PACIFIC JOURNAL OF MATHEMATICS
Vol. 85, No. 1, 1979

INEQUALITIES INVOLVING DERIVATIVES

RAY REDHEFFER AND WOLFGANG WALTER

This paper deals with generalizations of classical results on real-valued functions of a real variable which are of the following type: *Bounds for the function and for its* mth *derivative imply bounds for the* kth *derivative* $0 < k < m$. **Our theorems extend these results in various directions, the most important being the extension to functions of** n **variables.**

(A) The Hadamard-Littlewood three-derivatives theorem states that if $u(t) = o(1)$ and $u''(t) = O(1)$ as $t \to \infty$, then $u'(t) = o(1)$. In Theorem 1, the more general version "$u(t) = o(1)$ and $u^{(m+1)}(t) = O(1)$ implies $u^{(k)}(t) = o(1)$ for $1 \leqq k \leqq m$" is generalized in three directions. The assumption that $u = o(1)$ is weakened, the functions considered are Banach-space valued, and the boundedness of $u^{(m+1)}$ is replaced by a condition on $u^{(m)}$ which is weaker than uniform continuity. A similar result for functions of several variables is given in Theorem 4.

(B) Let $u(t)$ be of class C^m in an unbounded interval J and let

$$U_k = \sup_{t \in J} |u^{(k)}(t)| .$$

Inequalities of the form

$$U_k \leqq A(m, k) U_0^{1-k/m} U_m^{k/m} , \qquad 0 \leqq k \leqq m ,$$

hold for such functions, as is well known. In Theorem 5 we extend these inequalities to Banach-space valued functions $u(x)$ defined in suitably restricted domains of R^n. Counterexamples show that the restrictions imposed on the domain are appropriate.

(C) If J is an interval of finite length $|J|$, the inequality (B) is no longer valid. (It can be saved by imposing homogeneous boundary conditions, but this will not be done here.) We shall show that an inequality

$$U_k \leqq A(m, k) U_0^{1-k/m} (U_m^*)^{k/m} , \qquad 0 \leqq k \leqq m ,$$

still holds, where

$$U_m^* = \max(U_0 |J|^{-m}, U_m) .$$

In Theorem 2 this result is presented for Banach-space valued functions in bounded or unbounded domains of R^n.

It is not our aim to obtain the best or even good constants. In the one-dimensional case, the problem of finding the optimal constants in the inequalities in (B) and (C) has a large literature. The complete solution for the case $J = R$ was given by Kolmogoroff (1939), for the case $J = R_+$ by Schoenberg and Cavaretta (1970). More information and biographic references with respect to the one-dimensional case can be found in the book by Mitrinović (1970, pp. 138–140) and in Kallman-Rota (1967).

The motivation for this research stems from certain problems in ordinary differerential equations, calculus of variations, and partial differential equations of parabolic type. Except for a simple example in the last section, such applications are not considered here.

2. Notation. Throughout this paper X denotes a real Banach space with dual X^*. The open ball in X with center at x_0 and radius r is denoted by

$$B(x_0, r) = \{x \in X : |x - x_0| < r\}$$

and its closure by $\bar{B}(x_0, r)$. As usual, the real line and Euclidean n space are denoted by R and R^n, respectively. (This notation was already used above.) We also set $R_+ = [0, \infty)$, and we denote various continuity classes by C^m; for example, $C^m(R, X)$ is the class of functions $R \to X$ with continuous mth derivatives. The letters m and k denote integers and θ and h denote real numbers, with

$$0 \leqq k \leqq m\,, \quad 0 < \theta < \frac{\pi}{2}\,, \quad h > 0\,.$$

Further notation is introduced as needed.

3. Functions of a real variable. In this section we prove a generalization of the Hadamard-Littlewood three-derivatives theorem.

DEFINITION 1. *For $v: R_+ \to X$ and $a \in X$ the equation*

$$\lim_{t\to\infty}{}^* v(t) = a$$

means that the outer Lebesgue measure of the set

$$M(t) = \{s \in [t, t+1] : |v(s) - a| > \varepsilon\}$$

converges to 0 as $t \to \infty$ for every $\varepsilon > 0$.

For convenience, we sometimes omit the subscript $t \to \infty$ in lim*.

It is easily seen that $\lim^* v(t)$ is unique if it exists; more generally, $\lim^* v(t) = \lim^* w(t)$ if v and w differ only on a set of finite measure. Also $\lim^*$ is linear, and $\lim v(t) = a$ implies $\lim^* v(t) = a$ though the converse is, of course, false.

DEFINITION 2. *The function $\omega(t)\colon R_+ \to R_+$ is said to be a modulus of continuity if ω is continuous and increasing and $\omega(0) = 0$.*

THEOREM 1. *Let $v \in C^m(R_+, X)$ satisfy $\lim^* v(t) = 0$ and the following hypothesis (C^m_ω):*

$$(C^m_\omega) = \begin{cases} \text{There exists a modulus of continuity } \omega \text{ such that} \\ \qquad |v^{(m)}(s) - v^{(m)}(t)| \leqq \omega(|s-t|)(1 + |v|_{m,s,t}) \\ \text{for } 0 \leqq s \leqq t \leqq s+1, \text{ where} \\ \qquad |v|_{m,s,t} = \sup |v^{(k)}(\tau)| \colon s \leqq \tau \leqq t \,, \quad 0 \leqq k \leqq m \,. \end{cases}$$

Then $\lim_{t\to\infty} v^{(k)}(t) = 0$, $0 \leqq k \leqq m$.

Proof. Let $h(t) = \max |v^{(k)}(t)|$ for $0 \leqq k \leqq m$ and assume that, contrary to the conclusion of the theorem,

$$h(t_i) \geqq \varepsilon > 0 \,, \quad t_{i+1} - t_i \geqq 2 \,, \quad t_i \longrightarrow \infty \,.$$

Let J_i be an interval around t_i of length 1 and choose $s_i \in J_i$ such that

$$M_i = \max(h(t) \,|\, J_i) = h(s_i) \geqq \varepsilon \,.$$

In what follows, i is fixed. For some k, $0 \leqq k \leqq m$, we have $|v^{(k)}(s_i)| = M_i$. Hence there exists $c \in X^*$, $|c| = 1$, such that $f(t) = c(v^{(k)}(t))$ satisfies

$$f(s_i) = |v^{(k)}(s_i)| = M_i \,.$$

If $k < m$, then

$$|f'(t)| = |c(v^{(k+1)}(t))| \leqq |v^{(k+1)}(t)| \leqq M_i \quad \text{in} \quad J_i \,,$$

hence

$$|f(t)| \geqq f(s_i) - |f(t) - f(s_i)| \geqq M_i - |t - s_i| M_i \geqq M_i/2 \,,$$

if $|t - s_i| \leqq 1/2$. Hence

$$f(t) \geqq \varepsilon/2 \quad \text{in} \quad J_i^* \subset J_i,\ |J_i^*| = 1/2 \,.$$

If $k = m$ then $f(t) = c(v^{(m)}(t))$ satisfies

$$|f(t)| \geqq f(s_i) - |f(t) - f(s_i)| \geqq M_i - \omega(|t - s_i|)(1 + M_i) \,.$$

Choose $\delta < 1/2$ such that $\omega(\delta) < \varepsilon/(2+2\varepsilon)$, and note that the latter expression is $\leqq M_i/(2+2M_i)$. We get

$$f(t) \geqq M_i/2 \geqq \varepsilon/2 \quad \text{in} \quad J_i^*, \quad \text{where} \quad |J_i^*| = \delta\,, \qquad J_i^* \subset J_i\,.$$

This statement holds for both cases $k < m$ and $k = m$, with $\delta > 0$ independent of i.

Now we use the following lemma which was given by Redheffer (1974).

LEMMA 1. *Let ε and δ be positive constants, and let u be a real-valued function which satisfies $|u^{(k)}(t)| \geqq \varepsilon$ on an interval of length δ. Then*

$$|u(t)| \geqq \delta^k \varepsilon / 2^{k(k+1)} \quad \textit{on a subinterval of length} \quad \delta/4^k\,.$$

The function $g(t) = c(v(t))$ satisfies, according to Lemma 1,

$$2|g(t)| \geqq \varepsilon\delta^m/2^{m(m+1)} \quad \text{in} \quad J_i^{**} \subset J_i^*\,, \quad |J_i^{**}| = \delta/4^m\,.$$

Since $|v(t)| \geqq |g(t)|$, the last inequality holds also for $|v(t)|$, in contradiction to the hypothesis $\lim^* v(t) = 0$.

4. **Remarks.** The hypothesis $\lim^* |v(t)| = 0$ holds if $|v(t)| \leqq \rho(t)$ where $\rho(t)$ satisfies the corresponding condition for functions $R_+ \to R_+$. As seen in [1] the latter class contains all functions in L^p, $0 < p < \infty$, as well as functions with limit 0. Hence, Theorem 1 generalizes not only the three-derivatives theorem which forms the point of departure, but also a number of theorems due to Boas and others for functions satisfying various integrability conditions. We can even allow functions ρ satisfying

$$\lim_{t\to\infty} \int_t^{t+1} \varphi(|\rho(\tau)|)d\tau = 0$$

where φ is strictly increasing and $\varphi(0) = 0$; for example, $\varphi(\rho) = \exp(-1/\rho^2)$. Since the class of functions ρ satisfying $\lim^* \rho(t) = 0$ is closed under the formation of sums and products (cf. [1]) the hypothesis $\lim^* |v(t)| = 0$ of Theorem 1 is more general than appears at first glance.

If $v^{(m)}$ is absolutely continuous we have

$$v^{(m)}(s) - v^{(m)}(t) = \int_s^t v^{(m+1)}(\tau)d\tau$$

and the condition (C_ω^m) can be deduced from corresponding hypotheses on $v^{(m+1)}$. For example if $|v^{(m+1)}| \leqq K$ then (C_ω^m) holds with $\omega(t) = Kt$. The formulation of Theorem 1 has the advantage that the hypothesis does not involve derivatives of higher order than those in the conclusion.

It should be emphasized that the assumption $\lim^* v(t) = 0$ does not imply that v is bounded, and the assumption (C^m_ω) does not imply that v or any of its derivatives is bounded. For instance, if $X = R$, the function $v(t) = e^t$ satisfies (C^m_ω), as does every polynomial. It is true that both hypotheses together imply that v and its derivatives are bounded but this is part of the conclusion, not part of the hypothesis. We return to this matter in §10.

5. **Cones in R^n.** For $x, y \in R^n$, we use the customary notation $xy = x_1y_1 + \cdots + x_ny_n$, $x^2 = xx = |x|^2$. A cone $C(\theta, h)$ with vertex at 0, opening 2θ and height h is the set of all x satisfying $e_0x \geqq |x| \cos\theta$ and $|x| \leqq h$, where e_0 is a unit vector defining the axis direction of the cone. The reader is reminded that $h > 0$, $0 < \theta < \pi/2$, as stated in §2.

DEFINITION 3. *A set $G \subset R^n$ belongs to the class $K(\theta, h)$ if for each $x \in G$ there exists a cone $C(\theta, h)$ such that $x + C(\theta, h) \subset G$. The set G is said to satisfy a cone condition if $G \in K(\theta, h)$ for some θ, h.*

LEMMA 2. *All sets considered here are subsets of R^n.*
(i) *If sets belong to $K(\theta, h)$ so does their union.*
(ii) *If a set belongs to $K(\theta, h)$ so does its closure.*
(iii) *$C(\theta, h)$ belongs to $K(\theta, h/4)$ for small θ, say, $0 < \theta < \pi/8$.*

Proof. (i) and (ii) are easily proved. For the proof of (iii), we assume without loss of generality that $h = 1$. In what follows, e, e_0, e_1 are unit vectors and e_0 is the axis of the cone $C(\theta, 1)$. Let $x = te$, $0 \leqq t \leqq 1$, $ee_0 \geqq \cos\theta$, be an arbitrary point of the cone. If $0 \leqq t \leqq 3/4$, then $x + C(\theta, 1/4) \subset C(\theta, 1)$, where $C(\theta, 1/4)$ is the cone with the same axis e_0. Indeed, If $y = se_1$, $e_0e_1 \geqq \cos\theta$, $0 \leqq s \leqq 1/4$, is an arbitrary point in $C(\theta, 1/4)$, then $|x + y| \leqq 1$ and $(x + y)e_0 \geqq (s + t)\cos\theta \geqq |x + y| \cos\theta$.

Now, since $ee_0 = \cos\theta$ implies $|e - e_0| = 2\sin\theta/2$, $C(\theta, 1)$ is contained in the convex hull of $\{0\} \cup \bar{B}(e_0, 2\sin\theta/2)$, and a similar statement holds for cones of height h. The cone $C(\theta, 1)$ being convex, it suffices therefore to prove that for $x = te$, $ee_0 \geqq \cos\theta$, $3/4 \leqq t \leqq 1$, there exists a ball $B(a, 2h\sin\theta/2) \subset C(\theta, 1)$ satisfying $|a - x| = h \geqq 1/4$. We choose $a = se_0$, $s = 2t/3$. If $ee_0 = \cos\pi/8$, then $|e - (3/4)e_0| = d < .43$. Hence, for $ee_0 \geqq \cos\pi/8$ and $x = te$, $3/4 \leqq t \leqq 1$, there exists always a point $a = se_0$, $1/2 \leqq s \leqq 3/4$, such that $1/4 \leqq |x - a| \leqq d$. Since $B(a, 2d\sin\theta/2) \subset B(a, 1/2\sin\theta) \subset C(\theta, 1)$ (note that $1/2\sin\theta < 1/4$), part (iii) of the lemma is proved.

LEMMA 3. *Let G be an open subset of R^n which belongs to $K(\theta, h)$ with $\theta < \pi/8$ and let G_0 be a compact subset of G. Then there exists a compact set G_1 belonging to $K(\theta, h/4)$ such that $G_0 \subset G_1 \subset G$.*

Proof. Let $x \in G_0$ and let $C(\theta, h)$ be a cone with axis e_0 satisfying $x + C(\theta, h) \subset G$. Let $\delta > 0$ be chosen in such a way that the cone $C_x = x - \delta e_0 + C(\theta, h)$ is still contained in G; this is possible since $x + C(\theta, h)$ is a compact subset of G. Since $x \in \operatorname{int} C_x$, the sets $\operatorname{int} C_x$, where x runs through G_0, cover G_0. Hence a finite number of the sets C_x cover G_0. Their union has all the desired properties: it is a closed, bounded subset of G, and it belongs, by Lemma 2, to $K(\theta, h/4)$.

COROLLARY. *If G is an open set belonging to $K(\theta, h)$, where $\theta < \pi/8$, then there exists an increasing sequence of compact subsets of class $K(\theta, h/4)$ with union G.*

6. **Functions of n variables.** We use the notation

$$D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n} , \quad |\alpha| = \alpha_1 + \cdots + \alpha_n ,$$

where the α_i are nonnegative integers. For $u \in C^m(G, X)$ we define

$$U_k = \sup \{|D^\alpha u(x)|: |\alpha| = k, x \in G\} .$$

The following theorem is the n-dimensional version of the inequality quoted in (C).

THEOREM 2. *Let G be a bounded or unbounded, open subset of R^n belonging to $K(\theta, h)$, and let $u \in C^m(G, X)$, where $m \geqq 1$. There exists a constant $A = A(m, n, \theta)$ (independent of u, X and h) such that*

$$U_k \leqq A U_0^{1-k/m} (U_m^*)^{k/m} , \quad 0 \leqq k \leqq m ,$$

where

$$U_m^* = \max(U_m, U_0 h^{-m}) .$$

Proof. It suffices to establish the inequality for $h = 1$. Indeed, if $G \in K(\theta, h)$ and $u \in C^m(G, X)$, then the set $H = (1/h)G = \{x/h: x \in G\}$ is of class $K(\theta, 1)$, and $v(x) = u(hx) \in C^m(H, X)$. If V_k denotes the supremum of $|D^\alpha v(x)|$ for $|\alpha| = k$ and $x \in H$, and if the inequality

$$V_k \leqq A V_0^{1-k/m} [\max(V_m, V_0)]^{k/m}$$

is already established, then the inequality of the theorem follows immediately since $V_k = h^k U_k$.

For the sake of clarity we use $|\cdot|_e$ to denote the Euclidean distance in R^n in contrast to $|\cdot|$ which denotes the absolute value in R and the norm in X. We assume $m > 1$, $U_0 < \infty$, $U_m < \infty$; if one of these conditions fails, the result is trivial.

The case $m = 2$ is treated first. Let $y \in G$ be fixed, let $C = C(\theta, 1)$ be the cone belonging to y and let $c \in X^*$ with $|c| = 1$ and i be chosen in such a way that $|u_{x_j}(y)| \leqq |u_{x_i}(y)| = c(u_{x_i}(y))$ for $j = 1, \cdots, n$. Let $f(x) = c(u(x))$ and let $x = y + te$, $0 \leqq t \leqq 1$, $|e| = 1$, be a point in $y + C$, where $e \in C$ is chosen in such a way that $|f_x(y)e| \geqq |f_x(y)|_e \sin\theta$. (Here f_x denotes the gradient of f.) We have

$$|f(x) - f(y)| \leqq |u(x) - u(y)| \leqq 2U_0$$

and

$$f(x) - f(y) = (x - y)f_x(\xi) = (x - y)(f_x(y) + f_x(\xi) - f_x(y)) ,$$

where $\xi = y + \lambda te$, $0 < \lambda < 1$. Since

$$|f_{x_j}(\xi) - f_{x_j}(y)| \leqq t|\operatorname{grad} f_{x_j}|_e \leqq t\sqrt{n}\max_k |f_{x_j x_k}| \leqq t\sqrt{n}U_2 ,$$

hence $|f_x(\xi) - f_x(y)|_e \leqq tnU_2$, we obtain

$$\begin{aligned} 2U_0 \geqq |f(x) - f(y)| &\geqq |(x - y)f_x(y)| - |x - y|_e|f_x(\xi) - f_x(y)|_e \\ &\geqq t|f_x(y)|_e \sin\theta - t^2 nU_2 . \end{aligned}$$

Observing that

$$|f_x(y)|_e \geqq |f_{x_i}(y)| = |u_{x_i}(y)| \geqq |u_{x_j}(y)| , \quad j = 1, \cdots, n ,$$

we get

$$|u_{x_j}(y)| \sin\theta \leqq \frac{2}{t}U_0 + tnU_2 .$$

If $U_0 < U_2$, we choose $t = \sqrt{U_0/U_2}$, otherwise $t = 1$. Since y is an arbitrary point in G and j an arbitrary index, the inequality

$$U_1 \leqq A(U_0 \max(U_0, U_2))^{1/2} , \quad A = A(2, n, \theta) = \frac{2 + n}{\sin\theta}$$

follows.

The general case is proved by induction on m. We fix n and θ, write A_m for $A(m, n, \theta)$ and assume that the inequality of the theorem, which is denoted by (H_m), is true for the integer $m \geqq 2$. Let $u \in C^{m+1}(G, X)$ and assume for the moment that U_k is finite for $0 \leqq k \leqq m + 1$. To get (H_{m+1}) we distinguish three cases.

Case I. $U_m \leqq U_0$. Here (H_m) gives $U_k \leqq A_m U_0$ for $0 \leqq k \leqq m$. This gives (H_{m+1}) for any $A_{m+1} \geqq A_m$ (note that $A_m \geqq 1$).

Case II. $U_m > U_0$, $U_{m-1} > U_{m+1}$. By (H_2) and (H_m),

$$U_m \leqq A_2 U_{m-1} \quad \text{and} \quad U_{m-1} \leqq A_m U_0^{1/m} U_m^{(m-1)/m} ,$$

hence $U_m \leqq (A_2 A_m)^m U_0$, which is the case $k = m$ of (H_{m+1}). By (H_m) again,

$$U_k \leqq A_m U_0^{1-k/m} (A_2^m A_m^m U_0)^{k/m} , \quad 0 \leqq k \leqq m - 1 ,$$

hence (H_{m+1}) with $A_{m+1} \geqq (A_2 A_m)^m$.

Case III. $U_m > U_0$, $U_{m-1} \leqq U_{m+1}$. By (H_2) and (H_m),

$$U_m^2 \leqq A_2^2 U_{m-1} U_{m+1} \leqq A_2^2 U_{m+1} A_m U_0^{1/m} U_m^{(m-1)/m} ,$$

hence

$$U_m^{m+1} \leqq A_2^{2m} A_m^m U_0 U_{m+1}^m .$$

This is the case $k = m$ of (H_{m+1}). Using this relation and (H_m), we get

$$U_k \leqq A_m U_0^{1-k/m} U_m^{k/m} \leqq A_m U_0^{1-k/m} (A_2^{2m} A_m U_0 U_{m+1}^m)^{k/m(m+1)} .$$

This gives (H_{m+1}) for $0 \leqq k \leqq m$ and finishes the induction proof. An admissible constant A_{m+1} is given by $A_{m+1} = (A_2 A_m)^m$.

The additional assumption in the above proof that the U_k are finite can easily be disposed of. Let U_0 aud U_{m+1} be finite and let $C = x + C(\theta, h)$ be an arbitrary cone in G. Since C is a compact subset of G of class $K(\theta, h/4)$, inequality (H_{m+1}) holds with respect to C (and h replaced by $h/4$). This gives a bound for $|D^\alpha u|$, $|\alpha| \leqq m$, in C, which depends only on U_0 and U_{m+1}. Since C is arbitrary, it follows that the U_k are finite. (Alternatively, use §5, Corollary).

THEOREM 3. *Let $G \subset R^n$ be an open set of class $K(\theta, h)$, bounded or unbounded, and let $u \in C^m(G, X)$. In addition assume that u is bounded and that the following hypothesis (C_ω^m) holds:*

$$(C_\omega^m)\begin{cases} \text{There exists a modulus of continuity } \omega \text{ such that for } |\beta| = m \\ |D^\beta u(x) - D^\beta u(y)| \leqq \omega(|x - y|)(1 + |u|_{m,x,y}) \\ \text{whenever } \lambda x + (1 - \lambda)y \in G,\ |x - y| \leqq h,\ 0 \leqq \lambda \leqq 1, \text{ where} \\ |u|_{m,x,y} = \max |D^\alpha u(\lambda x + (1 - \lambda)y)| : 0 \leqq \lambda \leqq 1,\ |\alpha| \leqq m . \end{cases}$$

Then there exists a modulus of continuity $\delta(s)$ depending only on m, n, θ, h, ω (independent of u, X, $G \in K(\theta, h)$) such that

$$U_0 + U_1 + \cdots + U_m \leqq \delta(U_0) .$$

In particular, all U_k are finite.

Proof. We may assume without loss of generality that $m \geqq 1$, $h = 1$ and that all the U_k are finite; cf. the reasoning at the beginning and end of the proof of Theorem 2.

Let $k(1 \leqq k \leqq m)$, γ with $|\gamma| = k$ and $y \in G$ be fixed, and let $C = C(\theta, 1)$ be the cone belonging to y. There exists $c \in X^*$, $|c| = 1$ such that $f(x) = c(D^\gamma u(x))$ satisfies $f(y) = |D^\gamma u(y)|$. Let β be obtained from γ by replacing one index $\gamma_i > 0$ by $\gamma_i - 1$, thus $|\beta| = k - 1$, and let $g(x) = c(D^\beta u(x))$, i.e., $f = g_{x_i}$. There is a unit vector $e \in C$ satisfying $|e \cdot g_x(y)| \geqq |g_x(y)|_e \sin\theta$. For $x = y + te$, $0 \leqq t \leqq 1$,

$$\begin{aligned} 2U_{k-1} &\geqq |D^\beta u(x) - D^\beta u(y)| \geqq |g(x) - g(y)| \\ &= |(x - y)(g_x(y) + g_x(\xi) - g_x(y))| \\ &\geqq t \sin\theta |g_x(y)|_e - t|g_x(\xi) - g_x(y)|_e , \end{aligned}$$

where $\xi = y + \lambda te$, $0 < \lambda < 1$. We distinguish two cases

(i) $k < m$: $|g_x(\xi) - g_x(y)|_e \leqq tnU_{k+1}$

(ii) $k = m$: $|g_x(\xi) - g_x(y)|_e \leqq \sqrt{n}\omega(t)(1 + U)$, $U = U_0 + \cdots + U_m$

(cf. the proof of Theorem 2). Using $|D^\gamma u(y)| = |f(y)| = |g_{x_i}(y)| \leqq |g_x(y)|_e$, we obtain

(i) $t(\sin\theta)|D^\gamma u(y)| \leqq 2U_{k-1} + t^2 nU_{k+1}$

(ii) $t(\sin\theta)|D^\gamma u(y)| \leqq 2U_{k-1} + \sqrt{n}t\omega(t)(1 + U)$

in the two cases, respectively. Since $y \in G$ and γ with $|\gamma| = k$ are arbitrary, the left hand sides of these inequalities can be replaced by $U_k t \sin\theta$. Hence,

$$(1) \qquad \begin{aligned} U_k \sin\theta &\leqq \frac{2}{t} U_{k-1} + tnU_{k+1} \quad \text{for} \quad 1 \leqq k \leqq m - 1 , \\ U_m \sin\theta &\leqq \frac{2}{t} U_{m-1} + \sqrt{n}\omega(t)(1 + U) , \end{aligned}$$

where $0 \leqq t \leqq 1$. Let $V_k = U_k/(1 + U)$ and $t = \sqrt{V_{k-1}}$. This gives

$$V_k \leqq A\sqrt{V_{k-1}} , \quad A = \frac{2 + n}{\sin\theta} \quad (k = 1, \cdots, m - 1)$$
$$V_m \leqq A\omega(\sqrt{V_{m-1}}) ;$$

in the first case we used the fact that $V_{k+1} < 1$, in the second case we assumed $\omega(t) \geqq t$, which can be done without loss of generality. It follows from these inequalities that

$$(2) \qquad \begin{aligned} &V_k \leqq A_k V_0^{2^{-k}}, \ A_k = A^{2-2^{1-k}} \quad (0 \leqq k \leqq m - 1) \\ &V_m \leqq A\omega(BV_0^{2^{-m}}), \ B = A^{1-2^{1-m}} \quad (\omega(t) \geqq t) \end{aligned}$$

and hence that

$$V_0 + V_1 + \cdots + V_m = \frac{U}{1 + U} \leqq d(V_0) ,$$

where

$$d(s) = s + A_1 s^{1/2} + \cdots + A_{m-1} s^{2^{1-m}} + A\omega(Bs^{2^{-m}}) .$$

Let $\varepsilon > 0$ be such that $d(\varepsilon) = 1/2$. If $U_0 \leqq \varepsilon$, then

$$\frac{U}{1+U} \leqq d(V_0) \leqq d(U_0) \leqq \frac{1}{2} ,$$

hence $(1/2)U \leqq U/(1+U)$, which gives the desired inequality

$$U_0 + \cdots + U_m = U \leqq 2d(U_0) .$$

If $U_0 > \varepsilon$, let λ be defined by $\lambda U_0 = \varepsilon$. Since $\lambda < 1$, the function λu satisfies the assumptions of the theorem, i.e.,

$$\lambda(U_0 + \cdots + U_m) \leqq 2d(\lambda U_0) = 1 .$$

If δ is defined by

$$\delta(s) = \begin{cases} 2d(s) & \text{for} \quad 0 \leqq s \leqq \varepsilon \\ s/\varepsilon & \text{for} \quad s > \varepsilon , \end{cases}$$

then δ is a modulus of continuity satisfying

$$U_0 + \cdots + U_m \leqq \delta(U_0) .$$

This completes the proof.

7. **Remarks.** The hypothesis (C^m_ω) of Theorem 3 is required only when $x \in y + C$ where C is the cone belonging to y. Hence, by the mean-value theorem, we can replace this hypothesis by a condition on the next higher derivatives, $|D^\alpha u|$ with $|\alpha| = m + 1$. In particular, if these derivatives are bounded, the hypothesis holds with $\omega(s) =$ (const)s.

If we have a Hölder condition, $\omega(t) = Kt^\rho$ with $0 < \rho \leqq 1$, the choice $t = (V_{m-1})^{1/(1+\rho)}$ in (1) gives

$$(\sin\theta) V_m \leqq (2 + \sqrt{n}K)(V_{m-1})^{\rho/(1+\rho)} . \tag{3}$$

Using (2) with $k = m - 1$ for V_{m-1} in (3) we get an estimate of form

$$V_m \leqq (\text{const})\, V_0^\eta,\ \eta = 2^{1-m}\rho/(1+\rho) .$$

By (2) sharper estimates hold for V_k, $k \leqq m - 1$, and hence an estimate of the same form holds for the sum $V_0 + V_1 + \cdots + V_m$. Passing from V to U as in the proof of Theorem 3, we get the following corollary:

COROLLARY. *If u satisfies the conditions of Theorem* 3 *with*

$\omega(t) = Kt^\rho$, where K and ρ are constant, with $0 < \rho \leqq 1$, then there exists a constant L such that

$$U_0 + U_1 + \cdots + U_m \leqq (LU_0)^\eta \quad \text{for} \quad U_0 \leqq 1/L ,$$
$$U_0 + U_1 + \cdots + U_m \leqq LU_0 \quad \text{for} \quad U_0 > 1/L$$

where $\eta = 2^{1-m}\rho/(1 + \rho)$.

8. Two theorems for unbounded domains. First, we extend Theorem 1 to functions of n variables. Let v be a function $G \to X$ where G is an unbounded domain in R^n, let $a \in X$, and let $h > 0$ be constant. We write

$$\lim_{|x|\to\infty}{}^* v(x) = a$$

if the outer Lebesgue measure of the set

$$G(x) = \{y \in G: |y - x| < h, |v(y) - a| > \varepsilon\}$$

converges to 0 as $|x| \to \infty$ for every $\varepsilon > 0$. This definition is analogous to Definition 1.

THEOREM 4. *Let G be an unbounded open subset of R^n belonging to $K(\theta, h)$, and let u be a function in $C^m(G, X)$, $m \geqq 1$, which satisfies the condition (C^m_ω) of Theorem 3 and*

$$\lim_{|x|\to\infty}{}^* u(x) = 0 .$$

Then

$$\lim_{|x|\to\infty} D^\gamma u(x) = 0 \quad \text{for} \quad |\gamma| \leqq m .$$

Proof. Assume that U_0 is finite and that $u(x) \to 0$ as $|x| \to \infty$. Then U_k is finite for $0 \leqq k \leqq m$ according to Theorem 3. Now let G_r be the set of points in G such that $|x| > r$ and let G^*_r be the union of all cones $x + C(\theta, h)$ belonging to points in G_r. For large r we have seen that $|u|$ is small in G^*_r, hence the corresponding quantity U^*_0 computed relatively to G^*_r is small, and U^*_k is small by Theorem 3. This gives Theorem 4 when U_0 is finite and $\lim u(x) = 0$.

The assumption that $U_0 = \infty$ leads to a contradiction in the following way. Let $\delta(s)$ be the modulus of continuity corresponding to $\omega(t)$ and $h/4$, according to Theorem 3. The function $\delta(s)$ is linear for large s, say, $\delta(s) = Ks$ for $s \geqq K$ (cf. the proof of Theorem 3). Assume that $|u(y_p)| \geqq K$, $|y_p| \to \infty$ as $p \to \infty$. Then, with respect to the cone $C_p = y_p + C(\theta, h) \subset G$, which is of class $K(\theta, h/4)$, we have $U^*_1 \leqq KU^*_0$ where U^*_k is taken with respect to C_p. Hence

$$|u(x)| \geqq |u(x_p)| - |u(x) - u(x_p)| ,$$

where $x, x_p \in C_p$ and $|u(x_p)| = U_0^*$. If $|x - x_p| \leqq 1/(2\sqrt{n}K)$, we get

$$|u(x)| \geqq U_0^* - |x - x_p|\sqrt{n}\, U_1^* \geqq U_0^* \geqq K/2 ,$$

which contradicts $\lim^* u = 0$. Now that we have $U_1 < \infty$, a similar argument gives a contradiction if $|u(y_p)| \geqq K$ for any $K > 0$ as $|y_p| \to \infty$. This completes the proof of Theorem 4.

In the next theorem we assume that each point in G is the vertex of an infinite cone lying in G. A cone $C(\theta, \infty)$ with vertex at 0 is the set of all $x \in R^n$ satisfying $xe_0 \geqq |x| \cos\theta$, where e_0 is a fixed unit vector. The set $G \subset R^n$ belongs to $K(\theta, \infty)$ if to each $x \in G$ there corresponds a cone $C(\theta, \infty)$ such that $x + C(\theta, \infty) \subset G$.

THEOREM 5. *Let $G \subset R^n$ be an open, unbounded set belonging to $K(\theta, \infty)$, and let $u \in C^m(G, X)$. Then there exists a constant $A = A(m, n, \theta)$ (independent of u, X, $G \in K(\theta, \infty)$) such that*

$$U_k \leqq A U_0^{1-k/m} U_m^{k/m} \quad \text{for} \quad 0 \leqq k \leqq m .$$

In particular, all U_k are finite if U_0 and U_m are finite.

This follows immediately from Theorem 2 for $h \to \infty$.

9. Remarks and counterexamples. Let $X = R$ and $n = 2$. The function $u(x, y) = xy$, considered in G: $x > 1$, $0 < y < 1/x$, yields $U_0 = 1$, $U_1 = \infty$, $U_2 = 1$. Hence Theorems 2, 3 and 5 are not valid for $m = 2$ without a cone condition. An even simpler counterexample to Theorem 5, $m = 2$, is given by $u(x, y) = y$, $G = R \times (0, 1)$, $U_0 = 1$, $U_1 = 1$, $U_2 = 0$. The functions $u = xy^{m-1}$ and $u = y^{m-1}$, considered in the same regions, serve as counterexamples to Theorems 2, 3 and 5 for arbitrary $m \geqq 2$.

As an application to differential equations, consider the equation

$$u^{(m+1)}(t) = f(t, u, u', \cdots, u^{(m)}) \qquad (t > 0)$$

for $u: R_+ \to X$ and assume that $\lim^*_{t \to \infty} u(t) = a$ and

$$|f(t, z_0, \cdots, z_m)| \leqq L(1 + |z_0| + \cdots + |z_m|) .$$

It is easily seen that the function $v(t) = u(t) - a$ satisfies

$$|v^{(m)}(s) - v^{(m)}(t)| \leqq L|s - t| \max_{s \leqq \tau \leqq t} (1 + |a| + |v(\tau)| + \cdots + |v^{(m)}(\tau)|) .$$

Hence, by Theorem 1 with $\omega(s) = L(2 + |a| + m)s$

$$u(t) \longrightarrow a,\ u^{(k)}(t) \longrightarrow 0 \qquad (k = 1, \cdots, m) \quad \text{as} \quad t \longrightarrow \infty .$$

The behavior of $u^{(m+1)}(t)$ as $t \to \infty$ can now be determined by looking at the differential equation.

Other applications to ordinary and partial differential equations will be given elsewhere.

10. Interrelations among the theorems. It is evident that, in the original one-dimensional setting, the three statements in (A), (B), (C) are not independent of each other. Indeed, without considering the optimal constants, (B) follows from (C) by letting $|J| \to \infty$, and (A) follows from (B) or (C). In the same manner, Theorem 5, the n-dimensional analog of (B), follows from Theorem 2, the n-dimensional analog of (C). But it seems to be impossible to obtain Theorem 1, our generalized one-dimensional version of (A), from either (B) or (C), even if $\lim^* v(t) = 0$ is replaced by the sharper assumption $\lim_{t\to\infty} v(t) = 0$. It should be noted in this connection that assumption (C_ω^m) does not simply replace the boundedness of the derivative $v^{(m+1)}$ by the uniform continuity of $v^{(m)}$. Indeed, the modulus of continuity ω is multiplied by a factor which becomes large if v or one of its derivatives becomes large. These remarks apply also Theorem 4, the n-dimensional version of Theorem 1.

Theorem 3 states that all derivatives of u up to the mth order are small if u itself is small. The situation is similar to the one described above in connection with Theorem 1. If the $(m+1)$th derivatives are bounded, then the conclusion of Theorem 3 is a consequence of Theorem 2. The importance of Theorem 3 lies in the fact that the same conclusion follows from the much weaker assumption (C_ω^m) on the mth derivatives, which is the n-dimensional analog of the same assumption in Theorem 1.

Acknowledgment. At first we defined $\lim^* v(t) = a$ to mean that the outer Lebesgue measure of the set $\{t \in R_+: |v(t) - a| > \varepsilon$ is finite. The more general formulation given in Definition 1 is due to Professor P. Volkmann. The proof of Lemma 3 given here is due to Professor R. Lemmert; our proof was more difficult. The fact the results [1] should extend to functions $R_+ \to X$ was pointed out to one of us by Professor P. Hartman in 1975.

REFERENCES

1. Ray Redheffer, *A note on the Littlewood three-derivatives theorem*, J. London Math. Soc., (2), **9** (1974), 9-15.

2. D. S. Mitrinović, *Analytic Inequalities*, Springer-Verlag, Berlin-Heidelberg-New York, 1970.

3. A. Kolmogoroff, *On inequalities between upper bounds of consecutive derivatives of an arbitrary function defined on an infinite interval*, (Russian), Učen. Zap. Moskov.

Gos. Univ. Mat., **30** (1939), 3-16.
4. R. R. Kallman and G.-C. Rota, *On the inequality* $\|f'\|^2 < 4\|f\|\cdot\|f''\|$, Inequalities, II, Proc. Second Symposium, U. S. Air Force Academy, Colorado, 1967, Academic Press, New York, 1970, 187-192.
5. I. J. Schoenberg and A. Cavaretta, *Solution of Landau's Problem Concerning higher derivatives on a halfline*, MRCT S. R. 1050, Madison, Wis., 1970.

Received July 17, 1978. The first author was supported by NSF Grant MCS 77-03570.

UNIVERSITY OF CALIFORNIA
LOS ANGELES, CA 90024