PACIFIC JOURNAL OF MATHEMATICS
Vol. 85, No. 2, 1979

THE PROJECTIVITY OF EXT (T, A) AS A MODULE OVER $E(T)$

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Let A and T be abelian groups. Then $\mathrm{Ext}(T, A)$ can be considered as a right module over $E(T)$, the ring of endomorphisms of T. In this paper necessary and sufficient conditions are developed for $\mathrm{Ext}(T, A)$ to be $E(T)$-projective whenever T is reduced torsion and A is reduced.

In this paper A and T will be abelian groups and $\mathrm{Ext}(T, A)$ will be considered as a right $E(T)$-module. (See [5].) We consider the question of when $\mathrm{Ext}(T, A)$ is a projective $E(T)$-module. Theorems 1 and 2 provide necessary and sufficient conditions for $\mathrm{Ext}(T, A)$ to be $E(T)$-projective whenever T is reduced torsion and A is reduced. It is interesting to note (Theorem 3) that if B is any reduced group, a necessary condition for $\mathrm{Ext}(B, A)$ to be $E(B)$-projective is that $\mathrm{Ext}(B, A) \simeq \mathrm{Ext}(T(B), A)$. Hence if $\mathrm{Ext}(B, A)$ is $E(B)$-projective, $\mathrm{Ext}(B, A) \simeq \mathrm{Ext}(T(B), A)$ and $\mathrm{Ext}(T(B), A)$ may be considered as an $E(T(B))$-module, where $T(B)$ is, of course, reduced torsion.

We shall employ the following notations and conventions: The word "group" will always mean "abelian group." We reserve the letter T for a torsion group, and in this case, T_p will be the p-primary component of T. For an arbitrary group A, $T_p(A)$ is the p-primary component of the torsion part of A. For a ring R and a left R-module M, $r_R(M)$ will refer to the rank of M as defined in [4], $hd_R(M)$ and $id_R(M)$ will refer, respectively, to the homological and injective dimensions of M as defined in [6]. An isomorphism of R-modules M and N will be denoted by: $M \overset{R}{\simeq} N$. Other notations will follow [2]. Importantly, whenever we speak of $\mathrm{Ext}(T, A)$ as a right $E(T)$-module we may assume without loss of generality that A is reduced as a group. Finally, if $A \overset{Z}{\simeq} (v) \oplus A'$, and if $a \in A$, we will write, conveniently, when defining an endomorphism α of A: $\alpha(v) = a$, $\alpha = 0$ otherwise. We mean, more precisely, that: $\alpha(v) = a$, $\alpha|_{A'} = 0$. We now state our main theorems:

THEOREM 1. *Let T be a reduced p-primary group and let A be a reduced group. Then* $\mathrm{Ext}(T, A)$ *is a projective right* $E(T)$-*module if and only if either* $\mathrm{Ext}(T, A) = 0$, *or all of the following conditions hold*:

(i) *T is bounded, with minimal annihilator p^k, say.*

(ii) *$A[p^k]$ is either zero or is a direct sum of cyclic groups of order p^k.*

(iii) *If D is a divisible hull of $T(A)$ and E is a divisible hull of $A/T(A)$, and if*

$$\max\left\{r_p\left(\frac{D}{T(A)}\right),\quad r_p\left(\frac{E}{A/T(A)}\right)\right\} = m\ ,$$

m an infinite cardinal, then T is either finite, or in a decomposition of T into cyclic groups, there are at least m summands isomorphic to $Z(p^k)$.

THEOREM 2. *Let T be a reduced torsion group and let A be a reduced group. Then* $\operatorname{Ext}(T, A)$ *is a projective $E(T)$-module if and only if for every p,* $\operatorname{Ext}(T_p, A)$ *is a projective $E(T_p)$-module.*

THEOREM 3. *Let A and B be reduced groups. Then a necessary condition for* $\operatorname{Ext}(B, A)$ *to be $E(B)$-projective is that* $\operatorname{Ext}(B, A) \overset{Z}{\simeq} \operatorname{Ext}(T(B), A)$.

Proofs of the theorems. The proof of Theorem 1 will require numerous preliminary results. We postpone its proof. Theorem 2 follows easily from Lemmas 1 and 2 below. We now prove Theorem 3:

Proof of Theorem 3. Since B is reduced, it is easily verified that $E(B)$ is reduced. Now, from the Z-exact sequence: $0 \to T(B) \overset{i}{\to} B \to B/T(B) \to 0$, we obtain the Z-exact sequence: $0 \to \operatorname{Ker} i^* \to \operatorname{Ext}(B, A) \overset{i^*}{\to} \operatorname{Ext}(T(B), A) \to 0$. Since $\operatorname{Ker} i^*$ is a subgroup of $\operatorname{Ext}(B/T(B), A)$, and since $B/T(B)$ is torsionfree, $\operatorname{Ker} i^*$ is divisible. (See [2].) Since $\operatorname{Ext}(T(B), A)$ is reduced (see [2]), it follows that $\operatorname{Ker} i^*$ is the maximal divisible subgroup of $\operatorname{Ext}(B, A)$. Now, since $E(B)$ is reduced as a group, any free $E(B)$-module is reduced as a group. So if $\operatorname{Ext}(B, A)$ is to be $E(B)$-projective, we must have $\operatorname{Ker} i^* = 0$.

We will now aim at proving Theorem 1.

LEMMA 1. *Let $M = \prod_{i \in I} M_i$ where each M_i is an R_i-module, $R = \prod_{i \in I} R_i$, and M is an R-module via the coordinatewise action of $\prod_{i \in I} R_i$. Then M is R-projective* (resp. *injective*) *if and only if M_i is R_i-projective* (resp. *injective*) *for all $i \in I$.*

Proof. The proof is easy and is omitted.

LEMMA 2. *Let* $F: Ab \times Ab \to Ab$ *be either of the functors* Hom *or* Ext. *Then*:

(i) *If* $A = \bigoplus_{i \in I} A_i$ *where the* A_i *are fully invariant subgroups of* A, *then* $F(A, B) \overset{E(A)}{\simeq} \prod_{i \in I} F(A_i, B)$.

(ii) *If* $B = \prod_{i \in I} B_i$ *where the* B_i *are fully invariant subgroups of* B, *then* $F(A, B) \overset{E(B)}{\simeq} \prod_{i \in I} F(A, B_i)$.

Proof. The isomorphism in (i) is given by: $F(A,B) \overset{\psi}{\simeq} \prod_{i \in I} F(A_i,B)$ where, for $f \in F(A, B)$, $\psi(f) = [f\alpha_i]_{i \in I}$ where $\alpha_i \in E(A)$ is defined by: $\alpha_i |_{A_i} = 1_{A_i}$, $\alpha_i = 0$ otherwise. It is easily verified that ψ is an $E(A)$-homomorphism.

The isomorphism for (ii) is similar.

Lemma 3 computes the injective dimension over $E(T)$ of Ext (T, A) when T is torsion and A is torsionfree:

LEMMA 3. *Let* T *be torsion and let* A *be torsionfree. Suppose* S *is the set of primes for which* A *is p-divisible. Then*:

(i) $\mathrm{id}_{E(T)}(\mathrm{Ext}\,(T, A)) = 0$ *if and only if for every prime* $p \notin S$, T_p *is either bounded or has an unbounded basic subgroup.*

(ii) *Otherwise*, $\mathrm{id}_{E(T)}(\mathrm{Ext}\,(T, A)) = 1$.

Proof. If D is a divisible hull of A, then D/A is torsion and Hom $(T, D/A) \overset{E(T)}{\simeq}$ Ext (T, A). By Lemma 2, it suffices to prove the result in the case in which T is a p-group, and we may assume $(D/A)_p \neq 0$, since otherwise Ext $(T, A) = 0$. Assuming this, we note that by [8, Lemma 2], Hom $(T, D/A)$ is $E(T)$-injective if and only if T is $E(T)$-flat. From [9] we know that this holds if and only if the condition (i) of the lemma holds (where T is a p-group, and $p \notin S$.) Otherwise, from [1], we know T has dimension one as an $E(T)$-module, and if we take a projective resolution of T and dualize it, applying [8, Lemma 2] again, we obtain an injective resolution for Hom $(T, D/A)$, establishing part (ii) of the lemma.

LEMMA 4. *Let* A *be a reduced group with* $T_p(A)$ *unbounded. Then if* M *is a right* $E(A)$-*module with* $\mathrm{Hom}_Z(A, Z(p^\infty)) \subseteq M$, *then* M *is not* $E(A)$-*projective.*

Proof. We will show that there is no $E(A)$-monic map ψ:

$$0 \longrightarrow \mathrm{Hom}\,(A, Z(p^\infty)) \xrightarrow{\ \psi\ } \bigoplus_{b \in B} E(A)_b$$

for any indexing set B. This will complete the proof. Consider

$I = \{1, 2, 3, \cdots\}$. Since $T_p(A)$ is reduced and unbounded, for each $i \in I$, we may choose $\nu_i \in T_p(A)$ with the property that (ν_i) is a cyclic summand of $T_p(A)$ and such that $O(\nu_i) < O(\nu_{i+1})$, $i = 1, 2, 3, \cdots$. Say $(\nu_i) = Z(p^{n_i})$ for $i = 1, 2, 3, \cdots$. Now, let $h_i \in \operatorname{Hom}(A, Z(p^\infty))$ be defined by:

$$h_i(\nu_i) = \frac{1}{p^{n_i}}, \quad h_i = 0 \text{ otherwise} .$$

Let:

$$\psi(h_i) = \alpha_{b_{1i}} + \alpha_{b_{2i}} + \cdots + \alpha_{b_{k_i i}}$$

where $\alpha_{b_{ji}} \in E(A)_{b_{ji}}$ for all $j = 1, 2, \cdots, k_i$. Define $\beta_i \in E(A)$ by:

$$\beta_i(\nu_i) = 0, \quad \beta_i = 1 \text{ otherwise} .$$

Then the computation:

$$0 = \psi(0) = \psi(h_i\beta_i) = \alpha_{b_{1i}}\beta_i + \alpha_{b_{2i}}\beta_i + \cdots + \alpha_{b_{k_i i}}\beta_i$$

shows that $\alpha_{b_{ji}}\beta_i = 0$ for all $j = 1, 2, \cdots, k_i$, and hence that $\alpha_{b_{ji}} = 0$, except possibly on ν_i, for all $j = 1, 2, \cdots, k_i$ and for all $i = 1, 2, 3, \cdots$. Suppose $\alpha_{b_{ji}}(\nu_i) = t_{ji}$. Then $t_{ji} \in T_p(A)$, and not all t_{ji} are zero for a fixed i, where $j = 1, 2, \cdots, k_i$. By defining $\delta_i \in E(A)$ by:

$$\delta_i(\nu_{i-1}) = p^{n_i - n_{i-1}}\nu_i$$
$$\delta_i = 0 \text{ otherwise}$$

for $i = 2, 3, 4, \cdots$, the computation:

$$\begin{aligned}\psi(h_i\delta_i) &= \psi(h_{i-1}) = \alpha_{b_{1i-1}} + \alpha_{b_{2i-1}} + \cdots + \alpha_{b_{k_{i-1}i-1}} \\ &= \psi(h_i)\delta_i = \alpha b_{1i}\delta_i + \alpha_{b_{2i}}\delta_i + \cdots + \alpha_{b_{k_i i}}\delta_i\end{aligned}$$

shows that we may assume $k_1 = k_2 = \cdots = k$, say, and that:

$$\alpha_{b_{ji-1}} = \alpha_{b_{ji}}\delta_i \text{ for all } j = 1, 2, \cdots, k .$$

Now, since not all t_{ji} are zero for a fixed i, where $j = 1, 2, \cdots, k$, assume that:

$$\alpha_{b_{s1}}(\nu_1) = t_{s1} \neq 0 \text{ where } s \in \{1, 2, \cdots, k\} .$$

From this, we easily obtain the relations:

$$\begin{aligned} t_{s1} &= p^{n_2 - n_1} t_{s2} \\ t_{s2} &= p^{n_3 - n_2} t_{s3} \\ t_{s_3} &= p^{n_4 - n_3} t_{s4} \\ &\vdots \end{aligned}$$

However, this is a contradiction, since the subgroup of A generated by the t_{si}, $i = 1, 2, 3, \cdots$ is isomorphic to $Z(p^\infty)$, and A was assumed to be a reduced group.

COROLLARY 1. *Let T be a reduced torsion group. Then the following statements are equivalent:*

(i) *T is a projective left $E(T)$-module.*

(ii) *$\mathrm{Hom}\,(T, Q/Z)$ is a projective right $E(T)$-module.*

(iii) *Every p-primary component of T is bounded.*

Proof. In [7] it is shown that a torsion group T is a projective left $E(T)$-module $\langle = \rangle T_p$ is bounded for all p.

Now, to prove the equivalence of (i) and (ii), we note first that by Lemma 1 we may assume that T is p-primary. Let T be bounded with minimal annihilator p^k and let ν generate a cyclic summand of T of order p^k. Then $T \overset{Z}{\simeq} (\nu) \oplus T'$, the isomorphism being one of abelian groups. Hence:

$$\mathrm{Hom}\,(T, (\nu)) \overset{E(T)}{\simeq} \mathrm{Hom}\,(T, Z(p^\infty)) ,$$

is seen to be an $E(T)$ direct summand in $E(T)_{E(T)}$. If T is not bounded, then Lemma 4 completes the proof.

COROLLARY 2. *Let T be a torsion group. Then* $\mathrm{Hom}\,(T, Q/Z)$ *is a projective right $E(T)$-module if and only if for every prime p, T_p is either bounded or has an abelian group summand isomorphic to $Z(p^\infty)$.*

Proof. We may assume that T is p-primary. If T is reduced, the result follows from Corollary 1. If T is not reduced, then $T = Z(p^\infty) \oplus T'$ for some group T', and it is clear that $\mathrm{Hom}\,(T, Z(p^\infty))$ is an $E(T)$-direct summand in $E(T)_{E(T)}$.

COROLLARY 3. *Let A be a torsionfree group of finite rank, and let T be a torsion group. Further, let S be the set of primes for which A is p-divisible. Then* $\mathrm{Ext}\,(T, A)$ *is a projective right $E(T)$-module if and only if for every prime $p \notin S$, T_p is either bounded or has an abelian group summand isomorphic to $Z(p^\infty)$.*

Proof. Let D be a divisible hull of A. Then: $D/A \simeq \bigoplus_{p \in P'} D_p$, where D_p is a divisible torsion group of finite rank, and where $P' = P - S$. Then:

$$\mathrm{Ext}\,(T, A) \overset{E(T)}{\simeq} \mathrm{Hom}\left(T, \frac{D}{A}\right) \overset{E(T)}{\simeq} \prod_{p \in P'} \mathrm{Hom}\,(T_p, D_p) .$$

The proof is completed by Lemma 1 and Corollary 2, recalling also that a direct sum of projective modules is projective.

LEMMA 5. *Let T be a bounded p-primary group with minimal annihilator p^k and let n be any cardinal. Further assume that in a decomposition of T into cyclic groups, there are at least n summands isomorphic to $Z(p^k)$. Then for any indexing set I with $|I| \leqq n$,* $\mathrm{Hom}(T, \bigoplus_{i \in I} Z(p^\infty)_i)$ *is a cyclic projective $E(T)$-module.*

Proof. There is a set $\{v_j\}_{j \in J}$ where v_j generates a cyclic abelian group summand of T of order p^k, and where $|J| = |I|$. Then $T \overset{Z}{\simeq} \bigoplus_{j \in J}(v_j) \oplus T'$, where the isomorphism is one of abelian groups. Thus, $\mathrm{Hom}(T, \bigoplus_{j \in J}(v_j)) \overset{E(T)}{\simeq} \mathrm{Hom}(T, \bigoplus_{i \in I} Z(p^\infty)_i)$ is seen to be an $E(T)$-direct summand in $E(T)_{E(T)}$.

LEMMA 6. *Let V be a vector space of infinite dimension over a field k, and $E = \mathrm{End}(V)$. Let $H = \mathrm{Hom}(V, \bigoplus_{i \in I} V)$. Then H is not projective as an E-module if $|I| > \dim(V)$.*

Proof. We first note that if F is a countable subset of H, then F is contained in a cyclic submodule of H. To see this, let W be a subspace of $\bigoplus_{i \in I} V$ containing $f(V)$ for all $f \in F$, and such that $\dim(W) = \dim(V)$. We may regard $\mathrm{Hom}(V, W)$ as an E-submodule of H and this submodule certainly contains F. Since $W \simeq V$, $\mathrm{Hom}(V, W) \simeq E$.

We next note that any module with the above property cannot have an infinite direct sum decomposition (clearly). Now if H were projective, it would be a direct sum of countably generated submodules (by Kaplansky's theorem in [3]). Since H is clearly not countably generated, this would mean that it had an infinite direct sum decomposition, which, as we have just seen, it does not.

COROLLARY 4. *Let T be a bounded p-group, of exponent p^k, and such that in a direct sum decomposition of T there are n summands of order p^k where n is an infinite cardinal* (i.e., $n = \dim(T/T[p^{k-1}])$, *where $T/T[p^{k-1}]$ is viewed as a vector-space over Z/pZ). Let $H = \mathrm{Hom}(T, U)$, regarded as an $E(T)$-module, where U is a direct sum of m copies of $Z(p^\infty)$, for some m, $m > n$. Then H is not a projective $E(T)$-module.*

Proof. Let $E = \mathrm{End}(T/T[p^{k-1}])$. There is a natural map $\mathrm{End}(T) \to E$ (since $T[p^{k-i}]$ is a fully invariant submodule of T) which is clearly surjective. If I is the kernel of this map of rings, (so $I=\{f \in \operatorname{End}(T)\colon f(T) \leqq T[p^{k-1}]\}$), then one easily identifies H/HI with $\operatorname{Hom}(T/T[p^{k-1}], U[p^k]/U[p^{k-1}])$. If H is projective as an End (T)-module, then H/HI must be projective as an E-module, which, according to the previous lemma, it is not.

LEMMA 7. *Let T be a bounded p-group which is infinite, but such that it's highest nonzero Ulm invariant is finite. Let U be the direct sum of a countable number of copies of $Z(p^\infty)$, and let $E = \operatorname{End}(T)$. Then $H = \operatorname{Hom}(T, U)$ is not a projective E-module.*

Proof. If there is a split mono $\mu\colon H \to \bigoplus_{i \in I} E$, then it induces a split mono $H/H[p^{k-1}] \to \bigoplus_{i \in I} E/E[p^{k-1}]$, where we choose k such that $p^kT = 0$, $p^{k-1}T \neq 0$ (i.e., p^k is the exponent of T). We note that $H/H[p^{k-1}]$ is infinite dimensional and all of the terms on the right above are finite dimensional over Z/pZ. We now let $f\colon T/pT \to U[p]$ be a surjective homomorphism. If $g\colon T \to U[p]$ is any homomorphism with $T[p^{k-1}]$ in its kernel, then there is an endomorphism $\varepsilon_g\colon T \to T$ such that $g = f\varepsilon_g$. It follows that if $h \in H$, then for some endomorphism ϕ of E, $p^{k-1}h = f\phi$. Now if π_i is the projection onto the ith summand in the above free E-module, then $\pi_i\mu(f) \neq 0$ for only a finite number of indicies i. Let this finite subset of I be J. It follows that $p^{k-1}\pi_i\mu(h) = 0$ unless $i \in J$, for all $h \in H$. Hence the image of the induced map

$$H/H[p^{k-1}] \longrightarrow \bigoplus_{i \in I} E/E[p^{k-1}]$$

is actually in the submodule $\bigoplus_{i \in J} E/E[p^{k-1}]$. This is a contradiction, since this is finite dimensional, and $H/H[p^{k-1}]$ is not.

COROLLARY 5. *Let A be torsionfree, and let D be a divisible hull of A. Let S be the set of primes for which A is p-divisible, and let T be a reduced torsion group. Then $\operatorname{Ext}(T, A)$ is a projective $E(T)$-module if and only if for every $p \notin S$ the following two conditions hold:*

(i) *Whenever $r_p(D/A)$ is finite, T_p is bounded.*

(ii) *Whenever $r_p(D/A) = m$, m being an infinite cardinal, T_p is either finite, or T_p is bounded of exponent p^k and in a decomposition of T_p into cyclic groups, there are at least m summands isomorphic to $Z(p^k)$.*

Proof. We note first that for any finite group T, and any index set I, and groups $A_i (i \in I)$, there is a natural isomorphism: $\operatorname{Hom}(T, \bigoplus_{i \in I} A_i) \simeq \bigoplus_{i \in I} \operatorname{Hom}(T, A_i)$. Hence if T is finite and p-primary, $\operatorname{Hom}(T, \bigoplus_{i \in I} Z(p^\infty)_i)$ is a projective $E(T)$-module. The proof follows from this fact and from Lemma 7 and Corollary 4.

LEMMA 8. *Let* T *be a reduced primary group. Then* Hom $(T, Z(p^\infty))$ *is an indecomposable* $E(T)$*-module.*

Proof. Suppose Hom $(T, Z(p^\infty)) \overset{E(T)}{\simeq} M_1 \oplus M_2$ where $M_1 \neq 0$ and $M_2 \neq 0$. Now let ν and ω generate cyclic summands of T, where $o(\nu) \leqq o(\omega)$, say, and where ν and w need not be distinct. Suppose there exists $h_1 \in M_1$, $h_2 \in M_2$ with $h_1 \neq 0$, $h_2 \neq 0$, and having:

$$h_1(\nu) = \frac{r}{p^s},\ h_2(\omega) = \frac{u}{p^z}$$

where

$$(r, p) = (u, p) = 1 .$$

We consider the case where $s \leqq z$, $z - s = d \geqq 0$. The case $s > z$ is similar. Define $\alpha, \beta \in E(T)$ by:

$$\alpha(\omega) = x\nu \qquad \beta(\omega) = p^d y\omega$$

$$\alpha = 0 \text{ otherwise } \beta = 0 \text{ otherwise}$$

where x and y are nonzero solutions of the linear congruence:

$$rx - uy \equiv 0 \pmod{p^s} .$$

Then $h_1\alpha = h_2\beta \neq 0$, a contradiction. Thus we may suppose that for any $h \in M_1$, say, and any generator ν of a cyclic summand of T, that $h(\nu) = 0$. Since, if T is bounded this implies that $h = 0$, the proof is complete in the case of T bounded. For T not bounded, let $h \in M_1, h \neq 0$. Say $h(t) \neq 0$, for some $t \in T$, where $o(t) = p^k$. Choose ν to be a generator of a cyclic summand of T of order $p^r \geqq p^s$, and define $\alpha \in E(T)$ by: $\alpha(\nu) = t, \alpha = 0$ otherwise. Then $(h\alpha)(\nu) \neq 0$ — a contradiction.

LEMMA 9. *Let* T *be a reduced unbounded p-primary group, and let* A *be a reduced group. Then* Ext (T, A) *is a projective* $E(T)$*-module if and only if* Ext $(T, A) = 0$.

Proof. We show first that if $k \geqq 1$, and k is finite, Ext $(T, Z(p^k))$ is not $E(T)$-projective. For this, consider an injective resolution of $Z(p^k)$: $0 \to Z(p^k) \overset{i}{\to} Z(p^\infty) \overset{\beta}{\to} Z(p^\infty) \to 0$. This induces: Hom $(T, Z(p^\infty)) \overset{\beta^*}{\to}$ Ext $(T, Z(p^k)) \to 0$. Since T is unbounded, β_* is not an $E(T)$-isomorphism, and hence it follows from Lemma 8 that Ext $(T, Z(p^k))$ is not $E(T)$-projective. Now, if $T_p(A) \neq 0$, A has a cyclic abelian group summand isomorphic to $Z(p^k)$ for some $k \geqq 1$, and the lemma follows. Hence, suppose that $T_p(A) = 0$. Then the sequence of abelian groups: $0 \to T(A) \to A \to A/T(A) \to 0$ yields the $E(T)$-isomorphism: $\operatorname{Ext}(T, A) \overset{E(T)}{\simeq} \operatorname{Ext}(T, A/T(A))$. Since $A/T(A)$ is torsionfree, Corollary 5 completes the proof.

LEMMA 10. *Let T be a reduced torsion group. Then* $\operatorname{Ext}(T, Z(p^r))$ *is a projective $E(T)$-module if and only if T_p is bounded with minimal annihilator p^k where $k \leqq r$.*

Proof. By Lemma 9, it is necessary that T_p be bounded in order that $\operatorname{Ext}(T, Z(p^r))$ be $E(T)$-projective. Consider the injective resolution of $Z(p^r)$:

$$0 \longrightarrow Z(p^r) \xrightarrow{\ i\ } Z(p^\infty) \xrightarrow{\ \pi\ } \frac{Z(p^\infty)}{Z(p^r)} \longrightarrow 0\ .$$

This induces:

$$\operatorname{Hom}(T, Z(p^\infty)) \xrightarrow{\pi_*} \operatorname{Hom}\left(T, \frac{Z(p^\infty)}{Z(p^r)}\right) \xrightarrow{\ \Delta\ } \operatorname{Ext}(T, Z(p^r)) \longrightarrow 0\ .$$

Now if $k > r$, let ν generate a cyclic summand of T of order p^k. Define $h \in \operatorname{Hom}(T, Z(p^\infty))$ by: $h(\nu) = 1/p^k$, $h = 0$ otherwise. Then $\pi_* h \neq 0$, and so $\ker \Delta \neq 0$. Lemma 8 completes the proof in this case, since

$$\operatorname{Hom}\left(T, \frac{Z(p^\infty)}{Z(p^r)}\right) \overset{E(T)}{\simeq} \operatorname{Hom}(T, Z(p^\infty))\ .$$

If $k \leqq r$, we have: $\operatorname{Hom}(T, Z(p^\infty)) \overset{E(T)}{\simeq} \operatorname{Ext}(T, Z(p^r))$, and Corollary 1 completes the proof of the lemma.

We are now in a position to complete the proof of Theorem 1.

Proof of Theorem 1. If $A[p^k]$ is homogeneous (i.e., if $A[p^k] \simeq \bigoplus_{i \in I} Z(p^k)_i$ for some indexing set I), and D is a divisible hull for A, then it is clear that $D[p^k] \leqq A$, whence, it is also clear that if $p^k T = 0$, that the map $\operatorname{Hom}(T, D) \to \operatorname{Hom}(T, D/T(A))$ is the zero map. Since $\operatorname{Ext}(T, D/T(A)) = 0$, this means that $\operatorname{Hom}(T, D/T(A)) \overset{E(T)}{\simeq}$ $\operatorname{Ext}(T, T(A))$. Since T is bounded, an earlier result immediately says

$$\operatorname{Ext}(T, T(A)) \oplus \operatorname{Ext}(T, A/T(A)) \overset{E(T)}{\simeq} \operatorname{Ext}(T, A)\ .$$

The statement of the theorem for such A follows immediately from Corollaries 4 and 5 and from Lemmas 5 and 7.

If $A[p^k]$ is not homogeneous, it is routine that A has a cyclic summand of order p^r for some $r < k$, and the result follows from Lemma 10.

The author wishes to thank the referee, whose suggestions considerably shortened the work of proving Theorem 1.

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Received December 14, 1977 and in revised form August 16, 1978.

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