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EMBEDDING LATTICES INTO LATTICES OF IDEALS

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A lattice L is transferable iff, whenever L can be embedded in the ideal lattice of a lattice M, then L can be embedded in M. This concept was introduced by the first author in 1965 who also proved in 1966 that in a transferable lattice there are no doubly reducible elements. In fact, he proved that every lattice can be embedded in the ideal lattice of a lattice containing no doubly reducible elements. In a recent paper of the first two authors, the idea emerged that one should study transferability via classes K of lattices with the property that every lattice is embeddable in the ideal lattice of a lattice in K. This approach was used to establish that transferable lattices are semi-distributive. This investigation is carried further in this paper. Our main result shows that every lattice can be embedded in the ideal lattice of a lattice satisfying the two semi-distributive properties and two variants of Whitman's condition.

1. Introduction. It was shown by G. Grätzer ([6], [7]) that every transferable lattice L satisfies the condition

(X) L has no doubly reducible element.

In fact, he proved a stronger result, namely, that every lattice can be embedded in the ideal lattice of a lattice satisfying (X).

In general, if (P) is a lattice-theoretic property which is preserved by sublattices and which satisfies the assertion

$\mathscr{E}(P)$: every lattice can be embedded in the ideal lattice of a lattice satisfying (P),

then (P) is a property of all transferable lattices. In addition to (X), properties of a lattice L for which this assertion is known to hold include

(SF) L is sectionally finite (that is, all principal ideals are finite);

($SD_\wedge$) for $a, b, c \in L$, $a \wedge b = a \wedge c$ implies that $a \wedge b = a \wedge (b \vee c)$;

($SD_\vee$) for $a, b, c \in L$, $a \vee b = a \vee c$ implies that $a \vee b = a \vee (b \wedge c)$.

That $\mathscr{E}(SF)$ holds is a consequence of P. M. Whitman's embedding theorem [10] and the observation that the partition lattice on a set S is isomorphic to the ideal lattice of the lattice of all finite partitions of S; that $\mathscr{E}(SD_\wedge)$ and $\mathscr{E}(SD_\vee)$ hold is the content of a recent paper of G. Grätzer and C. R. Platt [8].

Consider the properties

(W) for $a, b, c, d \in L$, $a \wedge b \leqq c \vee d$ implies that $[a \wedge b, c \vee d] \cap \{a, b, c, d\} \neq \varnothing$;

(W_l) for $a, b, c, d \in L$, $c \leqq a \wedge b \leqq c \vee d$ implies that $[a \wedge b, c \vee d] \cap \{a, b, c, d\} \neq \varnothing$;

(W_u) for $a, b, c, d \in L$, $a \wedge b \leqq c \vee d \leqq a$ implies that $[a \wedge b, c \vee d] \cap \{a, b, c, d\} \neq \varnothing$.

K. Baker and A. W. Hales [2] proved that if a lattice satisfies (W), then so does its ideal lattice. Hence $\mathscr{E}(W)$ fails; however, in this paper, we will show that $\mathscr{E}(W_l)$ and $\mathscr{E}(W_u)$ hold. In fact, we will prove that every lattice can be embedded in the ideal lattice of a lattice satisfying the four properties ($SD_\vee$), ($SD_\wedge$), (W_l), and (W_u) simultaneously. More succinctly, our main result is

THEOREM. *$\mathscr{E}((SD_\vee)$ & $(SD_\wedge)$ & (W_l) & $(W_u))$ holds.*

It follows from the theorem and the preceding remarks that every transferable lattice is sectionally finite and satisfies ($SD_\vee$), ($SD_\wedge$), (W_l), and (W_u). By a result of R. Antonius and I. Rival [1], we conclude:

COROLLARY. *Every transferable lattice satisfies* (W).

The proof of the theorem is contained in §2. In §3 we shall settle the truth or falsity of $\mathscr{E}(P)$ for most remaining combinations (P) of the above properties. In particular, it will be shown that $\mathscr{E}((SD_\vee)$ & (SF) & $(X))$ holds and that $\mathscr{E}((SF)$ & $(SD_\wedge))$ and $\mathscr{E}((SF)$ & $(W_u))$ fail. With these results, we can determine the status of $\mathscr{E}(P)$ for all but two combinations (P) of the properties (X), (SF), ($SD_\vee$), ($SD_\wedge$), (W_l), and (W_u). These two will be given at the end of the paper.

2. Proof of the theorem.

DEFINITION 1. Let L be a lattice and let $\langle a, b, c, d\rangle$ be an ordered quadruple of elements of L. Then we will say that

(i) $\langle a, b, c, d\rangle$ is a *(W_l)-failure* if $c \leqq a \wedge b \leqq c \vee d$ and $[a \wedge b, c \vee d] \cap \{a, b, c, d\} = \varnothing$;

(ii) $\langle a, b, c, d\rangle$ is a *(W_u)-failure* if $a \wedge b \leqq c \vee d \leqq a$ and $[a \wedge b, c \vee d] \cap \{a, b, c, d\} = \varnothing$;

(iii) $\langle a, b, c, d\rangle$ is an *$(SD_\wedge)$-failure* if $a \wedge b = a \wedge c = d$ and $a \wedge (b \vee c) \neq d$;

(iv) $\langle a, b, c, d\rangle$ is an *$(SD_\vee)$-failure* if $a \vee b = a \vee c = d$ and $a \vee (b \wedge c) \neq d$;

(v) $\langle a, b, c, d\rangle$ is a *failure* if it is any of the above four types of failures.

DEFINITION 2. Let L be a lattice, let $\langle a, b, c, d\rangle$ be a failure in L, and let φ be a homomorphism from a lattice M onto L. Then φ *repairs* $\langle a, b, c, d\rangle$, or $\langle a, b, c, d\rangle$ *is repaired in* M *by* φ, iff $\langle a', b', c', d'\rangle$ is never a failure in M of the same type as $\langle a, b, c, d\rangle$, for any $a' \in \varphi^{-1}(a)$, $b' \in \varphi^{-1}(b)$, $c' \in \varphi^{-1}(c)$, and $d' \in \varphi^{-1}(d)$.

LEMMA 3. *Let K, L, and M be lattices, let $\varphi_1\colon M \to L$ and $\varphi_2\colon L \to K$ be onto homomorphisms, and let $\langle a, b, c, d\rangle$ be a failure in K. If $\langle a, b, c, d\rangle$ is repaired in L by φ_2, then it is repaired in M by $\varphi_2 \circ \varphi_1$.*

Proof. Each of the four conditions $(SD_\vee)$, $(SD_\wedge)$, (W_l), and (W_u) can be expressed in the form $P(x, y, z, w) \Rightarrow Q(x, y, z, w)$, where P and Q are disjunctions of polynomial equations and hence are preserved under homomorphisms. Since $\langle a, b, c, d\rangle$ is a failure in K, there exist appropriate P and Q such that $P(a, b, c, d)$ holds but $Q(a, b, c, d)$ fails. Suppose that $\langle a, b, c, d\rangle$ is not repaired in M by $\varphi_2 \circ \varphi_1$; then there are elements $a', b', c', d' \in M$ such that $(\varphi_2 \circ \varphi_1)(x') = x$ for $x \in \{a, b, c, d\}$, $P(a', b', c', d')$ holds, and $Q(a', b', c', d')$ fails. Consequently, $P(\varphi_1(a'), \varphi_1(b'), \varphi_1(c'), \varphi_1(d'))$ holds in L. Since $\langle a, b, c, d\rangle$ is repaired in L by φ_2, this implies that $Q(\varphi_1(a'), \varphi_1(b'), \varphi_1(c'), \varphi_1(d'))$ holds in L. But now $Q(a, b, c, d) = Q(\varphi_2(\varphi_1(a')), \varphi_2(\varphi_1(b')), \varphi_2(\varphi_1(c')), \varphi_2(\varphi_1(d')))$ holds in K, a contradiction.

Part of the proof of our theorem involves showing how to repair all failures in a lattice. Before describing the constructions by which this is accomplished, we make some observations.

Denote the lattice of ideals of a lattice L by $\mathscr{I}(L)$. Let L and K be lattices and let φ be a homomorphism of L onto K. For $I \in \mathscr{I}(K)$, consider the set

$$\varphi^{-1}(I) = \{x \in L \mid \varphi(x) \in I\} .$$

$\varphi^{-1}(I)$ is an ideal of L, and hence φ^{-1} is a map of $\mathscr{I}(K)$ into $\mathscr{I}(L)$

which is easily seen to be order preserving and one-to-one. Moreover, since meets of ideals are defined by set intersection, φ^{-1} is also meet preserving.

LEMMA 4. *The map* $\varphi^{-1}: \mathscr{I}(K) \to \mathscr{I}(L)$ *is an embedding if and only if* φ *satisfies the condition*

(*) *if* $y \in L$, $x_1, x_2 \in K$, *and* $\varphi(y) \leqq x_1 \vee x_2$, *then* $y \leqq y_1 \vee y_2$ *for some* $y_1, y_2 \in L$ *satisfying* $\varphi(y_1) = x_1$, $\varphi(y_2) = x_2$.

Proof. To prove the "if" direction, by the above remarks we need only show that for $I, J \in \mathscr{I}(K)$, $\varphi^{-1}(I \vee J) \subseteqq \varphi^{-1}(I) \vee \varphi^{-1}(J)$. Let $x \in \varphi^{-1}(I \vee J)$; then $\varphi(x) \in I \vee J$, so there exist $x_1 \in I$, $x_2 \in J$ such that $\varphi(x) \leqq x_1 \vee x_2$. By (*) there are $y_1, y_2 \in L$ such that $\varphi(y_1) = x_1$, $\varphi(y_2) = x_2$, and $x \leqq y_1 \vee y_2$. But then $y_1 \in \varphi^{-1}(I)$, $y_2 \in \varphi^{-1}(J)$, so $x \in \varphi^{-1}(I) \vee \varphi^{-1}(J)$, as desired.

Conversely, suppose that φ^{-1} is an embedding, and let $y \in L$ and $x_1, x_2 \in K$ be such that $\varphi(y) \leqq x_1 \vee x_2$. Then $(\varphi(y)] \leqq (x_1] \vee (x_2]$ in $\mathscr{I}(K)$, and since φ^{-1} is join preserving we have that $y \in \varphi^{-1}((\varphi(y)]) \subseteqq \varphi^{-1}((x_1]) \vee \varphi^{-1}((x_2])$. Thus there exist $y_1 \in \varphi^{-1}((x_1])$, $y_2 \in \varphi^{-1}((x_2])$ such that $y \leqq y_1 \vee y_2$. Clearly we may assume that $\varphi(y_1) = x_1$ and $\varphi(y_2) = x_2$.

The next three propositions allow us to repair all failures in a lattice. The constructions used in these results are slight modifications of constructions that have appeared elsewhere; that of Proposition 5 is taken from Theorem 4.4 of H. S. Gaskill, G. Grätzer, and C. R. Platt [5], and that of Propositions 6 and 7 is taken from Theorem 3.1 of T. G. Kucera and B. Sands [9]. We have included Figures 1 and 2 to illustrate the constructions in Propositions 5 and 6 respectively.

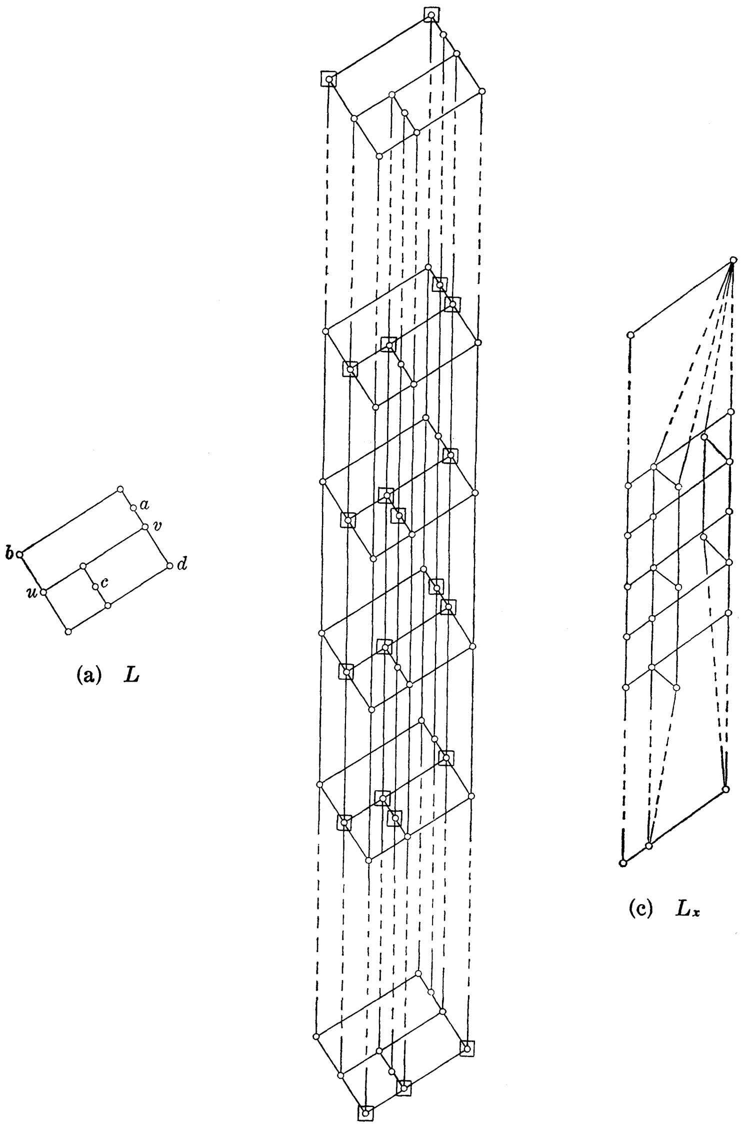
PROPOSITION 5. *Let* $\boldsymbol{x} = \langle a, b, c, d \rangle$ *be a failure of* (W_l) *or* (W_u) *in the lattice* L. *There exists a lattice* L_x *and a homomorphism* φ_x *of* L_x *onto* L *satisfying* (*) *such that* $\boldsymbol{x}$ *is repaired in* L_x *by* φ_x.

REMARK. One method of repairing failures of (W_l) or (W_u) is already in the literature; namely, the "interval construction" of A. Day [3]. However, it will be crucial for the proof of our main theorem that the homomorphisms we use to repair failures satisfy (*), and it is easy to verify that the homomorphism associated with the interval construction does not enjoy this necessary property.

Proof of Proposition 5. Let $\boldsymbol{Z}$ be the integers with their natural order and let $\boldsymbol{E}$ and $\boldsymbol{O}$ denote the sets of even and odd integers, respectively. Extend Z to $Z_b = Z \cup \{-\infty, \infty\}$ where $-\infty$ is the least and ∞ the greatest element of Z_b. Setting $u = a \wedge b$ and $v = c \vee d$, we define a subset L_x of $L \times Z_b$ by

$$L_x = ([b) \times \{\infty\}) \cup ((d] \times \{-\infty\}) \cup (((v] - (d]) \times E) \\ \cup ((((L - (v]) \cup [u)) - [b)) \times O) .$$

(Figure 1(c) shows L_x for the case when L is the lattice of Figure 1(a) and $x = \langle a, b, c, d \rangle$. Figure 1(b) shows L_x as a subset of $L \times Z_b$.) It is not hard to verify that each element of $L \times Z_b$ that



(a) L

(c) L_x

(b) $L \times Z_b$

FIGURE 1

is not of the form $\langle y, -\infty\rangle$ for $y \nleqq d$ has a least upper bound in L_x. This and a dual observation shows that L_x, with the partial order inherited from $L \times \boldsymbol{Z}_b$, is a lattice. Also, the projection $\pi_1: L \times \boldsymbol{Z}_b \to L$ restricts to a homomorphism φ_x of L_x onto L.

We first claim that φ_x satisfies (*). Let $\langle y, t\rangle \in L_x$ and $x_1, x_2 \in L$ be such that $y = \varphi_x(\langle y, t\rangle) \leqq x_1 \vee x_2$. There exist $i, j \in \boldsymbol{Z}_b$ such that $y_1 = \langle x_1, i\rangle$ and $y_2 = \langle x_2, j\rangle$ are in L_x. If $y \leqq d$ then $t = -\infty$, so $\langle y, t\rangle \leqq y_1 \vee y_2$, as desired. Also, if $x_1 \vee x_2 \geqq b$ then $y_1 \vee y_2 = \langle x_1 \vee x_2, \infty\rangle$, whence again $\langle y, t\rangle \leqq y_1 \vee y_2$. Thus we may assume $y \nleqq d$ and $x_1 \vee x_2 \ngeqq b$, and without loss of generality we have both y and x_1 in $L - ((d] \cup [b))$, implying that $t, i \in Z$. Since, for any $x \in L$ and $n \in Z$, $\langle x, n\rangle \in L_x$ implies $\langle x, n \pm 2\rangle \in L_x$, we may choose $i \geqq t$, and so $\langle y, t\rangle \in y_1 \vee y_2$ holds in any case, showing that φ_x satisfies (*).

Secondly, we show that $\boldsymbol{x}$ is repaired in L_x by φ_x. Let $a', b', c', d' \in L_x$ be such that $\varphi_x(a') = a$, $\varphi_x(b') = b$, $\varphi_x(c') = c$, and $\varphi_x(d') = d$. It follows that $a' = \langle a, i\rangle$ where $i \in \boldsymbol{O}$, $c' = \langle c, j\rangle$ where $j \in \boldsymbol{E}$, $b' = \langle b, \infty\rangle$, and $d' = \langle d, -\infty\rangle$. Thus $a' \wedge b' = \langle a \wedge b, i\rangle$ and $c' \vee d' = \langle c \vee d, j\rangle$, whence if $a' \wedge b' \leqq c' \vee d'$ we have $i \leqq j$. If $\boldsymbol{x}$ is failure of (W_l), assume that $\langle a', b', c', d'\rangle$ is a failure of (W_l) in L_x; then $c' \leqq a' \wedge b'$, yielding $j \leqq i$ and thus $i = j$, which is impossible since i is odd and j is even. Thus $\langle a', b', c', d'\rangle$ cannot be a failure of (W_l). Similarly, if $\boldsymbol{x}$ is a failure of (W_u), $\langle a', b', c', d'\rangle$ is not a failure of (W_u) in L_x. Hence $\boldsymbol{x}$ is repaired in L_x by φ_x.

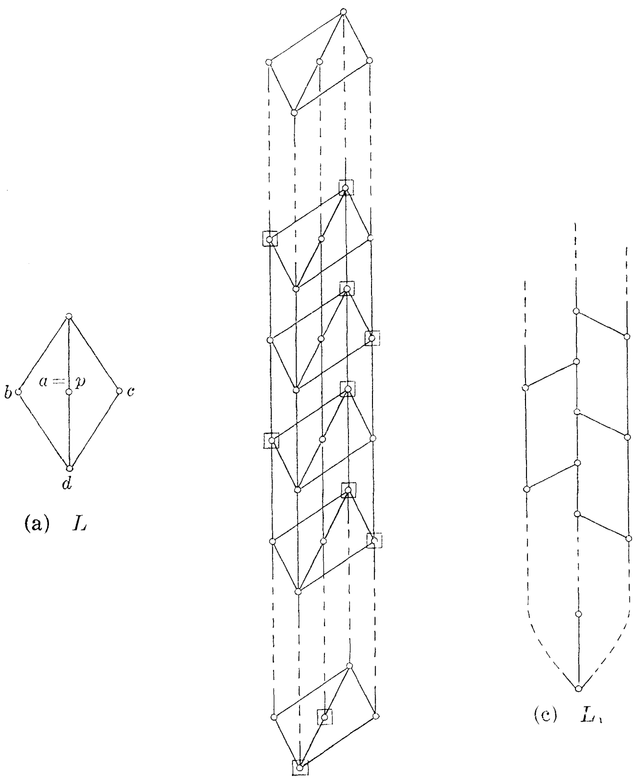
PROPOSITION 6. *Let $\boldsymbol{x} = \langle a, b, c, d\rangle$ be a failure of $(SD_\vee)$ in the lattice L. There exists a lattice L_x and a homomorphism φ_x of L_x onto L satisfying (*) such that $\boldsymbol{x}$ is repaired in L_x by φ_x.*

Proof. Let $\boldsymbol{Z}_b$, $\boldsymbol{E}$, and $\boldsymbol{O}$ be as in Proposition 5, and set $p = a \vee (b \wedge c)$. Define a subset L_x of $L \times \boldsymbol{Z}_b$ by

$$L_x = ((p] \times \{-\infty\}) \cup ((L - ((p] \cup (b])) \times \boldsymbol{E}) \cup ((L - ((p] \cup (c])) \times \boldsymbol{O}) .$$

Figure 2(c) shows L_x when L in the lattice of 2(a) and $\boldsymbol{x} = \langle a, b, c, d\rangle$, and Figure 2(b) shows L_x as a subset of $L \times \boldsymbol{Z}_b$. It is easy to see that L_x is a join-semilattice, and that each element of $L \times \boldsymbol{Z}_b$ that is not of the form $\langle y, \infty\rangle$ has a greatest lower bound in L_x; whence L_x, with the partial order inherited from $L \times \boldsymbol{Z}_b$, is a lattice. Furthermore, the projection $\pi_1: L \times \boldsymbol{Z}_b \to L$ again restricts to a homomorphism φ_x of L_x onto L.

To show that φ_x satisfies (*), let $\langle y, t\rangle \in L_x$ and $x_1, x_2 \in L$ be such that $y = \varphi_x(\langle y, t\rangle) \leqq x_1 \vee x_2$. There exist $i, j \in \boldsymbol{Z}_b$ such that $y_1 = \langle x_1, i\rangle$ and $y_2 = \langle x_2, j\rangle$ are in L_x. If $y \leqq p$, then $t = -\infty$, so $\langle y, t\rangle \leqq$



(a) L (b) $L \times \boldsymbol{Z}_b$ (c) L_x

FIGURE 2

$y_1 \vee y_2$. On the other hand, if $y \not\leqq p$, then without loss of generality we may let $x_1 \not\leqq p$, and both t and i are in $\boldsymbol{Z}$. We may now choose i such that $\langle x_1, i\rangle \in L_x$ and $t \leqq i$, and thus $\langle y, t\rangle \leqq y_1 \vee y_2$ follows in either case, proving that φ_x satisfies (*).

To show that $\boldsymbol{x}$ is repaired in L_x by φ_x, let $a', b', c', d' \in L_x$ be such that $\varphi_x(a') = a$, $\varphi_x(b') = b$, $\varphi_x(c') = c$, and $\varphi_x(d') = d$. Then $b' = \langle b, i\rangle$ where $i \in \boldsymbol{O}$, $c' = \langle c, j\rangle$ where $j \in \boldsymbol{E}$, and since $a \leqq p$, $a' = \langle a, -\infty\rangle$. Thus $a' \vee b' = \langle a \vee b, i\rangle$ and $a' \vee c' = \langle a \vee c, j\rangle$. Since i is odd and j is even, $a' \vee b' \neq a' \vee c'$, so $\langle a', b', c', d'\rangle$ is not a failure of $(SD_\vee)$ in L_x.

PROPOSITION 7. *Let $\boldsymbol{x} = \langle a, b, c, d\rangle$ be a failure of $(SD_\wedge)$ in the lattice $\boldsymbol{L}$. There exists a lattice L_x and a homomorphism φ_x of L_x onto L satisfying (*) such that $\boldsymbol{x}$ is repaired in L_x by φ_x.*

Proof. Let $p = a \wedge (b \vee c)$, and define a subset L_x of $L \times \boldsymbol{Z}_b$ by

$$L_x = ([p) \times \{\infty\}) \cup ((\boldsymbol{L} - ([p) \cup [b))) \times \boldsymbol{E})$$
$$\cup ((\boldsymbol{L} - ([p) \cup [c))) \times \boldsymbol{O}) .$$

This construction is just the dual of the one in the previous proposition, so L_x is a lattice and we have the natural homomorphism φ_x of L_x onto L. An argument dual to that in Proposition 6 shows that x is repaired in L_x by φ_x, so we need only show that φ_x satisfies (*). Let $\langle y, t\rangle \in L_x$ and $x_1, x_2 \in L$ be such that $\varphi_x(\langle y, t\rangle) \leqq x_1 \vee x_2$. There exist $i, j \in Z_b$ such that $y_1 = \langle x_1, i\rangle$ and $y_2 = \langle x_2, j\rangle$ are in L_x. If $x_1 \vee x_2 \geqq p$ then $y_1 \vee y_2 = \langle x_1, i\rangle \vee \langle x_2, j\rangle = \langle x_1 \vee x_2, \infty\rangle \geqq \langle y, t\rangle$; therefore, we assume $x_1 \vee x_2 \ngeqq p$, which implies that $t, i, j \in \boldsymbol{Z}$. Now we can choose $i \in \boldsymbol{Z}$ such that $\langle x_1, i\rangle \in L_x$ and $i \geqq t$, whence $\langle y, t\rangle \leqq y_1 \vee y_2$ follows.

Before continuing with the proof of the theorem, we recall the following construction.

DEFINITION 8. Let $(L_i \mid i \in I)$ be a family of lattices, let L be a lattice, and let $\varphi_i\colon L_i \to L$ be a lattice homomorphism for each $i \in I$. Form the direct product $\Pi(L_i \mid i \in I)$, and consider the subset

$$K = \{\boldsymbol{x} \in \Pi L_i \mid \varphi_i(\boldsymbol{x}(i)) = \varphi_j(\boldsymbol{x}(j)) \text{ for all } i, j \in I\} .$$

Then K is a sublattice of ΠL_i, and is called the *pullback* of the family $(\varphi_i \mid i \in I)$. Letting $\pi_i\colon K \to L_i$ be the restriction of the projection of ΠL_i onto L_i, we have $\varphi_i \circ \pi_i = \varphi_j \circ \pi_j$ for all $i, j \in I$; hence there is a natural homomorphism $\varphi = \varphi_i \circ \pi_i$ of K into L. If φ_i is onto for all $i \in I$, then φ is onto.

PROPOSITION 9. *For any lattice L, there exists a lattice L^* and a homomorphism φ^* of L^* onto L satisfying* (*) *that repairs all failures in L.*

Proof. Let $\mathscr{F}(L)$ be the set of all failures in L. From Propositions 5, 6, and 7, we obtain a family $(L_x \mid \boldsymbol{x} \in \mathscr{F}(L))$ of lattices and a family $(\varphi_x\colon L_x \to L \mid \boldsymbol{x} \in \mathscr{F}(L))$ of onto homomorphisms satisfying (*) such that for each $\boldsymbol{x} \in \mathscr{F}(L)$, $\boldsymbol{x}$ is repaired in L_x by φ_x. Let L^* be the pullback of $\{\varphi_x \mid \boldsymbol{x} \in \mathscr{F}(L)\}$ and let φ^* be the natural homomorphism of L^* onto L. Then by Lemma 3, φ^* repairs all failures in L. To show that φ^* satisfies (*), let $\boldsymbol{p} \in L^*$ and $u, v \in L$ be such that $\varphi^*(\boldsymbol{p}) \leqq u \vee v$. Letting p_x denote the $\boldsymbol{x}$th component of $\boldsymbol{p}$, for each $\boldsymbol{x} \in \mathscr{F}(L)$, we have that $\varphi_x(p_x) = \varphi^*(\boldsymbol{p}) \leqq u \vee v$. For each $\boldsymbol{x} \in \mathscr{F}(L)$, since φ_x satisfies (*), there exist $u_x, v_x \in L_x$ such that $\varphi_x(u_x) = u$, $\varphi_x(v_x) = v$, and $p_x \leqq u_x \vee v_x$ in L_x. Then the elements $\boldsymbol{u} = (u_x \mid \boldsymbol{x} \in \mathscr{F}(L))$ and $\boldsymbol{v} = (v_x \mid \boldsymbol{x} \in \mathscr{F}(L))$ are clearly in L^*; moreover $\varphi^*(\boldsymbol{u}) = u$, $\varphi^*(\boldsymbol{v}) = v$, and $\boldsymbol{p} \leqq \boldsymbol{u} \vee \boldsymbol{v}$.

Finally, we are in a position to prove our main result.

THEOREM 10. *Every lattice can be embedded in the ideal lattice of a lattice satisfying* $(SD_\vee)$, $(SD_\wedge)$, (W_l), *and* (W_u).

Proof. Let L be a lattice. Set $L_0 = L$, and inductively let $L_{n+1} = L_n^*$ and $\varphi_n^*: L_n^* \to L_n$, $n \geqq 0$, be the lattice and homomorphism of Proposition 9. Let L_∞ be the inverse limit of the system of lattices $(L_n \mid n < \omega)$ and homomorphisms $(\varphi_n^* \mid n < \omega)$, and let φ_n: $L_\infty \to L_n$ be the natural projection for each n.

We first claim that L_∞ satisfies $(SD_\vee)$, $(SD_\wedge)$, (W_l), and (W_u). Again note that each of these four conditions is expressible in the form $P(x, y, z, w) \Rightarrow Q(x, y, z, w)$, where P and Q are disjunctions of polynomial equations. Let $\langle \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}\rangle$ be a failure in L_∞; then there exist appropriate P and Q such that $P(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d})$ holds but $Q(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d})$ fails. It follows that $P(\varphi_n(\boldsymbol{a}), \varphi_n(\boldsymbol{b}), \varphi_n(\boldsymbol{c}), \varphi_n(\boldsymbol{d}))$ holds for all $n \in \omega$, and, since L_∞ is a sublattice of $\Pi(L_n \mid n \in \omega)$, $Q(\varphi_m(\boldsymbol{a}), \varphi_m(\boldsymbol{b}), \varphi_m(\boldsymbol{c}), \varphi_m(\boldsymbol{d}))$ fails for some $m \in \omega$. Therefore $\langle \varphi_m(\boldsymbol{a}), \varphi_m(\boldsymbol{b}), \varphi_m(\boldsymbol{c}), \varphi_m(\boldsymbol{d})\rangle$ is a failure in L_m. But by construction $\langle \varphi_{m+1}(\boldsymbol{a}), \varphi_{m+1}(\boldsymbol{b}), \varphi_{m+1}(\boldsymbol{c}), \varphi_{m+1}(\boldsymbol{d})\rangle$ is not a failure in L_{m+1}, which contradicts Lemma 3. Thus there can be no failures in L_∞; that is, L_∞ satisfies $(SD_\vee)$, $(SD_\wedge)$, (W_l), and (W_u).

Next we prove that the homomorphism φ_0 of L_∞ onto L satisfies (*). Let $\boldsymbol{x} \in L_\infty$ and $u_0, v_0 \in L = L_0$ be such that $\varphi_0(\boldsymbol{x}) \leqq u_0 \vee v_0$. Then $\varphi_0(\boldsymbol{x}) = \varphi_0^*(\varphi_1(\boldsymbol{x})) \leqq u_0 \vee v_0$, and since φ_0^* satisfies (*) there exist $u_1, v_1 \in L_1$ such that $\varphi_0^*(u_1) = u_0$, $\varphi_0^*(v_1) = v_0$, and $\varphi_1(\boldsymbol{x}) \leqq u_1 \vee v_1$. Proceeding by induction, assume that we have $u_n, v_n \in L_n$ such that $\varphi_n(\boldsymbol{x}) \leqq u_n \vee v_n$. Then $\varphi_n(\boldsymbol{x}) = \varphi_n^*(\varphi_{n+1}(\boldsymbol{x})) \leqq u_n \vee v_n$, and since φ_n^* satisfies (*) there exist $u_{n+1}, v_{n+1} \in L_{n+1}$ such that $\varphi_n^*(u_{n+1}) = u_n$, $\varphi_n^*(v_{n+1}) = v_n$, and $\varphi_{n+1}(\boldsymbol{x}) \leqq u_{n+1} \vee v_{n+1}$. Now let $\boldsymbol{u} = \langle u_n \mid n < \omega\rangle$ and $\boldsymbol{v} = \langle v_n \mid n < \omega\rangle$. It follows that $\boldsymbol{u}, \boldsymbol{v} \in L_\infty$, $\varphi_0(\boldsymbol{u}) = u$, $\varphi_0(\boldsymbol{v}) = v$, and $\boldsymbol{x} \leqq \boldsymbol{u} \vee \boldsymbol{v}$, whence φ_0 satisfies (*).

From Lemma 4, $\mathscr{I}(L)$ is embedded in $\mathscr{I}(L_\infty)$. Since L is embedded in $\mathscr{I}(L)$, the theorem is proved.

As mentioned earlier, we have the following corollary.

COROLLARY 11. *Every transferable lattice satisfies* (W).

REMARK. The use of homomorphisms, pullbacks, and inverse limits to repair failures stems from a proof in a recent paper of A. Day, namely, the proof (see Theorem 3.2 in [4]) that every lattice is a bounded homomorphic image of a lattice satisfying (W).

3. Additional results. In this section we investigate the status of $\mathscr{E}(P)$ for most other combinations (P) of the properties defined in the introduction. First, we shall indicate how certain techniques in a paper of G. Grätzer and C. R. Platt [8] can be modified so as to prove that $\mathscr{E}((SD_\vee)$ & (SF) & $(X))$ holds.

Let L be a lattice. It has already been observed that there is a lattice K satisfying (SF) such that L is embeddable in $\mathscr{I}(K)$. Hence we need only show that for every lattice K satisfying (SF) there is a lattice M satisfying $(SD_\vee)$, (SF), and (X) such that $\mathscr{I}(K)$ is embeddable in $\mathscr{I}(M)$.

Let K be a lattice satisfying (SF). In [8], Grätzer and Platt construct a lattice $L(K_I)$ satisfying $(SD_\vee)$ such that K can be embedded in $\mathscr{I}(L(K_I))$. From Lemma 3 and their proof it is clear that they in fact embed $\mathscr{I}(K)$ in $\mathscr{I}(L(K_I))$. The lattice $L(K_I)$ consists of certain subsets (called *closed* subsets) of $K \times \boldsymbol{Z}$, ordered by inclusion.

Now we replace $\boldsymbol{Z}$ by ω, and consider the set $L_f(K_I)$ of all *finitely generated* closed subsets of $K \times \omega$, that is, all closed subsets which are closures of finite subsets of $K \times \omega$. Since K satisfies (SF), each element of $L_f(K_I)$ is finite. Hence $L_f(K_I)$, ordered by inclusion, is a lattice; in fact $L_f(K_I)$ is embeddable in $L(K_I)$ and therefore satisfies $(SD_\vee)$. Furthermore, $L_f(K_I)$ is sectionally finite. Next, it can be proved as in [8] that $\mathscr{I}(K)$ is embeddable in $\mathscr{I}(L_f(K_I))$, and moreover the image under this embedding of each ideal in K is a nonprincipal ideal of $L_f(K_I)$. Therefore (G. Grätzer [7]) the elements of $L_f(K_I)$ may all be "split" to yield a lattice M satisfying (X) such that $\mathscr{I}(K)$ is embeddable in $\mathscr{I}(M)$. It is easy to see that M will still satisfy $(SD_\vee)$ and (SF). Thus we have:

THEOREM 12. *$\mathscr{E}((SD_\vee)$ & (SF) & $(X))$ holds.*

In contrast to the above, we now establish two negative results.

LEMMA 13. *If a lattice L satisfies (SF) and $(SD_\wedge)$, then $\mathscr{I}(L)$ satisfies $(SD_\wedge)$.*

Proof. Let L satisfy (SF) and $(SD_\wedge)$, and let $A, B, C \in \mathscr{I}(L)$ satisfy $A \cap B = A \cap C$. Let $p \in A \cap (B \vee C)$. There exist $b \in B$ and $c \in C$ such that $p \leqq b \vee c$. By (SF), there exist largest elements $b_0 \in B$ and $c_0 \in C$ such that $b_0, c_0 \leqq b \vee c$. Since $p \in A$, $p \wedge b_0 \in A \cap B$ and $p \wedge c_0 \in A \cap C = A \cap B$. Thus the element $q = (p \wedge b_0) \vee (p \wedge c_0) \in A \cap B$, and by the choice of b_0, $p \wedge c_0 \leqq q \leqq b_0$. Hence $p \wedge c_0 \leqq p \wedge b_0$; by symmetry we have that $p \wedge b_0 = p \wedge c_0$. Since L satisfies $(SD_\wedge)$, $p \wedge b_0 = p \wedge (b_0 \vee c_0) = p \wedge (b \vee c) = p$. We conclude that $p \in A \cap B$, and so $A \cap (B \vee C) = A \cap B$, showing that $\mathscr{I}(L)$ satisfies $(SD_\wedge)$.

COROLLARY 14. *$\mathscr{E}$((SF) & ($SD_\wedge$)) fails.*

LEMMA 15. *If a lattice L satisfies (SF) and (W_u), then $\mathscr{I}(L)$ satisfies (W_u).*

Proof. Let L satisfy (SF) and (W_u), and suppose that $\langle A, B, C, D\rangle$ is a (W_u)-failure in $\mathscr{I}(L)$. Then there exists an element $x \in A \cap B$ such that $x \notin C$ and $x \notin D$, and an element $b \in B$ such that $b \geqq x$ and $b \notin C \vee D$. Since L satisfies (SF), there exists a largest element $x_0 \in A \cap B$ such that $x_0 \leqq b$; note that $x_0 \geqq x$ and so $x_0 \notin C$, $x_0 \notin D$. Since $x_0 \in A \cap B \subseteqq C \vee D$, there exist $c \in C$, $d \in D$ such that $x_0 \leqq c \vee d$. However, $b \notin C \vee D$, so $b \nleqq c \vee d$. Finally, since $A \supset C \vee D$, we may choose $a \in A$ such that $a > c \vee d$. But now $a \wedge b \geqq (c \vee d) \wedge b \geqq x_0$, and by the maximality of x_0 we obtain that $a \wedge b = x_0$. Hence the quadruple $\langle a, b, c, d\rangle$ is a failure of (W_u) in L, contradicting the hypothesis. We conclude that $\mathscr{I}(L)$ satisfies (W_u).

COROLLARY 16. *$\mathscr{E}$((SF) & (W_u)) fails.*

To end this paper we ask two questions that are still open:
(i) Does $\mathscr{E}$((SF) & (W_l)) hold?
(ii) Does $\mathscr{E}$((SF) & (W_l) & ($SD_\vee$)) hold?

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