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VECTOR VALUED ERGODIC THEOREMS FOR OPERATORS SATISFYING NORM CONDITIONS

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A new approach is developed in the theory of pointwise ergodic theorems. Our consideration is based upon $\Omega_\mu^p(0 \leqq p < \infty)$, which is a linear space containing properly the linear span of $U_{p>1}L_p(X;\mathscr{X})$, where $(X, \mathscr{F}, \mu)$ is a σ-finite measure space and $\mathscr{X}$ is a reflexive Banach space. Some weak and strong type inequalities are proved as vector valued generalizations of the Dunford and Schwartz's results, and then, used to study the integrability of the ergodic maximal function. These results do make it possible to extend the Chacon's vector valued ergodic theorem. We have analogous extensions for the case of continuous semi-groups, and the local ergodic theorem is shown to hold on Ω_μ^0. The results include two applications to the random ergodic theorem and the "strong differentiability" theorem.

1. Introduction. In [6] Hopf proved an ergodic theorem for positive operators satisfying certain norm conditions and acting in spaces of real valued functions. This result was generalized by Dunford and Schwartz [3] to include nonpositive operators in spaces of complex valued functions. The principle of proof adopted by Dunford and Schwartz consisted in majorizing the operator in question by a positive one so that the Hopf's result could be brought to bear on the problem. Chacon [2] proved a maximal ergodic lemma for operators which are not necessarily positive and which act in spaces of functions taking their values in a Banach space, and then, used the result to obtain a vector valued ergodic theorem as a generalization of the Dunford and Schwartz's theorem. In this paper we intend to generalize the vector valued ergodic theorem of Chacon to operators acting in a function space which is wider than the usual Banach spaces. Let $(X, \mathscr{F}, \mu)$ be a σ-finite measure space and $(\mathscr{X}, |||\cdot|||)$ a reflexive Banach space throughout this paper. If for $0 \leqq p < \infty$ we denote by Ω_μ^p the class of all functions f which are defined on X and take their values in $\mathscr{X}$, such that

$$\int_{\{|||f(x)|||>t\}} \frac{|||f(x)|||}{t}\left(\log \frac{|||f(x)|||}{t}\right)^p d\mu < \infty$$

for every $t > 0$, then these classes constitute a generalized descending sequence of linear spaces containing properly the linear span of $U_{q>1}L_q(X; \mathscr{X})$. We prove some weak and strong type inequalities which enable us to investigate the integrability of the ergodic maxmal function. One of these inequalities will permit to extend the Chacon's theorem to functions f in the class Ω_μ^0. We also consider the analogous extensions for the case of continuous semigroups of operators and the local behavior of operator averages. Further, our results have the additional advantage that they are sufficiently general to obtain some extensions of the Beck and Schwartz's random ergodic theorem [1] and of the "strong differentiability" theorem of Jessen, Marcinkiewicz and Zygmund [7] in its one-parameter form.

2. **Preliminaries.** Let $L_p(X; \mathscr{X}) = L_p(X, \mathscr{F}, \mu; \mathscr{X})$, $1 \leqq p < \infty$, denote the space of all strongly measurable $\mathscr{X}$-valued functions f defined on X for which the norm is given by

$$\|f\|_p = \left(\int_X |||f(x)|||^p \, d\mu\right)^{1/p} < \infty ;$$

and let $L_\infty(X; \mathscr{X}) = L_\infty(X, \mathscr{F}, \mu; \mathscr{X})$ denote the space of all strongly measurable $\mathscr{X}$-valued functions f defined on X for which the norm is given by

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x \in X} |||f(x)||| < \infty .$$

We shall suppress the argument of a function, writing f for $f(x)$ when convenient. Furthermore, the relevant equations are understood to hold almost everywhere. Following Chacon [2], we define for $\lambda > 0$,

$$f^{\lambda+}(x) = [\operatorname{sgn} f(x)][\max(\lambda, |||f(x)|||) - \lambda]$$
$$f^{\lambda-}(x) = [\operatorname{sgn} f(x)]\min(\lambda, |||f(x)|||) ,$$

where $\operatorname{sgn} f(x) = f(x)/|||f(x)|||$ if $f(x) \neq 0$, and $\operatorname{sgn} f(x) = 0$ if $f(x) = 0$. Let T be a linear operator in $L_1(X; \mathscr{X})$ such that $\|T\|_1 \leqq 1$, $\sup\{\|T^n\|_\infty : n \geqq 1\} \leqq K$ for some constant $K \geqq 1$. Then T can easily be extended to a linear operator, written by the same notation, which maps $L_p(X; \mathscr{X})$ into $L_p(X; \mathscr{X})$ for $1 < p < \infty$, and $\|T^n\|_p \leqq K$ for $n \geqq 1$. Let $\{T_t : t \geqq 0\}$ be a strongly continuous one-parameter semigroup of linear operators in $L_1(X; \mathscr{X})$ such that $\|T_t\|_1 \leqq 1$ for $t \geqq 0$ and $\sup\{\|T_t\|_\infty : t \geqq 0\} \leqq K$ for some constant $K \geqq 1$. Then $\{T_t : t \geqq 0\}$ may be regarded as a strongly continuous semigroup in $L_p(X; \mathscr{X})$ with $1 < p < \infty$, and $\|T_t\|_p \leqq K$ for $t \geqq 0$. For any $\lambda > 0$, let us define $E(\lambda) = \{x : |||f(x)||| > \lambda\}$, $E_d^*(\lambda) = \{x : f_d^*(x) > \lambda\}$ and $E_c^*(\lambda) = \{x : f_c^*(x) > \lambda\}$, where

$$f_d^*(x) = \sup_{n \leqq 1} \left|\left|\left| \frac{1}{n} \sum_{k=0}^{n-1} T^k f(x) \right|\right|\right|$$
$$f_c^*(x) = \sup_{\alpha > 0} \left|\left|\left| \frac{1}{\alpha} \int_0^\alpha T_t f(x) dt \right|\right|\right| .$$

For the meaning of the integral pertaining to the semigroup $\{T_t: t \geqq 0\}$, see, for instance, [3] and [11].

THEOREM (*Chacon*). *Let T be a linear operator in $L_1(X; \mathscr{X})$ with $\|T\|_1 \leqq 1$ and $\|T\|_\infty \leqq 1$.*

(i) *If $f \in L_p(X; \mathscr{X})$, $1 \leqq p < \infty$, and $\lambda > 0$, then*

$$\int_{E_d^*(\lambda)} [\lambda - |||f^{\lambda-}(x)|||] d\mu \leqq \int_X |||f^{\lambda+}(x)||| d\mu .$$

(ii) *If $f \in L_p(X; \mathscr{X})$, $1 \leqq p < \infty$, then the limit*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} T^k f(x)$$

exists strongly for almost all $x \in X$.

(iii) *If $1 < p < \infty$, then there exists a function $f^{**} \in L_p(X; \mathscr{X})$ such that*

$$\left\| \frac{1}{n} \sum_{k=0}^{n-1} T^k f(x) \right\| \leqq |||f^{**}(x)||| \quad \text{a.e. } (n \geqq 1) .$$

The continuous versions of (ii) and (iii) appearing in the Chacon's theorem were included in the author [11]. Suppose the conditions $\|T\|_1 \leqq 1$ and $\sup\{\|T^n\|_\infty: n \geqq 1\} \leqq K$ for some constant $K \geqq 1$. Then for $f \in L_p(X; \mathscr{X})$, $1 \leqq p < \infty$, and $\lambda > 0$,

$$(*) \qquad \int_{E_d^*(\lambda K)} [\lambda - |||f^{\lambda-}(x)|||] d\mu \leqq \int_X |||f^{\lambda+}(x)||| d\mu .$$

This fact (*) can be obtained by duplication of the Chacon's proof of (i) in the above theorem with trivial change. Using the approximation argument of Dunford and Schwartz, (*) also holds with $E_c^*(\lambda K)$ instead of $E_d^*(\lambda K)$ (cf. Hasegawa, Sato and Tsurumi [5]).

We denote by $\Omega_\mu^p(0 \leqq p < \infty)$ the class of all $\mathscr{X}$-valued functions f defined on X such that

$$\int_{\{|||f(x)|||>t\}} \frac{|||f(x)|||}{t} \left(\log \frac{|||f(x)|||}{t}\right)^p d\mu < \infty$$

for every $t > 0$. Such classes were considered by Fava [4] in case where $p = 0, 1, 2, \cdots$. Let $L_1(X; \mathscr{X}) + L_\infty(X; \mathscr{X})$ denote the class of all functions f which can be written as the sum of g in $L_1(X; \mathscr{X})$ and h in $L_\infty(X; \mathscr{X})$. Let $L(X; \mathscr{X})[\log^+ L(X; \mathscr{X})]^p$ denote the class of all functions f for which

$$\int_X |||f(x)|||[\log \max(1, |||f(x)|||)]^p d\mu < \infty .$$

PROPOSITION 1. *For each real* $p \geqq 0$, *the class* Ω_μ^p *is a linear space*:

(i) *If* $f \in \Omega_\mu^p$ *and* λ *is a scalar, then* $\lambda f \in \Omega_\mu^p$.

(ii) *If* $f, g \in \Omega_\mu^p$ *then* $f + g \in \Omega_\mu^p$.

PROPOSITION 2. *The following inclusion relations hold*:

(i) $L_1(X; \mathscr{X}) \subsetneqq \Omega_\mu^0 \subset L_1(X; \mathscr{X}) + L_\infty(X; \mathscr{X})$.

(ii) $\Omega_\mu^\beta \subset \Omega_\mu^\alpha$ *for any* α, β *with* $0 \leqq \alpha \leqq \beta$.

(iii) $L_q(X; \mathscr{X}) \subsetneqq \Omega_\mu^p \subset L(X; \mathscr{X})[\log^+ L(X; \mathscr{X})]^p \subset L_1(X; \mathscr{X}) + L_\infty(X; \mathscr{X})$, $p \geqq 0$, $q > 1$.

(iv) $\Omega_\mu^p = L(X; \mathscr{X})[\log^+ L(X; \mathscr{X})]^p (p \geqq 0)$ *if and only if* $\mu(X) < \infty$.

(v) $\Omega_\mu^p \supsetneqq$ *linear span* $[U_{q>1} L_q(X; \mathscr{X})](p \geqq 0)$.

The proofs of these propositions are simple exercises (cf. Fava [4]).

3. **The results.** According to our convenience in what follows, we shall write $f^*(x)$ (resp. $E^*(\lambda)$) for $f_d^*(x)$ (resp. $E_d^*(\lambda)$) in the discrete time case and for $f_c^*(x)$ (resp. $E_c^*(\lambda)$) in the continuous time case. We begin by giving a simple proof of the maximal ergodic lemma.

LEMMA 1. *Let* T *be a linear operator in* $L_1(X; \mathscr{X})$ *with* $\|T\|_1 \leqq 1$ *and* $\sup\{\|T^n\|_\infty : n \geqq 1\} \leqq K$ *for some constant* $K \geqq 1$. *Let* $\{T_t : t \geqq 0\}$ *be a strongly continuous semigroup of linear operators in* $L_1(X; \mathscr{X})$ *such that* $\|T_t\|_1 \leqq 1$ *for* $t \geqq 0$ *and* $\sup\{\|T_t\|_\infty : t \geqq 0\} \leqq K$ *for some constant* $K \geqq 1$. *Let* $\lambda > 0$ *and* $0 < t < 1$. *Then for every* $f \in L_p(X; \mathscr{X})$ *with* $1 \leqq p < \infty$,

$$\mu(E^*(\lambda K)) \leqq \frac{1}{\lambda \min(t, 1-t)} \int_{E(\lambda t)} |||f(x)||| d\mu .$$

Proof. Using the inequality (*) in § 2 and its continuous version, it follows that

$$\begin{aligned} &(1-t)\lambda\mu(E^*(\lambda K) \backslash E(\lambda t)) \\ &\leqq \int_{E^*(\lambda K)} [\lambda - |||f^{\lambda-}(x)|||] d\mu \\ &\leqq \int_X |||f^{\lambda+}(x)||| d\mu . \end{aligned}$$

Therefore, we have

$$\mu(E^*(\lambda K)) \leqq \mu(E(\lambda t)) + \frac{1}{\lambda(1-t)} \int_X |||f^{\lambda+}(x)||| d\mu$$

$$\leqq \frac{1}{\lambda t}\int_{E(\lambda t)} |||f^{\lambda-}(x)|||d\mu$$
$$+ \frac{1}{\lambda(1-t)}\int_{E(\lambda t)} |||f^{\lambda+}(x)|||d\mu$$
$$\leqq \frac{1}{\lambda \min(t, 1-t)}\int_{E(\lambda t)} |||f(x)|||d\mu ,$$

as required.

Lemma 1 generalizes both Lemma 7 in §3 and Lemma 6 in §4 of Dunford and Schwartz [3] who considered the case $K = 1$, $t = 1/2$ for complex valued functions.

THEOREM 1. *On the hypothesis of Lemma* 1, *let* $\lambda > 0$ *and* $0 < t < 1$.

(i) *If* $1 < p < \infty$ *and* $f \in L_p(X; \mathscr{X})$, *then*

$$\int_X [f^*(x)]^p d\mu \leqq \frac{pK^p}{t^{p-1}(p-1)\min(t, 1-t)}\int_X |||f(x)|||^p d\mu .$$

(ii) *If* $\mu(X) < \infty$ *and* $f \in L_1(X; \mathscr{X}) \cap L(X; \mathscr{X})[\log^+ L(X; \mathscr{X})]$, *then*

$$\int_X f^*(x)d\mu \leqq \frac{K}{t}\left\{ \mu(X) + \frac{1}{\min(t, 1-t)}\int_X |||f(x)|||[\log^+|||f(x)|||]d\mu \right\} ,$$

where $\log^+ u$ *is defined for* $u > 0$ *and* $\log^+ u = \log\max(1, u)$.

Proof. (i). In view of Lemma 1 we have

$$\int_X [f^*(x)]^p d\mu = pK^p \int_X \int_0^{(1/K)f^*(x)} \lambda^{p-1} d\lambda d\mu$$
$$= pK^p \int_X \int_0^{\infty} \lambda^{p-1} 1_{E^*(\lambda K)}(x) d\lambda d\mu$$
$$= pK^p \int_0^{\infty} \lambda^{p-1} \mu(E^*(\lambda K)) d\lambda$$
$$\leqq \frac{pK^p}{\min(t, 1-t)}\int_0^{\infty} \lambda^{p-2}\left[\int_{E(\lambda t)} |||f(x)|||\, d\mu\right] d\lambda$$
$$= \frac{pK^p}{\min(t, 1-t)}\int_0^{\infty}\int_X \lambda^{p-2} 1_{E(\lambda t)}(x) |||f(x)||| d\mu d\lambda$$
$$= \frac{pK^p}{\min(t, 1-t)}\int_X |||f(x)||| \left[\int_0^{(1/t)|||f(x)|||} \lambda^{p-2} d\lambda\right] d\mu$$
$$= \frac{pK^p}{t^{p-1}(p-1)\min(t, 1-t)}\int_X |||f(x)|||^p d\mu ,$$

where $1_E(x)$ denotes the indicator function of the set E (cf. [5]).

(ii) This case is also treated similarly as follows.

$$\begin{aligned}\int_X f^*(x)d\mu &= K\int_X\int_0^{(1/K)f^*(x)} d\lambda d\mu \\ &= K\int_X\int_0^\infty 1_{E^*(\lambda K)}(x)d\lambda d\mu \\ &= K\int_0^\infty \mu(E^*(\lambda K))d\lambda \\ &\leqq \frac{K}{t}\mu(X) + K\int_{1/t}^\infty \mu(E^*(\lambda K))d\lambda .\end{aligned}$$

On the other hand, by Lemma 1 again

$$\begin{aligned}&\int_{1/t}^\infty \mu(E^*(\lambda K))d\lambda \\ &\leqq \frac{1}{\min(t, 1-t)}\int_{1/t}^\infty \lambda^{-1}\left[\int_{E(\lambda t)} |||f(x)|||\, d\mu\right]d\lambda \\ &= \frac{1}{t\min(t, 1-t)}\int_1^\infty \lambda^{-1}\left[\int_{E(\lambda)} |||f(x)|||\, d\mu\right]d\lambda \\ &= \frac{1}{t\min(t, 1-t)}\int_1^\infty\int_X \lambda^{-1}1_{E(\lambda)}(x)\, |||f(x)|||\, d\mu d\lambda \\ &= \frac{1}{t\min(t, 1-t)}\int_X |||f(x)|||\left[\int_1^{\max(1,|||f(x)|||)} \lambda^{-1}d\lambda\right]d\mu \\ &= \frac{1}{t\min(t, 1-t)}\int_X |||f(x)|||[\log^+ |||f(x)|||]d\mu .\end{aligned}$$

Hence, combining these two parts, we get (ii) and complete the proof of Theorem 1.

Theorem 1 generalizes both Theorem 8 in § 3 and Theorem 7 in § 4 of Dunford and Schwartz [3] who considered the same case as before.

Now let us write $L(X)[\log^+ L(X)]^p$ for $L(X; \mathscr{X})[\log^+ L(X; \mathscr{X})]^p$ if $\mathscr{X}$ is the real or complex linear space.

THEOREM 2. *Let $\mu(X) < \infty$ and $f \in L_1(X; \mathscr{X})$. Then for every $\alpha \geqq 0$, $f \in L(X; \mathscr{X})[\log^+ L(X; \mathscr{X})]^{\alpha+1}$ implies $f^* \in L(X)[\log^+ L(X)]^\alpha$.*

Proof. According to Lemma 1, there holds

$$\mu(E^*(\lambda K)) \leqq \frac{2}{\lambda}\int_{E(\lambda/2)} |||f(x)|||\, d\mu .$$

Define $F^*(\lambda) = \mu(E^*(\lambda))$ for $\lambda > 0$. Then we have, for $\alpha > 0$.

$$\begin{aligned}\int_X f^*(x)[\log^+ f^*(x)]^\alpha d\mu &= -\int_0^\infty \lambda[\log^+\lambda]^\alpha dF^*(\lambda)\\ &\leqq C_1\Big[1+\int_{M\|f\|_1}^\infty \mu(E^*(\lambda K))\{\log^+(\lambda K)\}^\alpha d\lambda\Big]\\ &\leqq C_1\Big[1+\int_{M\|f\|_1}^\infty (\log^+(\lambda K))^\alpha\Big\{\frac{2}{\lambda}\int_{E(\lambda/2)} |||f(x)||\,d\mu\Big\}d\lambda\Big]\\ &= C_1\Big[1+\int_{MK\|f\|_1}^\infty (\log^+\lambda)^\alpha\Big\{\frac{2K}{\lambda}\int_{E(\lambda/2K)} |||f(x)|||\,d\mu\Big\}d\lambda\Big]\\ &\leqq C_1\Big[1+2K\int_{E(\|f\|_1)} |||f(x)|||\Big\{\int_{MK\|f\|_1}^{MK|||f(x)|||}\frac{(\log^+\lambda)^\alpha}{\lambda}d\lambda\Big\}d\mu\Big]\\ &\leqq C_2\Big[1+\int_{E(\|f\|_1)} |||f(x)|||(\log^+ MK|||f(x)|||)^{\alpha+1}d\mu\Big]\\ &\leqq C_3\Big[1+\int_{E(MK)} |||f(x)|||(\log^+|||f(x)|||)^{\alpha+1}\,d\mu\Big]\end{aligned}$$

for some constants $M(\geqq \max(3/K, 2))$, $C_i = C_i(\alpha, f, K)(i = 1, 2, 3)$. The conclusion of Theorem 2 follows directly from this and Theorem 1.

It is interesting problem to investigate the converse of Theorem 2. Such a problem has been studied by Ornstein [8] for an ergodic automorphism and by Petersen [9] for an ergodic measurable flow.

THEOREM 3. *Let T be a linear operator on $L_1(X; \mathscr{X}) + L_\infty(X; \mathscr{X})$ such that $\|T\|_1 \leqq 1$ and $\sup\{\|T^n\|_\infty : n \geqq 1\} \leqq K$ for some constant $K \geqq 1$. Then for every $f \in \Omega_\mu^0$, the limit*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} T^k f(x)$$

exists strongly for almost all $x \in X$.

In proving Theorem 3, we make use of the following two lemmas.

LEMMA 2. *Let T be as in Theorem 3. For any $f \in L_1(X; \mathscr{X})+ L_\infty(X; \mathscr{X})$, put*

$$f_d^*(x) = \sup_{n\geqq 1}\Big|\Big|\Big|\frac{1}{n}\sum_{k=0}^{n-1} T^k f(x)\Big|\Big|\Big| .$$

Then there holds

$$\mu\{f_d^* \geqq 2Kt\} \leqq \frac{C}{t}\int_{\{|||f(x)|||\geqq t\}} |||f(x)|||\,d\mu$$

for every $t > 0$, where C is a constant independent of f and t.

Proof. We may consider only the case that for $f \in L_1(X; \mathscr{X})+$

$L_\infty(X; \mathscr{X})$ and $t > 0$, f is (B)-integrable over the set $\{|||f(x)||| \geqq t\}$, because if the right hand side is infinite then the lemma holds trivially. Thus, putting

$$f^t(x) = (f1_{\{|||f(x)||| \geqq t\}})(x),\ f_t(x) = (f1_{\{|||f(x)||| < t\}})(x)\,,$$

it follows that $f = f^t + f_t$ and $f_d^* \leqq (f^t)_d^* + Kt$. Therefore by Lemma 1 we have

$$\begin{aligned}\mu\{f_d^* \geqq 2Kt\} &\leqq \mu\{(f^t)_d^* \geqq Kt\}\\ &\leqq \frac{C}{t}\int_X |||f^t(x)|||\, d\mu\\ &= \frac{C}{t}\int_{\{|||f(x)||| \geqq t\}} |||f(x)|||\, d\mu\,.\end{aligned}$$

Hence the lemma follows.

LEMMA 3. *Let T be a linear operator on $L_1(X; \mathscr{X})$ with $\|T\|_1 \leqq 1$ and $\sup\{\|T^n\|_\infty : n \geqq 1\} \leqq K$. Then for every $f \in L_p(X; \mathscr{X})$ with $1 \leqq p < \infty$, the strong limit*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} T^k f(x)$$

exists almost everywhere.

Proof. Note that it follows from the Riesz convexity theorem that $\sup\{\|T^n\|_p : n \geqq 1\} \leqq K$. For $1 < p < \infty$, $L_p(X; \mathscr{X})$ is reflexive and thus, from Corollary 1.4 of [3] it results that the limit in question exists strongly in $L_p(X; \mathscr{X})$. So, by virtue of the Kakutani and Yosida's mean ergodic theorem, the linear manifold $\mathscr{L}$ generated by vectors of the form $f = g + (h - Th)$ with $g \in L_p(X; \mathscr{X})$, $Tg = g$, $h \in L_p(X; \mathscr{X}) \cap L_\infty(X; \mathscr{X})$, is dense in $L_p(X; \mathscr{X})$. For such a function f, one has

$$\left\| \frac{1}{n}\sum_{k=0}^{n-1} T^k f(x) - g(x) \right\| \leqq \frac{2K}{n}\|h\|_\infty \longrightarrow 0$$

as $n \to \infty$ almost everywhere, and hence

$$\mathscr{X} - \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} T^k f(x) = g(x) \quad \text{a.e.}$$

This guarantees that for every $f \in \mathscr{L}$, the limit in question converges almost everywhere. If $f \in L_p(X; \mathscr{X})$, $1 \leqq p < \infty$, then by Proposition 2 and Lemma 2 we have

$$\mu\{f_d^* \geqq 2Kt\} \leqq \frac{C}{t}\int_{\{|||f(x)|||\geqq t\}} |||f(x)|||\,d\mu$$
$$\leqq \frac{C}{t^p}\int_X |||f(x)|||^p d\mu \longrightarrow 0$$

as $t \to \infty$ and so, $f_d^*(x) < \infty$ a.e. Therefore, for any $f \in L_p(X; \mathscr{X})$ with $1 < p < \infty$, the almost everywhere convergence of $(1/n)\sum_{k=0}^{n-1} T^k f(x)$ follows from the Banach convergence theorem. Moreover, since $L_p(X; \mathscr{X}) \cap L_1(X; \mathscr{X})$ is dense in $L_1(X; \mathscr{X})$, we may apply the Banach convergence theorem again to obtain the almost everywhere convergence in question for every f in $L_1(X; \mathscr{X})$.

Proof of Theorem 3. For any $f \in L_1(X; \mathscr{X}) + L_\infty(X; \mathscr{X})$, define

$$\omega(f)(x) = \limsup_{n,m\to\infty} \left|\left|\left| \frac{1}{n}\sum_{k=0}^{n-1} T^k f(x) - \frac{1}{m}\sum_{k=0}^{m-1} T^k f(x) \right|\right|\right| .$$

It is then clear that ω is subadditive and that $\omega(f) \leqq 2f_d^*$. Now for $f \in \Omega_\mu^0$, we choose a sequence $\{f_n\}$ of simple functions having support of finite measure, such that $\lim_{n\to\infty} |||f_n(x) - f(x)||| = 0$ a.e. and $|||f(x) - f_n(x)||| \leqq 2\,|||f(x)|||$ for each n. Since $\omega(f_n) = 0$ by Lemma 3, one gets

$$\omega(f) \leqq \omega(f - f_n) \leqq 2(f - f_n)_d^* .$$

Thus, in view of Lemma 2, we have

$$\mu\{\omega(f) \geqq 8Kt\} \leqq \mu\{(f - f_n)_d^* \geqq 4Kt\}$$
$$\leqq \frac{C}{t}\int_{\{|||f(x)-f_n(x)|||\geqq 2t\}} |||f(x) - f_n(x)|||\,d\mu$$
$$\leqq \frac{C}{t}\int_{\{|||f(x)|||\geqq t\}} |||f(x) - f_n(x)|||\,d\mu$$

for every $t > 0$, which tends to zero letting $n \to \infty$ by the Lebesgue dominated convergence theorem. Consequently the theorem follows at once from this.

COROLLARY 1. *On the hypothesis of Theorem* 3, *if* $\mu(X) < \infty$ *then for every* $f \in L(X; \mathscr{X})[\log^+ L(X; \mathscr{X})]^p$ *with* $0 \leqq p < \infty$, *the limit*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} T^k f(x)$$

exists strongly for almost all $x \in X$.

4. The case of continuous semigroups. Let $\{T_t : t \geqq 0\}$ be a semigroup of linear operators on $L_1(X; \mathscr{X}) + L_\infty(X; \mathscr{X})$, such that

$\|T_t\|_1 \leqq 1$ for $t \geqq 0$ and $\sup\{\|T_t\|_\infty : t \geqq 0\} \leqq K$ for some constant $K \geqq 1$. We assume that

(i) T_t is strongly continuous when restricted to $L_1(X; \mathscr{X})$.

(ii) T_t is strongly integrable over every finite interval when restricted to $L_\infty(X; \mathscr{X})$. Define

$$\int_0^a T_t f dt = (L_1)\int_0^a T_t g dt + (L_\infty)\int_0^a T_t h dt$$

for $f = g + h$ with $g \in L_1(X; \mathscr{X})$ and $h \in L_\infty(X; \mathscr{X})$. It is easy to see that this definition is consistent. Choosing scalar representations $(T_t g)(x)$ and $(T_t h)(x)$ of $T_t g$ and $T_t h$ respectively, we obtain a scalar representation $(T_t f)(x)$ of $T_t f$

$$(T_t f)(x) = (T_t g)(x) + (T_t h)(x) ;$$

and the Bochner integral

$$\int_0^a (T_t f)(x) dt = \int_0^a (T_t g)(x) dt + \int_0^a (T_t h)(x) dt$$

as a function of x is a scalar representation of $\int_0^a T_t f dt$ (cf. Fava [4]).

LEMMA 4. *Let $\{T_t : t \geqq 0\}$ be a semigroup of linear operators on $L_1(X; \mathscr{X}) + L_\infty(X; \mathscr{X})$ with $\|T_t\|_1 \leqq 1 (t \geqq 0)$, $\sup\{\|T_t\|_\infty : t \geqq 0\} \leqq K$ for some constant $K \geqq 1$, which satisfies the conditions* (i) *and* (ii). *For $f \in L_1(X; \mathscr{X}) + L_\infty(X; \mathscr{X})$, put*

$$f_c^*(x) = \sup_{\alpha > 0} \left|\left|\left| \frac{1}{\alpha}\int_0^\alpha T_t f(x) dt \right|\right|\right| .$$

Then

$$\mu\{f_c^* \geqq 2Kt\} \leqq \frac{C}{t}\int_{\{|||f(x)||| \geqq t\}} |||f(x)|||\, d\mu$$

for every $t > 0$, where C is a constant independent of f and t.

The proof of Lemma 4 is exactly the same as that of Lemma 2.

LEMMA 5. *Let $\{T_t : t \geqq 0\}$ be a strongly continuous semigroup of linear operators on $L_1(X; \mathscr{X})$ with $\|T_t\|_1 \leqq 1 (t \geqq 0)$ and $\sup\{\|T_t\|_\infty : t \geqq 0\} \leqq K$. Then for every $f \in L_p(X; \mathscr{X})$ with $1 \leqq p < \infty$, the limit*

$$\lim_{\alpha \to \infty} \frac{1}{\alpha}\int_0^\alpha T_t f(x) dt$$

exists strongly almost everywhere.

Proof. For $\alpha \geqq 1$ and $f \in L_p(X; \mathscr{X})$ with $1 \leqq p < \infty$, it holds that

$$\frac{1}{\alpha}\int_0^1 T_t f dt = \frac{[\alpha]}{\alpha}\left\{\frac{1}{[\alpha]}\sum_{k=0}^{[\alpha]-1} T_1^k \left(\int_0^r T_t f dt\right)\right.$$
$$\left. + \frac{[\alpha]+1}{[\alpha]}\frac{1}{[\alpha]+1}\left(\sum_{k=0}^{[\alpha]} T_1^k - \sum_{k=0}^{[\alpha]-1} T_1^k\right)\left(\int_0^r T_t f dt\right)\right\},$$

where $\alpha = [\alpha] + r$, $0 \leqq r < 1$. While, a priori

$$\left\|\int_0^1 T_t f dt\right\|_p^p \leqq \|f\|_p^p, \quad \left\|\int_0^r T_t f dt\right\|_p^p \leqq \|f\|_p^p.$$

Then these relations serve to ensure the assertion by reducing the present lemma to Lemma 3.

LEMMA 6. *On the hypothesis of Lemma* 5, *let*

$$\mathfrak{m} = \left\{\frac{1}{\beta}\int_0^\beta T_t f dt \colon 0 < \beta < 1, \quad f \in L_1(X; \mathscr{X})\right\}.$$

(i) $\mathfrak{m}$ *is dense in* $L_1(X; \mathscr{X})$.
(ii) *For every* $F \in \mathfrak{m}$, *the equality*

$$\lim_{\alpha \to 0+} \frac{1}{\alpha}\int_0^\alpha T_t F(x) dt = F(x)$$

holds strongly almost everywhere.

The proof of Lemma 6 may be done similarly as in Terrell [10] (cf. Yoshimoto [11]). Making use of Lemma 6, we have

LEMMA 7. *Let* $\{T_t \colon t \geqq 0\}$ *be as in Lemma* 5. *Then for any* f *in* $L_1(X; \mathscr{X})$, *there holds*

$$\lim_{\alpha \to 0+} \frac{1}{\alpha}\int_0^\alpha T_t f(x) dt = f(x)$$

strongly for almost all $x \in X$.

Proof. Let f be in $L_1(X; \mathscr{X})$ and define

$$\omega(f)(x) = \limsup_{\alpha, \beta \to 0+} \left|\left|\left| \frac{1}{\alpha}\int_0^\alpha T_t f(x) dt - \frac{1}{\beta}\int_0^\beta T_t f(x) dt \right|\right|\right|.$$

Clearly ω is subadditive. Now we select a sequence $\{f_n\}$ of functions in $\mathfrak{m}$ with $\lim_{n\to\infty} \|f - f_n\|_1 = 0$. Then $\omega(f) \leqq 2(f - f_n)_c^* + \omega(f_n)$ and $\omega(f_n) = 0$ because of (ii) of Lemma 6. Thus by Lemma 4 we have

$$\mu\{\omega(f) \geqq 4Kt\} \leqq \mu\{(f - f_n)^*_c \geqq 2Kt\}$$
$$\leqq \frac{C}{t}\|f - f_n\|_1 \longrightarrow 0$$

for each $t > 0$, letting $n \to \infty$. On the other hand, the repeated use of Lemmas 4 and 6 yields that for each $t > 0$,

$$\mu\left\{\limsup_{\alpha\to 0+}\left|\left|\left|\frac{1}{\alpha}\int_0^\alpha T_t f(x)dt - f(x)\right|\right|\right| \geqq 4Kt\right\}$$
$$\leqq \mu\{|||f(x) - f_n(x)||| \geqq 2Kt\} + \mu\{(f - f_n)^*_c(x) \geqq 2Kt\}$$
$$\leqq \frac{M}{t}\|f - f_n\|_1 \longrightarrow 0$$

as $n \to \infty$, since

$$\limsup_{\alpha\leftarrow 0+}\left|\left|\left|\frac{1}{\alpha}\int_0^\alpha T_t f(x)dt - f(x)\right|\right|\right|$$
$$\leqq |||f(x) - f_n(x)||| + (f - f_n)^*_c(x)$$

almost everywhere. Accordingly, the conclusion of Lemma 7 follows at once from what the above fact shows.

LEMMA 8. *Let $\{T_t: t \geqq 0\}$ be as in Lemma 5. Then for every f in $L_p(X; \mathscr{X})$ with $1 < p < \infty$, the equality*

$$\lim_{\alpha\to 0+}\frac{1}{\alpha}\int_0^\alpha T_t f(x)dt = f(x)$$

holds strongly almost everywhere.

Proof. For any $f \in L_p(X; \mathscr{X})$ we choose a sequence $\{f_n\}$ of simple functions having support of finite measure such that $\lim_{n\to\infty}|||f(x) - f_n(x)||| = 0$ a.e. and $|||f(x) - f_n(x)||| \leqq 2\,|||f(x)|||$ everywhere for all $n \geqq 1$. Since $\omega(f_n) = 0$ by Lemma 7, we have $\omega(f) \leqq \omega(f - f_n) \leqq 2(f - f_n)^*_c$. Therefore $\omega(f) = 0$ by Lemma 4 using the same argument as in the proof of Theorem 3. This guarantees the existence of the limit in question. On the other hand, since we can select the functions f_n such that $\lim_{n\to\infty}\|f - f_n\|_p^p = 0$, we have that for each $t > 0$,

$$\mu\left\{\limsup_{\alpha\to 0+}\left|\left|\left|\frac{1}{\alpha}\int_0^\alpha T_t f(x)dt - f(x)\right|\right|\right| \geqq 4Kt\right\}$$
$$\leqq \mu\{|||f(x) - f_n(x)||| \geqq 2Kt\} + \mu\{(f - f_n)^*_c(x) \geqq 2Kt\}$$
$$\leqq \frac{D}{t^p}\|f - f_n\|_p^p \longrightarrow 0$$

as $n \to \infty$, which, together with the above fact, concludes that Lemma 8 follows immediately.

By combining the last two lemmas just observed above, we have established the following theorem.

THEOREM 4. *Let* $\{T_t: t \geqq 0\}$ *be a strongly continuous semigroup of linear operators on* $L_1(X; \mathscr{X})$ *with* $\|T_t\|_1 \leqq 1 (t \geqq 0)$ *and* $\sup \{\|T_t\|_\infty: t \geqq 0\} \leqq K$ *for some constant* $K \geqq 1$. *Then for every* $f \in L_p(X; \mathscr{X})$ *with* $1 \leqq p < \infty$, *the local ergodic equality*

$$\lim_{\alpha \to 0+} \frac{1}{\alpha} \int_0^\alpha T_t f(x) dt = f(x)$$

holds strongly for almost all $x \in X$.

Now, from Lemmas 4-6 and Theorem 4, we can derive the following theorems.

THEOREM 5. *Let* $\{T_t: t \geqq 0\}$ *be a semigroup of linear operators on* $L_1(X; \mathscr{X}) + L_\infty(X; \mathscr{X})$ *with* $\|T_t\|_1 \leqq 1 (t \geqq 0)$ *and* $\sup\{\|T_t\|_\infty: t \geqq 0\} \leqq K$ *for some constant* $K \geqq 1$, *which satisfies the conditions* (i) *and* (ii) *in the beginning of* §4. *Then for every* $f \in \Omega_\mu^0$ *the limit*

$$\lim_{\alpha \to \infty} \frac{1}{\alpha} \int_0^\alpha T_t f(x) dt$$

exists strongly for almost all $x \in X$.

The proof of Theorem 5 is omitted, since the argument is essentially the same as that in Theorem 3.

THEOREM 6. *On the hypothesis of Theorem* 5, *if* f *is in* Ω_μ^0 *then the local ergodic equality*

$$\lim_{\alpha \to 0+} \frac{1}{\alpha} \int_0^\alpha T_t f(x) dt = f(x)$$

holds strongly for almost all $x \in X$.

Proof. With the semigroup T_t and a function f in Ω_μ^0, we define a subadditive operator ω as in Lemma 7. For $f \in \Omega_\mu^0$, choose a sequence $\{f_n\}$ of simple functions having support of finite measure, such that $\lim_{n\to\infty} |||f(x) - f_n(x)||| = 0$ a.e. and $|||f(x) - f_n(x)||| \leqq 2 |||f(x)|||$ everywhere for all $n \geqq 1$. Then $\omega(f) \leqq 2(f - f_n)_c^* + \omega(f_n)$ and $\omega(f_n) = 0$ on account of Theorem 4. Thus, after a simple calculation using Lemma 4, we have $\omega(f) = 0$, from which follows the existence of the limit in question. Moreover, noticing that by virtue of Theorem 4

$$\mathscr{X}-\lim_{\alpha\to 0+}\frac{1}{\alpha}\int_0^\alpha T_t f_n(x)dt=f_n(x)$$

for almost all $x\in X$ and every $n\geqq 1$, we have

$$\begin{aligned}\mu&\left\{\limsup_{\alpha\to 0+}\left|\left|\left|\frac{1}{\alpha}\int_0^\alpha T_t f(x)-f(x)\right|\right|\right|\geqq 8Kt\right\}\\ &\leqq\mu\{|||f(x)-f_n(x)|||\geqq 4Kt\}+\mu\{(f-f_n)^*_c(x)\geqq 4Kt\}\\ &\leqq\frac{L}{t}\int_{\{|||f(x)|||\geqq t\}}|||f(x)-f_n(x)|||\,d\mu\end{aligned}$$

which tends to zero as $n\to\infty$ by the Lebesgue dominated convergence theorem. This completes the proof of Theorem 6.

COROLLARY 2. *On the hypothesis of Theorem* 5, *suppose the measure is finite. Then both conclusions of Theorems* 5 *and* 6 *remain true for every* $f\in L(X;\mathscr{X})[\log^+L(X;\mathscr{X})]^p$ *with* $0\leqq p<\infty$.

5. **Applications.** The general results of §§3–4 can readily be applied to give some generalizations of the vector valued random ergodic theorem of Beck and Schwartz [1] and the "strong differentiability" theorem of Jessen, Marcinkiewicz and Zygmund [7] in its one-parameter form. We first state and sketch the proof of the random ergodic theorem.

THEOREM 7. *Let there be defined on* X *a strongly measurable function* U_x *with values in the B-space* $B(\mathscr{X})$ *of bounded linear operators on* $\mathscr{X}$. *Suppose that* $|||U_x|||\leqq 1$ *for all* $x\in X$. *Let* φ *be a measure preserving transformation in* $(X,\mathscr{F},\mu)$. *Then for every* $f\in\Omega^0_\mu$ *the strong limit*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n U_xU_{\varphi x}\cdots U_{\varphi^{k-1}x}f(\varphi^k x)$$

exists for almost all $x\in X$.

Proof. For every $f\in\Omega^0_\mu$, define

$$Uf(x)=U_xf(\varphi x)\ .$$

Then it can easily be seen that U satisfies the conditions of Theorem 3 and hence the conclusion follows at once from Theorem 3.

Let R be the one-dimensional Euclidean space equipped with the

Lebesgue measure m. The integral of a function f in $L_1(R; \mathscr{X})+L_\infty(R; \mathscr{X})$ is said to be strongly differentiable at the point $x \in R$ if the limit

$$\lim_{\alpha \uparrow \beta} \frac{1}{\beta - \alpha} \int_\alpha^\beta f(u) dm(u)$$

exists and is finite, where $\alpha < x < \beta$. The limit function is then called the strong derivative of the integral of f at x. Using the method of § 4, applied to concrete analytic situations, we have

THEOREM 8. (i). *For each $f \in \Omega_m^0$, the integral of f is strongly differentiable at almost every point $x \in R$ and the derivative is equal to $f(x)$ almost everywhere.* (ii). *Let I be the unit interval. Then for every $f \in L(I; \mathscr{X})[\log^+ L(I; \mathscr{X})]^p$ with $0 \leqq p < \infty$, the integral of f is strongly differentiable at almost every point $x \in I$ and the strong derivative is equal to $f(x)$ almost everywhere.*

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