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SOME GENERALIZATIONS OF CARLITZ'S THEOREM

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Recently, L. Carlitz extended certain known generating functions for Laguerre and Jacobi polynomials to the forms:

$$\sum_{n=0}^{\infty} c_n^{(\alpha+\lambda n)}\frac{t^n}{n!} \qquad \text{and} \qquad \sum_{n=0}^{\infty} d_n^{(\alpha+\lambda n,\beta+\mu n)}\frac{t^n}{n!}\,,$$

respectively, where $c_n^{(\alpha)}$ and $d_n^{(\alpha,\beta)}$ are general one- and two-parameter coefficients. In the present paper some generalizations of Carlitz's results of this kind are derived, and a number of interesting applications of the main theorem are given.

1. Introduction and the main results. Motivated by his generating function [2, p. 826, Eq. (8)]

$$\sum_{n=0}^{\infty} L_n^{(\alpha+\lambda n)}(x)t^n = \frac{(1+v)^{\alpha+1}}{1-\lambda v}\exp(-xv)\,, \tag{1.1}$$

where α, λ are arbitrary complex numbers and v is a function of t defined by

$$v = t(1+v)^{\lambda+1}\,, \qquad v(0)=0\,, \tag{1.2}$$

and by its subsequent generalization due to Srivastava and Singhal [9, p. 749, Eq. (8)]

$$\begin{aligned}&\sum_{n=0}^{\infty} P_n^{(\alpha+\lambda n,\beta+\mu n)}(x)t^n \\ &\quad = (1+\xi)^{\alpha+1}(1+\eta)^{\beta+1}[1-\lambda\xi-\mu\eta-(1+\lambda+\mu)\xi\eta]^{-1}\,,\end{aligned} \tag{1.3}$$

where ξ and η satisfy

$$(x+1)^{-1}\xi = (x-1)^{-1}\eta = \frac{1}{2}t(1+\xi)^{\lambda+1}(1+\eta)^{\mu+1}\,, \tag{1.4}$$

Carlitz [3] has recently derived generating functions for certain general one- and two-parameter coefficients [op. cit., p. 521, Theorem 1 and Eq. (2.10)]. Our proposed generalizations of Carlitz's main results in [3] are contained in the following

THEOREM. *Let $A(z)$, $B(z)$ and $z^{-1}C(z)$ be arbitrary functions which are analytic in the neighborhood of the origin, and assume that*

$$A(0) = B(0) = C'(0) = 1\,. \tag{1.5}$$

Define the sequence of functions $\{f_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ *by means of*

$$A(z)[B(z)]^{\alpha} \exp (xC(z)) = \sum_{n=0}^{\infty} f_n^{(\alpha)}(x)\frac{z^n}{n!}\,, \tag{1.6}$$

where α *and* x *are arbitrary complex numbers independent of* z.

Then, for arbitrary parameters λ *and* y *independent of* z,

$$\sum_{n=0}^{\infty} f_n^{(\alpha+\lambda n)}(x + ny)\frac{t^n}{n!} = \frac{A(\zeta)[B(\zeta)]^{\alpha} \exp (xC(\zeta))}{1 - \zeta\{\lambda[B'(\zeta)/B(\zeta)] + yC'(\zeta)\}}\,, \tag{1.7}$$

where

$$\zeta = t[B(\zeta)]^{\lambda} \exp (yC(\zeta))\,. \tag{1.8}$$

More generally, if the functions $A(z)$, $B_i(z)$ *and* $z^{-1}C_j(z)$ *are analytic about the origin such that*

$$A(0) = B_i(0) = C_j'(0) = 1\,, \qquad i = 1, \cdots, r;\ j = 1, \cdots, s\,, \tag{1.9}$$

and if

$$A(z) \prod_{i=1}^{r} \{[B_i(z)]^{\alpha_i}\} \exp \left(\sum_{j=1}^{s} x_j C_j(z)\right) = \sum_{n=0}^{\infty} g_n^{(\alpha_1, \cdots, \alpha_r)}(x_1, \cdots, x_s)\frac{z^n}{n!}\,, \tag{1.10}$$

then, for arbitrary α's, λ's, x's *and* y's *independent of* z,

$$\begin{aligned} &\sum_{n=0}^{\infty} g_n^{(\alpha_1+\lambda_1 n, \cdots, \alpha_r+\lambda_r n)}(x_1 + ny_1, \cdots, x_s + ny_s)\frac{t^n}{n!} \\ &= \frac{A(\zeta) \prod_{i=1}^{r} \{[B_i(\zeta)]^{\alpha_i}\} \exp \left(\sum_{j=1}^{s} x_j C_j(\zeta)\right)}{1 - \zeta\left\{\sum_{i=1}^{r} \lambda_i[B_i'(\zeta)/B_i(\zeta)] + \sum_{j=1}^{s} y_j C_j'(\zeta)\right\}}\,, \end{aligned} \tag{1.11}$$

where

$$\zeta = t \prod_{i=1}^{r} \{[B_i(\zeta)]^{\lambda_i}\} \exp \left(\sum_{j=1}^{s} y_j C_j(\zeta)\right). \tag{1.12}$$

REMARK 1. For $x = y = 0$, our generating function (1.7) would evidently reduce to Carlitz's result given by his Theorem 1 [3, p. 521].

REMARK 2. The general result (1.11) with $r = 2$ and $x_j = y_j = 0$, $j = 1, \cdots, s$, is essentially the same as a known result on generating functions for certain two-parameter coefficients, which is due also to Carlitz [3, p. 521, Eq. (2.10)].

REMARK 3. Formula (1.7) with $\lambda = y = 0$ and its generalization

(1.11) with $\lambda_i = y_j = 0$, $i = 1, \cdots, r$; $j = 1, \cdots, s$, evidently correspond to the generating functions (1.6) and (1.10), respectively.

2. Proof of the theorem. By Taylor's theorem, (1.6) gives

$$f_n^{(\alpha)}(x) = D_z^n\{A(z)[B(z)]^\alpha \exp(xC(z))\}|_{z=0} , \tag{2.1}$$

whence

$$f_n^{(\alpha+\lambda n)}(x + ny) = D_z^n\{f(z)[\phi(z)]^n\}|_{z=0} , \tag{2.2}$$

where, for convenience,

$$f(z) = A(z)[B(z)]^\alpha \exp(xC(z)) , \qquad \phi(z) = [B(z)]^\lambda \exp(yC(z)) . \tag{2.3}$$

From (2.2) we have

$$\sum_{n=0}^{\infty} f_n^{(\alpha+\lambda n)}(x + ny)\frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{t^n}{n!} D_z^n\{f(z)[\phi(z)]^n\}\Big|_{z=0} , \tag{2.4}$$

where $f(z)$ and $\phi(z)$ are given by (2.3).

We now apply Lagrange's expansion in the form [6, p. 146, Problem 207]:

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} D_z^n\{f(z)[\phi(z)]^n\}\Big|_{z=0} = \frac{f(\zeta)}{1 - t\phi'(\zeta)} , \tag{2.5}$$

where the functions $f(z)$ and $\phi(z)$ are analytic about the origin, and ζ is given by

$$\zeta = t\phi(\zeta) , \qquad \phi(0) \neq 0 , \tag{2.6}$$

and the generating function (1.7) follows readily from (2.4) under the constraints (1.5) and (1.8).

The derivation of the multivariable (and multiparameter) generating function (1.11) runs parallel to that of (1.7) as described above, and we skip the details involved.

3. Applications to special polynomials. We begin by recalling the generating function [8, p. 78, Eq. (3.2)]

$$\sum_{n=0}^{\infty} G_n^{(\alpha)}(x^{1/r}, r, p, k)z^n = (1 - kz)^{-\alpha/k} \exp(px[1 - (1 - kz)^{-r/k}]) , \tag{3.1}$$

where $G_n^{(\alpha)}(x, r, p, k)$ are the polynomials considered by Srivastava and Singhal [8] in an attempt to present a unified study of the various known generalizations of the classical Laguerre and Hermite polynomials, the parameters α, p, k and r being arbitrary (with, of course, $k, r \neq 0$).

Compare (1.6) and (3.1), and we have

(3.2) $A(z) = 1\,, \quad B(z) = (1 - kz)^{-1/k}\,, \quad C(z) = p[1 - (1 - kz)^{-r/k}]\,,$

and

$$f_n^{(\alpha)}(x) \longrightarrow n!\, G_n^{(\alpha)}(x^{1/r}, r, p, k)\,.$$

It follows from (1.7) that

$$\sum_{n=0}^{\infty} G_n^{(\alpha+\lambda n)}([x + ny]^{1/r}, r, p, k)t^n = \frac{(1-\zeta)^{-\alpha/k} \exp(px[1 - (1-\zeta)^{-r/k}])}{1 - k^{-1}\zeta(1-\zeta)^{-1}[\lambda - rpy(1-\zeta)^{-r/k}]}\,, \tag{3.3}$$

where

$$\zeta = kt(1-\zeta)^{-\lambda/k} \exp(py[1 - (1-\zeta)^{-r/k}])\,. \tag{3.4}$$

Put $\zeta = w/(1 + w)$, so that

$$1 - \zeta = \frac{1}{1+w} \quad \text{and} \quad \frac{\zeta}{1-\zeta} = w\,. \tag{3.5}$$

Thus (3.3) can be put in its equivalent form:

$$\sum_{n=0}^{\infty} G_n^{(\alpha+\lambda n)}([x + ny]^{1/r}, r, p, k)t^n = \frac{(1+w)^{\alpha/k} \exp(px[1 - (1+w)^{r/k}])}{1 - k^{-1}w[\lambda - rpy(1+w)^{r/k}]}\,, \tag{3.6}$$

where

$$w = kt(1+w)^{1+\lambda/k} \exp(py[1 - (1+w)^{r/k}])\,. \tag{3.7}$$

Some special cases of (3.3) and (3.6) are worthy of mention. Indeed, the polynomials $G_n^{(\alpha)}(x, r, p, k)$ can be specialized to a number of familiar classes of polynomials by appealing to the relationships given, for example, by Srivastava and Singhal [8, p. 76]. First of all we make use of a relationship with Laguerre polynomials, viz [8, p. 76, Eq. (1.9)]

$$G_n^{(\alpha+1)}(x, 1, 1, 1) = L_n^{(\alpha)}(x)\,. \tag{3.8}$$

Thus, our formulas (3.3) and (3.6) with $r = p = k = 1$ reduce to the corresponding generating functions for the Laguerre polynomials. These generalizations of (1.1) were considered by Carlitz [3, p. 525].

Next we recall that [8, p. 76, Eq. (1.8)]

$$G_n^{(1-n)}(x, 2, 1, 1) = \frac{(-x)^n}{n!} H_n(x)\,. \tag{3.9}$$

By setting $\alpha = 1$, $\lambda = -1$, $r = 2$, and $p = k = 1$, (3.3) thus reduces to

$$(3.10)\qquad \sum_{n=0}^{\infty} H_n(\sqrt{x+ny})\frac{(t\sqrt{x+ny})^n}{n!} = \frac{\exp(x(\zeta^2+2\zeta)(1+\zeta)^{-2})}{1-2y\zeta(1+\zeta)^{-2}} ,$$

where

$$(3.11)\qquad \zeta = t(1+\zeta)\exp(y(\zeta^2+2\zeta)(1+\zeta)^{-2}) .$$

Similarly, (3.6) yields

$$(3.12)\qquad \sum_{n=0}^{\infty} H_n(\sqrt{x+ny})\frac{(t\sqrt{x+ny})^n}{n!} = \frac{\exp(x(2w-w^2))}{1-2yw(1-w)} ,$$

where

$$(3.13)\qquad w = t\exp(y(2w-w^2)) .$$

The generating functions (3.10) and (3.12) for Hermite polynomials are believed to be new. Notice, however, that if in (3.1) (with $\alpha = 0$, $r = 2$, $p = 1$, and $k = -1$) we replace x by x^2, use the relationship [8, p. 76, Eq. (1.8)]

$$(3.14)\qquad G_n^{(0)}(x, 2, 1, -1) = \frac{(-x)^n}{n!}H_n(x) ,$$

instead of (3.9), and then apply our theorem *directly*, we shall obtain a known generating function for Hermite polynomials [3, p. 524, Eq. (4.4)].

Yet another set of special cases of our generating functions (3.3) and (3.6) would follow if we put $p = r = 1$ and apply the easily verifiable relationship

$$(3.15)\qquad G_n^{(\alpha+1)}(x, 1, 1, k) = k^n Y_n^{\alpha}(x; k) ,$$

where $Y_n^{\alpha}(x; k)$ are one class of the *biorthogonal* polynomials introduced by Konhauser [4] for $\alpha > -1$ and $k = 1, 2, 3, \cdots$.
From (3.3) we thus find that

$$(3.16)\qquad \begin{aligned} &\sum_{n=0}^{\infty} Y_n^{\alpha+\lambda n}(x+ny; k)t^n \\ &\quad = \frac{(1-\zeta)^{-(\alpha+1)/k}\exp(x[1-(1-\zeta)^{-1/k}])}{1-k^{-1}\zeta(1-\zeta)^{-1}[\lambda - y(1-\zeta)^{-1/k}]} ,\end{aligned}$$

where

$$(3.17)\qquad \zeta = t(1-\zeta)^{-\lambda/k}\exp(y[1-(1-\zeta)^{-1/k}]) ,$$

while (3.6) gives us

$$\sum_{n=0}^{\infty} Y_n^{\alpha+\lambda n}(x+ny;k)t^n = \frac{(1+w)^{(\alpha+1)/k}\exp(x[1-(1+w)^{1/k}])}{1-k^{-1}w[\lambda-y(1+w)^{1/k}]}, \tag{3.18}$$

where

$$w = t(1+w)^{1+\lambda/k}\exp(y[1-(1+w)^{1/k}]) . \tag{3.19}$$

For $y=0$, the generating functions (3.16) and (3.18) reduce essentially to a result due to Calvez *et* Génin [1, p. A41, Eq. (2)]. Furthermore, since

$$Y_n^{\alpha}(x;1) = L_n^{(\alpha)}(x) , \tag{3.20}$$

in their special cases when $k=1$, (3.16) and (3.18) naturally yield the aforementioned Carlitz's results involving Laguerre polynomials.

Finally, we give a simple application of our multiparameter generating function (1.11). Indeed, for the Lauricella polynomials (cf. [5, p. 113])

$$F_D^s[-n,\beta_1,\cdots,\beta_s;\alpha;\gamma_1,\cdots,\gamma_s] = \sum_{m_1,\cdots,m_s=0}^{m_1+\cdots+m_s\leqq n} \frac{(-n)_{m_1+\cdots+m_s}(\beta_1)_{m_1}\cdots(\beta_s)_{m_s}}{(\alpha)_{m_1+\cdots+m_s}}\frac{\gamma_1^{m_1}}{m_1!}\cdots\frac{\gamma_s^{m_s}}{m_s!}, \tag{3.21}$$

where $(\alpha)_n = \Gamma(\alpha+n)/\Gamma(\alpha)$, it is readily observed that

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}F_D^s[-n,\beta_1,\cdots,\beta_s;\alpha;\gamma_1,\cdots,\gamma_s]z^n = (1-z)^{-\alpha}\prod_{j=1}^{s}\left(1+\frac{\gamma_j z}{1-z}\right)^{-\beta_j}, \qquad |z|<1 . \tag{3.22}$$

Compare (3.22) and (1.10) with $r=s+1$, and we get

$$A(z)=1 , \qquad B_1(z)=(1-z)^{-1} , \qquad B_{j+1}(z)=\left(1+\frac{\gamma_j z}{1-z}\right)^{-1}, \qquad x_j = 0 , \qquad j=1,\cdots,s , \tag{3.23}$$

and

$$g_n^{(\alpha,\beta_1,\cdots,\beta_s)}(0,\cdots,0) \longrightarrow (\alpha)_n F_D^s[-n,\beta_1,\cdots,\beta_s;\alpha;\gamma_1,\cdots,\gamma_s] .$$

It follows at once from (1.11) that

$$\sum_{n=0}^{\infty}\frac{(\alpha+\lambda n)_n}{n!}F_D^s[-n,\beta_1+\mu_1 n,\cdots,\beta_s+\mu_s n;\alpha+\lambda n;\gamma_1,\cdots,\gamma_s]t^n = \frac{(1-\zeta)^{-\alpha}\prod_{j=1}^{s}\left(1+\frac{\gamma_j\zeta}{1-\zeta}\right)^{-\beta_j}}{1-\zeta(1-\zeta)^{-1}\left[\lambda-\sum_{j=1}^{s}\gamma_j\mu_j(1-\zeta+\gamma_j\zeta)^{-1}\right]}, \tag{3.24}$$

where

$$\zeta = t(1-\zeta)^{-\lambda} \prod_{j=1}^{s} \left(1 + \frac{\gamma_j \zeta}{1-\zeta}\right)^{-\mu_j} . \tag{3.25}$$

Replacing α by $\alpha + 1$ and ζ by $\zeta/(1+\zeta)$, (3.24) may be rewritten in its equivalent form:

$$\begin{aligned} &\sum_{n=0}^{\infty} \binom{\alpha+(\lambda+1)n}{n} F_D^s[-n, \beta_1 + \mu_1 n, \cdots, \beta_s + \mu_s n ; \\ &\qquad \alpha + \lambda n + 1; \gamma_1, \cdots, \gamma_s] t^n \\ &= \frac{(1+\zeta)^{\alpha+1} \prod_{j=1}^{s} (1+\gamma_j\zeta)^{-\beta_j}}{1 - \zeta\left[\lambda - (1+\zeta) \sum_{j=1}^{s} \gamma_j \mu_j (1+\gamma_j\zeta)^{-1}\right]} , \end{aligned} \tag{3.26}$$

where ζ is now given by

$$\zeta = t(1+\zeta)^{\lambda+1} \prod_{j=1}^{s} (1+\gamma_j\zeta)^{-\mu_j} . \tag{3.27}$$

For $\mu_1 = \cdots = \mu_s = 0$, the multiparameter generating function (3.26) is derivable also as a special case of a known result [7, p. 1080, Eq. (6)] involving the generalized Lauricella functions of several variables.

A number of additional applications of our theorem can be given by using some of the examples considered earlier by Carlitz [3].

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