphi \\ \widetilde{T}(\varphi) = -\Delta\mu_L(\varphi)\,.\end{cases}$$

Moreover

$$(82)\qquad \widetilde{\Delta\mu}_L(\delta_g) - T(\sigma(g)a\delta_0) * \delta_g = \widetilde{\Delta\mu}_L(\delta_g) * a\delta_0 - \widetilde{T}(a\delta_0)$$

or, equivalently,

$$(83)\qquad {}^{\sigma(g)a}\mu_L * \delta_g = {}^a\mu_L\,.$$

If $\widetilde{T}$ restricted to A is inner (i.e., is a coboundary), then T is inner.

Conversely, if $t \in Z^1(A, \mathfrak{X}_L^{*,A})$ (or $Z^1(A, \mathfrak{X}^{*,A''})$) and $\mu_L \in \mathfrak{X}_L^{*,A}$ (or $\mathfrak{X}^{*,A''}$) are related according to (82) *or* (83), *then formulas* (80) *and* (81) *define an element $T \in Z^1(\mathfrak{A}, \mathfrak{X}_L^{*,A})$ (or $Z^1(\mathfrak{A}, \mathfrak{X}^{*,A''})$). If t is inner then T is inner.*

Finally, if $T(A) \subset \mathfrak{A}$, i.e., if T is crossed homomorphism on $\mathfrak{A}$, $\widetilde{T}(A) \subset M_1(G, A)$ and $\mu_L \in \mathfrak{A}$, and conversely.

DEFINITION. $T \in Z^1(\mathfrak{A}, \mathfrak{X}_L^{*,A})$ will be called special if $\widetilde{T}(A) \subset A$, i.e., $\widetilde{T}(a\delta_0) = t(a)\delta_0$ where t is a crossed homomorphism on A.

THEOREM 17^{bis}. *Let $T \in Z^1(\mathfrak{A}, \mathfrak{X}_L^{*,A})$ be special. Then T is equivalent to the couple (t, μ_L) where t is a crossed homomorphism on A and μ_L an element of $\mathfrak{X}_L^{*,A}$ related by* (82) *or* (83). *If t is inner, then T is inner. Finally $T(\mathfrak{A}) \subset \mathfrak{A}$* i.e., *$T$ is a special crossed homomorphism on $\mathfrak{A}$ if $\mu_L \in \mathfrak{A}$ and conversely. If $T \in Z^1(\mathfrak{A}\mathfrak{X}, {}^{*,A''})$ is special, $\mu_L \in \mathfrak{X}^{*,A''}$ and conversely.*

The equivalent of Theorem 18 gives now rather trivial results.

THEOREM 18^{bis}. *If* T *is such that* $\tilde{T}(\sigma(g^{-1})a\delta_0)*\delta_{g^{-1}} = \tilde{T}(a\delta_0)$ (*which in the case of a special crossed homomorphism means that* $t(\sigma(g)a)\delta_g = t(a)\delta_0$, i.e., $t = 0$) *then* ${}^a\mu_L = \tilde{T}(a\delta_0)$ *or, equivalently,* $\mu_L = \mu_L * a\delta_0$ *for any* a, i.e., $\mu_L = 0$, *and conversely.*

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