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ON THE CHARACTERIZATIONS OF THE BREAKDOWN POINTS OF QUASILINEAR WAVE EQUATIONS

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We consider the mixed initial and boundary value problem of the quasilinear wave equation:

$$(1)\qquad \begin{aligned} u_t - v_x &= 0\,, \\ v_t - Q^2(u)u_x &= 0\,; \end{aligned}$$

$$(2)\qquad \begin{aligned} &u(x, 0) = 0,\ v(x, 0) = v_0(x),\ 0 \leqq x \leqq 1\,, \\ &v(0, t) = v(1, t) = 0,\ t \geqq 0\,. \end{aligned}$$

In general the solution of the system (1), (2) eventually breaks down in the sense that some of its first derivatives become unbounded at a finite time. It is shown that there are only finitely many breakdown points and that at each of them there originates one or two shock curves.

The fact that solutions of (1), (2) eventually break down has been derived by many authors, e.g., see [12], [9], [5], [10], [6], [7], and [1]. It should be pointed out however, that even though a solution break down in finite time it can be extended to large t as a weak solution. See e.g., [3], [4], and [11].

We define $r = v + M(u)$ and $s = v - M(u)$, where $M(\xi) = \int_0^\xi Q(\zeta)d\zeta$. Then r and s are Riemann invariants of (1). Let $q(\eta) = Q(M^{-1}(\eta/2))$. After transforming (1) to equations with r and s as dependent variables we apply a hodograph transformation to invert the resulting equations to

$$(3)\qquad \begin{aligned} x_r - q(r - s)t_r &= 0\,, \\ x_s + q(r - s)t_s &= 0\,. \end{aligned}$$

Eliminating x in (3) gives

$$(4)\qquad t_{rs} = \rho(r - s)(t_r - t_s)$$

where $\rho(\eta) = Q'(M^{-1}(\eta/2))/4q^2(\eta)$.

We assume that

(5) $Q(\xi)$ is a positive analytic function over $(-\infty, \infty)$;

(6) $v_0(x) = f|_{[0,1]}(x)$ is concave over $[0, 1]$, where $f(x)$ is an odd periodic analytic function over $(-\infty, \infty)$ with period 2.

Let $a = \max_{0\leqq x\leqq 1} f(x) = f(b)(0 < b < 1)$, $\Omega = \{(r, s): |r| \leqq a; |s| \leqq a\}$, $\Omega_i = \{(r, s) \in \Omega: (-1)^{i-1}(r - s) \geqq 0\}(i = 1, 2)$, $a_1 = M(\infty)$ and $a_2 = -M(-\infty)$. We assume further that

(7) $\partial^i\rho/\partial r^i(r-s)=(-1)^i\partial^i\rho/\partial s^i(r-s)<0$ for $i\geqq 0$ and $(r,s)\in\Omega$;

(8) $$a<\min(a_1, a_2)\ .$$

We refer the part in (7) with $i=0$ as (7) $(i=0)$ and that in (7) with $i>0$ as (7) $(i>0)$. We refer a breakdown point as a b.d.p. We call a shock curve of the equations (1) propagating to the left an 1-shock and that propagating to the right a 2-shock.

In this paper applying a method developed in [2] we prove that the system (1), (2) has only finitely many b.d.p.'s. Each b.d.p. P_0 can be characterized as being of one of the following two kinds:

I. There is exactly one shock curve originating at P_0, which is either an 1-shock or a 2-shock.

II. There are exactly two shock curves originating at P_0; one is an 1-shock and another a 2-shock.

The methods used in this paper are applicable to some other problems of (1) (see Remark 1). The condition (7) is satisfied by the system which governs the motion of an isentropic polytropic gas (see Remark 2). I am grateful to Professor H. Weinberger and to the referee for several helpful comments.

The condition (7) $(i=0)$ is the genuine nonlinearity condition of a hyperbolic 2-conservation law. Without imposing (7) $(i>0)$ we find two other possible characterizations of a b.d.p. P_0.

III. There is a (infinite) sequence of shocks originating at a sequence of points $\{P_k\}$ convergent to P_0. Moreover, the set $\{P_k\}$ is the only set with the above property in a neighborhood of P_0 and $\{P_k\}$ contains no subsequence convergent to a point other than P_0.

IV. There is a sequence of points $\{P_k\}$ convergent to P_0 such that for each k there is a sequence of shocks originating at a sequence of points convergent to P_k.

The condition (8) is the first case of MacCamy and Mizel [10]. The second and third cases are $\min(a_1, a_2)\leqq a\leqq\max(a_1, a_2)$ and $a>\max(a_1, a_2)$. Under (7) $a_2=\infty$ and the third case does not occur. As will be seen in Section 2 a local solution of (1), (2) can be constructed and extended from some solutions $X(R)=(x(r,s), t(r,s))$ of (3) over Ω. The second case differs from (8) in the shapes of the images $X(\Omega)$ in the (x,t) plane. For this case applying the methods used in this paper we obtain the same kinds of characterizations of the b.d.p.'s.

From now on in this section and § 2 the results are derived without assuming (7). The following lemmas are Lemmas 1, 2, and 3 of [1].

LEMMA 1. *Suppose* $X(R)$ *is a solution of* (3) *in an open region* W *with the property that any two points in* W *can be connected by one of the following*: (i) *a horizontal segment or a vertical segment in* W, (ii) *one or two line segments in* W *of positive slopes,* (iii) *one or two line segments in* W *of negative slopes, and if* $t_r \cdot t_s \neq 0$ *in* W, *then* $X(R)$ *is a homeomorphism on* W.

LEMMA 2. *Suppose* $X(R)$ *is a solution of* (3). *If* W *is an open region in which* $X(R)$ *is one-to-one with nonvanishing Jacobian* $\det \nabla_R X = 2qt_r t_s$, *then*

$$U(X) = \left(M^{-1}\left(\frac{r(x, t) - s(x, t)}{2}\right), \frac{r(x, t) + s(x, t)}{2}\right) \tag{9}$$

is a solution of (1) *on* $X(W)$.

LEMMA 3. *If* X *satisfies the assumptions of Lemma* 2 *and is continuous on* $\bar{W}$, *and if* $t_r \to 0$ *or* $t_s \to 0$ *as* $R \to R_0 \in \partial W$, *then* $|u_x(X(R))| \to \infty$ *or* $|v_x(X(R))| \to \infty$ *as* $R \to R_0$.

2. The existence of a local solution. The equation (4) can be written

$$(\sqrt{q}\, t_r)_s = q^*(r - s)t_s\,, \text{ or} \tag{10}$$

$$(\sqrt{q}\, t_s)_r = -q^*(r - s)t_r \tag{11}$$

where $q^*(\eta) = -Q'(M^{-1}(\eta/2))/4q^{3/2}(\eta)$. Integrating (10) and (11) with the diagonal $r = s$ of Ω as initial curve and applying successive approximations we have

LEMMA 4. *Suppose* $t(r, s)$ *is a solution of* (4) *over* Ω. *For* $i = 1$ *or* 2, *if* $(-1)^i Q'(M^{-1}((r - s)/2)) > 0$ *in* Ω_i *and if* $(-1)^i t_r < 0$ *and* $(-1)^i t_s > 0$ *along the diagonal* $r = s$ *of* Ω, *then* $(-1)^i t_r < 0$ *and* $(-1)^i t_s > 0$ *in* Ω_i.

Assuming $Q'(\xi) > 0$ for $\xi < 0$ and $Q'(\xi) < 0$ for $\xi > 0$, MacCamy and Mizel [10] construct the local solution of (1), (2) as follows. Let $f_1(x)$ be the portion of $f(x)$ over $[-b, b]$ and $f_2(x)$ that over $[b, 2 - b]$. Solving (3) with the initial conditions

$$x_i^0(r, r) = f_i^{-1}(r),\ t_i^0(r, r) = 0 \tag{12}$$

one obtains a solution $X_i^0(R) = (x_i^0(r, s), t_i^0(r, s))$ over $\Omega_i(i = 1, 2)$. Differentiating (12) with respect to r and using (3) we find

$$(13)\quad (-1)^i t_{ir}^0(r, r) = (-1)^{i+1} t_{is}^0(r, r) = (-1)^i f_i^{-1}(r)/2q(0) < 0 \quad (i = 1, 2)\,.$$

It follows from Lemma 4 that

$$(14)\qquad (-1)^i t_{ir}^0 < 0 \quad \text{and} \quad (-1)^i t_{is}^0 > 0 \quad \text{in} \quad \Omega_i(i = 1, 2)\,.$$

By (6) and the uniqueness theorem of the initial value problem of (3) we can derive that

$$(15)\qquad X_i^0(R) = (1 + (-1)^i - x_i^0(-s, -r), t_i^0(-s, -r))(i = 1, 2)\,.$$

Solving (3) with the characteristic initial conditions

$$(16)\qquad X^1(a, s) = X_1^0(a, s), X^1(r, a) = X_2^0(r, a)$$

one obtains a solution $X^1(R) = (x^1(r, s), t^1(r, s))$ over Ω. Let $\{r = r_0\}$ and $\{s = s_0\}$ denote the line segments $r = r_0$ and $s = s_0$ in Ω. By (16) and (14) there exists a neighborhood Z of $\{r = a\} \cup \{s = a\}$ in Ω such that

$$(17)\qquad t_r^1 < 0 \quad \text{and} \quad t_s^1 < 0 \quad \text{in} \quad Z\,.$$

Let $D_i^0 = X_i^0(\Omega_i)$ $(i = 1, 2)$. By (14), (17), Lemmas 1 and 2 the function defined by $U(X) = U(X_i^0)$ for $X \in D_i^0(i = 1, 2)$ and $U(X) = U(X^1)$ for $X \in X^1(Z)$ as in (9) is a solution in $D_1^0 \cup X^1(Z) \cup D_2^0$ of (1) with initial conditions $u(x, 0) = 0$, $v(x, 0) = f(x)$ for $-b \leqq x \leqq 2 - b$. By (15) the function $U(X)$ satisfies $U(X) = (u(1 + (-1)^i - x, t), -v(1 + (-1)^i - x, t))$ for $X \in D_i^0(i = 1, 2)$. Thus $U(X)$ is a local solution of (1), (2). As we stated in Remark 2 of [1], without assuming any condition on the signs of Q', applying (13) and Lemma 1 we can also construct such a local solution.

From now on in this section we impose no condition on the signs of Q'. Constructing $X^1(R)$ as previously described we can solve a series of characteristic initial value problems as described in § 3 of [10] and construct the functions X_i^{2k}, X_i^{2k+1}, and X^{2k+1} over Ω. For later reference we list the characteristic initial conditions for the functions X_i^1, X_i^2 $(i = 1, 2)$, and X^3 as follows:

$$(18)\qquad \begin{aligned} &X_i^1(-a, s) = (x_2^0(-a, s) - 1 + (-1)^i, t_2^0(-a, s))\,, \\ &X_i^1(r, -a) = (x_1^0(r, -a) + 1 + (-1)^i, t_1^0(r, -a))\,; \\ &X_1^2(a, s) = X_1^1(a, s), X_1^2(r, -a) = X^1(r, -a)\,; \\ &X_2^2(-a, s) = X^1(-a, s), X_2^2(r, a) = X_2^1(r, a)\,; \\ &X^3(-a, s) = X_1^2(-a, s), X^3(r, -a) = X_2^2(r, -a)\,. \end{aligned}$$

Let $D^{2k+1} = X^{2k+1}(\Omega)$, $D_i^{2k+1} = X_i^{2k+1}(\Omega)$, and $D_i^{2k} = X_i^{2k}(\Omega)$ (see Fig. 1).

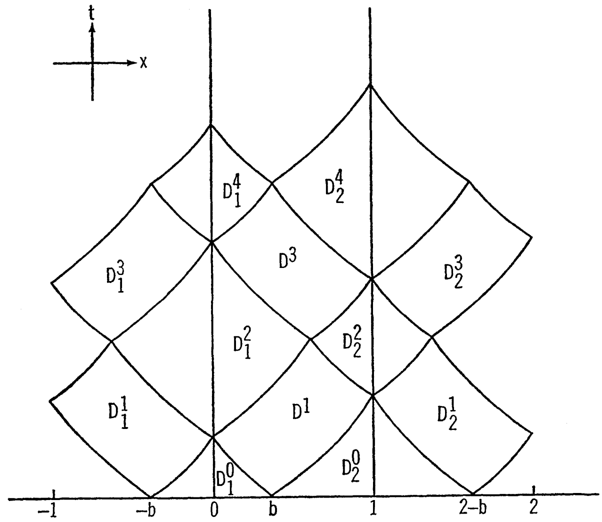

FIGURE 1

By (15), (16), (18), and the uniqueness theorem of the characteristic initial value problem of (3) we can derive that

$$(19)\qquad \begin{aligned} X^{2k+1}(R) &= (1 + (-1)^i - x_i^{2k+1}(-s,\ -r),\ t_i^{2k+1}(-s,\ -r))\ , \\ X_i^{2k+2}(R) &= (1 + (-1)^i - x_i^{2k+2}(-s,\ -r),\ t_i^{2k+2}(-s,\ -r)) \end{aligned}$$

for $k \geqq 0$ and $i = 1, 2$.

Let $\Psi(R) = t^1(R) + t^1(-s, -r) - t_1^0(R) - t_2^0(R)$. The following is Lemma 5 of [1].

LEMMA 5. *For* $k \geqq 0$ *and* $i = 1, 2$, $t^{4k+1}(R) = 2k\Psi(R) + t^1(R)$, $t_i^{4k+2}(R) = (2k + 1)\Psi(R) + t^0_{i-(-1)^i}(R)$, $t^{4k+3}(R) = (2k+1)\Psi(R) + t^1(-s, -r)$, $t_i^{4k+4}(R) = (2k + 2)\Psi(R) + t_i^0(R)$.

Let $\mathring{\Omega}$ denote the interior of Ω and V the set of four vertices $\{(a, a), (-a, a), (-a, -a), (a, -a)\}$ of Ω.

LEMMA 6. *For* $k \geqq 0$ *and* $i = 1, 2$ *the following hold.*

(i) *For* $j = 0, 2 (\mathrm{l} = 1, 3) t_{ir}^{4k+j}(t_r^{4k+\mathrm{l}})$ *is finite in* $(\mathring{\Omega} \cup \{s = a\} \cup \{s = -a\}) - V$ *and* $t_{is}^{4k+j}(t_s^{4k+\mathrm{l}})$ *is finite in* $(\mathring{\Omega} \cup \{r = a\} \cup \{r = -a\}) - V$.

(ii) *Along* $\{r = a\} \cup \{r = -a\}$

$$t_{1r}^{4k} = -t_{2r}^{4k} = -t_r^{4k+1} = -t_{1r}^{4k+2} = t_{2r}^{4k+2} = t_r^{4k+3} = \infty\ .$$

(iii) *Along* $\{s = a\} \cup \{s = -a\}$

$$t_{1s}^{4k} = -t_{2s}^{4k} = t_s^{4k+1} = -t_{1s}^{4k+2} = t_{2s}^{4k+2} = -t_s^{4k+3} = -\infty\ .$$

The parts with $k = 0$, $j = 0$, and $\mathrm{l} = 1$ of Lemma 6(i) and those with superscripts 0, 1 of Lemma 6 (ii), (iii) are Lemmas 6 and 7 of [1]. By Lemma 8 of [1] the function Ψ has finite first-order partial derivatives in Ω. From Lemma 5 the other parts of Lemma 6 follow.

3. The zero curves of t^0_{2r} and of t^0_{2s}. We now assume (7).

LEMMA 7. *Let $t(R)$ denote either $t^{2k}_i(R)$ or $t^{2k+1}(R)$ $(k \geqq 0;\ i = 1, 2)$. If $C_r(C_s)$ is a zero curve of $t_r(t_s)$, then $C_r(C_s)$ contains no horizontal line segment or vertical line segment.*

Proof. Since $Q(\xi)$ and $f(x)$ are analytic, the function $t(R)$ is analytic. For a fixed s_0, $t_r(r, s_0)$ is analytic in r. Now a real analytic function of one variable either is a constant function or does not take a constant value on any set with an accumulation point. It follows from Lemma 6 (ii) that $t_r(r, s_0)$ does not vanish on a closed interval. Hence C_r contains no horizontal segment. Similarly C_s contains no vertical segment. By (10) and (11) we then derive that C_r contains no vertical segment and that C_s contains no horizontal segment.

Applying (10) and (11) we can prove

LEMMA 8. *Let $t(R)$ denote either $t^{2k}_i(R)$ or $t^{2k+1}(R)$ $(k \geqq 0;\ i = 1, 2)$. If $t_r(r_1, s_1) = t_r(r_1, s_2) = 0$ along $\{r = r_1\}$, then $t_s(r_1, s_0) = 0$ for some s_0 between s_1 and s_2. If $t_s(r_3, s_3) = t_s(r_4, s_3) = 0$ along $\{s = s_3\}$, then $t_r(r_0, s_3) = 0$ for some r_0 between r_3 and r_4.*

Under (7) by Theorem 4(i) of [1] the solution U of (1), (2) breaks down. By Lemma 4, $t^0_{1r} > 0$ and $t^0_{1s} < 0$ in Ω_1. It follows from Lemmas 2 and 3 that U does not break down in D^0_1. Now $Q'(M^{-1}((r - s)/2)) < 0$ in Ω_2. The method of successive approximations used in showing Lemma 4 is not applicable to show that $t^0_{2r} < 0$ and $t^0_{2s} > 0$ in Ω_2. In this section we assume that $t^0_{2r} \cdot t^0_{2s}$ vanishes somewhere in Ω_2. That is, U breaks down in D^0_2. By (15)

$$t^0_{2r}(r, s) = -t^0_{2s}(-s, -r) \quad \text{for} \quad (r, s) \in \Omega_2 \,. \tag{20}$$

Then both t^0_{2r} and t^0_{2s} vanish somewhere in Ω_2.

From now on in this section let $X(R) = (x(R), t(R))$ denote $X^0_2(R)$. By (13) $t_r(r, r) = -t_s(r, r) < 0$. Let F be the family of subsets S of Ω_2 which satisfy: (i) S is open and connected with respect to the induced Enclidean topology on Ω_2, (ii) S contains the diagonal $r = s$, (iii) $t_r < 0$ and $t_s > 0$ in S. The set $N = \bigcup_{S \in F} S$ is the largest member of F; $N \subsetneqq \Omega_2$. Let $A_r = \{R \in \Omega_2\colon t_r(R) = 0\}$ and $A_s = \{R \in$

Ω_2: $t_s(R) = 0\}$.

LEMMA 9. (i) *In* N, $t_r < 0$ *and* $t_s > 0$. (ii) $\partial N \subseteqq A_r \cup A_s$.

For a fixed number r^0 we denote the curve $t(r, s) = t(r^0, -r^0)$ through the point $(r^0, -r^0)$ by $C_{t(r^0,-r^0)}$. Let $\alpha = \inf\{r$: $-a < r < 0$; $C_{t(r,-r)}$ lies in $N\}$ $(-a < \alpha < 0)$, $T_0 = t(\alpha, -\alpha)$, and $C_0 = C_{T_0}$. Let $r = r_0(s)$, $s^* \leqq s \leqq a$, describe C_0. From now on by an increasing (decreasing) function we mean a strictly increasing (decreasing) function. By a method similar to that used in showing Lemma 3.3 of [2] we can show

LEMMA 10. (i) *The function* $r_0(s)$ *is increasing.* (ii) $\partial N \cap C_0 \neq \phi$. (iii) *The interior of the region to the right of* C_0 *in* Ω_2 *is contained in* N.

We describe the X images of the region N, of a curve $C_{t(r^0,-r^0)} (r^0 > \alpha)$, and of the curve C_0 in Fig. 2. By Lemma 9 and Lemma 10 (iii)

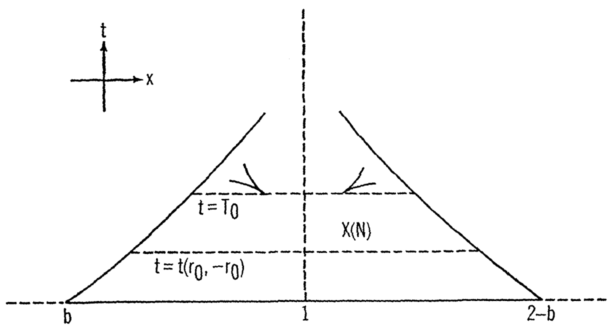

FIGURE 2

$$\partial N \cap C_0 = (A_r \cap C_0) \cup (A_s \cap C_0)\ . \tag{21}$$

LEMMA 11. *The set* $\partial N \cap C_0$ *is countable.*

Proof. For any arc γ of C_0, applying Lemma 7, integrating (10) and (11) with γ as initial curve, and applying successive approximations, we can prove that $A_r \cap \gamma \subsetneqq \gamma$ and $A_s \cap \gamma \subsetneqq \gamma$. Then the closed sets $\{s$: $s^* \leqq s \leqq a$; $t_r(r_0(s), s) = 0\}$ and $\{s$: $s^* \leqq s \leqq a$; $t_s(r_0(s), s) = 0\}$ contain no closed interval. Thus both of them are countable. It follows from (21) that $\partial N \cap C_0$ is countable.

We denote a zero curve $s = s(r)$ $(r = r(s))$ of $\partial^m t/\partial r^m (\partial^m t/\partial s^m)$ in Ω_2 by $C_{r^m}(C_{s^m}) (m \geqq 1)$ and $C_{r^1}(C_{s^1})$ by $C_r(C_s)$. We call a number x_0 a relative maximum (minimum) point of a certain function $\phi(x)$ if $\phi(x)$ has a relative maximum (minimum) at x_0. We call x_0 a relative extreme point of $\phi(x)$ if it is either a relative maximum point or a relative minimum point of $\phi(x)$.

DEFINITION 1. A point $R_0 = (r_0, s_0)$ is a bending maximum point of $C_{rm}(C_{sm})$ if the function $s = s(r)$ $(r = r(s))$ describing $C_{rm}(C_{sm})$ has exactly one relative extreme point $r_0(s_0)$ in a neighborhood of $r_0(s_0)$, which is a relative maximum point. We define the bending minimum point similarly. The point R_0 is a bending point of $C_{rm}(C_{sm})$ if it is either a bending maximum point or a bending minimum point of $C_{rm}(C_{sm})$. A $C_{rm}(C_{sm})$ is of the first kind through R_0 if it contains only finitely many bending points in a neighborhood of R_0.

In considering $A_r \cap C_0$ and $A_s \cap C_0$, by (21), Lemma 10 (ii), and Lemma 6(ii), (iii), we have the following five cases:

Case 1. $(A_r \cap C_0 \cap \mathring{\Omega}_2) - A_s \neq \phi$.
Case 2. $(A_r \cap C_0 \cap \{s = a\}) - (A_s \cup V) \neq \phi$.
Case 3. $(A_s \cap C_0 \cap \mathring{\Omega}_2) - A_r \neq \phi$.
Case 4. $(A_s \cap C_0 \cap \{r = -a\}) - (A_r \cup V) \neq \phi$.
Case 5. $A_r \cap A_s \cap C_0 \cap \mathring{\Omega}_2 \neq \phi$.

We consider whether there exists a C_r or a C_s through each point in these intersections and whether these intersections contain only isolated points of $\partial N \cap C_0$.

For Case 1, given $R_0 \in (A_r \cap C_0 \cap \mathring{\Omega}_2) - A_s$, by a method similar to that used in Case 1 of § 3 of [2], we can show that there is a unique C_r, which is of the first kind through R_0, with R_0 as a bending minimum point. We describe this C_r in Fig. 3. By (21) the point R_0 is an isolated point of $\partial N \cap C_0$.

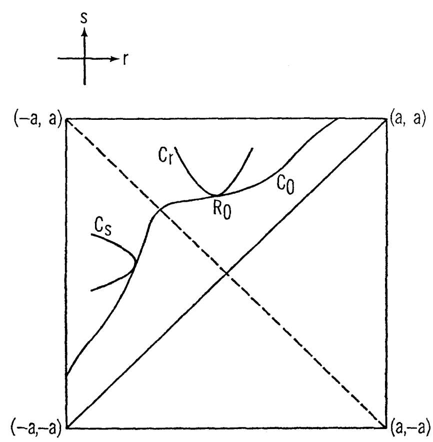

FIGURE 3

For Case 2 we prove

LEMMA 12. *Given* $R_0 = (r_0, a) \in (A_r \cap C_0 \cap \{s = a\}) - (A_s \cup V)$, *the following hold.*

(i) *There is a deleted neighborhood of* R_0 *in* Ω_2 *in which* $t_r < 0$.

(ii) *There is a unique zero curve* C_r *of* t_r^1, *which is of the first kind through* R_0, *with* R_0 *as a bending maximum point. Suppose* $s = s(r)$ *describes* C_r. *We also have that* $t_r^1(r, s) < 0 (>0)$ *for* $s > s(r)$ $(s < s(r))$ *and* $|r - r_0|$ *small.*

(iii) *The point* R_0 *is an isolated point of* $\partial N \cap C_0$.

Proof. By Lemma 6(iii) there is a convex neighborhood W of R_0 in Ω_2 such that $t_s > 0$ in W. Suppose there is a zero curve C_r of t_r through R_0. By Lemma 8 it is the only such curve through R_0. By Lemma 10 (iii) C_r lies to the left of C_0. In W we choose a point $R^0 = (r^0, s^0)$ in $\mathring{\Omega}_2$ lying strictly between C_r and C_0 such that the line segment $\{(r, s^0): r^0 \leqq r \leqq r_0(s^0)\}$ lies in N. Then $t(R^0) > t(r_0(s^0), s^0) = T_0$. By (10) we can show, however, that the curve $t(r, s) = t(R^0)$ through R^0 meets C_0. This is a contradiction. Thus there is no zero curve of t_r through R_0. This derives (i).

By (16) $t_r^1(r, a) = t_r(r, a) (-a \leqq r \leqq a)$. By (4), (7), and Lemma 6(iii) $t_{rs}^1 < 0$ in a neighborhood W_1 of R_0 in Ω_2. Then $t_r^1(r_0, s) > 0$ for small positive $a - s$. Since $t_r^1(r, a) < 0$ for r in a deleted neighborhood of r_0, we may choose W_1 sufficiently small such that $t_r^1 < 0$ along $(W_1 \cap \{s = a\}) - \{R_0\}$. It follows from (10) that there is a unique zero curve C_r of t_r^1 lying in $\mathring{W}_1 \cup \{R_0\}$ through R_0.

We prove that C_r is of the first kind through R_0.

Since $t_r^1(r, a)$ is analytic in r at r_0, there is an even number m such that

$$\text{(22)} \quad \partial^i t_r^1/\partial r^i(R_0) = 0 (0 \leqq i \leqq m - 1) \quad \text{and} \quad \delta = \partial^m t_r^1/\partial r^m(R_0) < 0 .$$

We claim that for R in some neighborbood W_2 of R_0,

$$\text{(23)} \qquad \partial^k t_s^1/\partial r^k(R) < 0 \quad (0 \leqq k \leqq m) .$$

For, by Lemma 6(iii) $t_s^1 < 0$ in a neighborhood of R_0. For $m \leqq 4$ or for $k \leqq 3$, by (4), (22), and (7) we can verify directly that (23) holds. For $m \geqq k \geqq 4$, by (4), (22), (7), and by induction we can prove that in a small neighborhood of R_0, the sign of the function $\partial^k t_s^1/\partial r^k$ is the same as that of the following function:

$$-\partial^{k-1}\rho/\partial r^{k-1} t_s^1 + \binom{k-1}{k-2} \partial^{k-2}\rho/\partial r^{k-2} \rho t_s^1 + \binom{k-1}{k-3} \partial^{k-3}\rho/\partial r^{k-3} (\rho_r - \rho^2) t_s^1$$

$$
\begin{aligned}
&+ \sum_{i=4}^{k} \binom{k-1}{k-i} \partial^{k-i}\rho/\partial r^{k-i} \Biggl\{ \partial^{i-2}\rho/\partial r^{i-2} \\
&+ \sum_{j=4}^{i} (-1)^{j-1} \Biggl[\binom{i-2}{i-j+1} \partial^{i-j+1}\rho/\partial r^{i-j+1}\rho^{j-3} \\
&+ \sum_{\mathfrak{l}=2}^{j-2} \Biggl(\sum_{i_2=0}^{i-j} \binom{i-2}{i_2} \partial^{i_2}\rho/\partial r^{i_2} \Biggl(\sum_{i_3=0}^{i-j-i_2} \binom{i-3-i_2}{i_3} \partial^{i_3}\rho/\partial r^{i_3} \cdots \\
&\times \Biggl(\sum_{i_{\mathfrak{l}}=0}^{i-j-i_2-\cdots-i_{\mathfrak{l}-1}} \binom{i-\mathfrak{l}-i_2-\cdots-i_{\mathfrak{l}-1}}{i_{\mathfrak{l}}} \partial^{i_{\mathfrak{l}}}\rho/\partial r^{i_{\mathfrak{l}}} \\
&\times \binom{i-\mathfrak{l}-1-i_2-\cdots-i_{\mathfrak{l}}}{i-j+1-i_2-\cdots-i_{\mathfrak{l}}} \partial^{i-j+1-i_2-\cdots-i_{\mathfrak{l}}}\rho/\partial r^{i-j+1-i_2-\cdots-i_1} \\
&\times \rho^{j-\mathfrak{l}-2} \Biggr) \cdots \Biggr) \Biggr] + (-1)^i \rho^{i-1} \Biggr\} t_s^1 \,.
\end{aligned}
$$

It follows from (7) that (23) holds.

By (22) we can choose W_2 sufficiently small in which $\partial^m t_r^1/\partial r^m < 0$. This and (23) imply that the curves $\partial^m t^1/\partial r^m(r, s) = \text{constant}$ in W_2 are decreasing. Then there is a unique zero curve C_{r^m} of $\partial^m t^1/\partial r^m$ through R_0, which is decreasing in W_2. Let $r = \phi(s)$ describe C_{r^m}. Expanding $\partial^m t^1/\partial r^m(r, a)$ into power series about r_0 gives $\partial^m t^1/\partial r^m(r, a) = \delta(r - r_0) + o(|r - r_0|)$. By (22) $\partial^m t^1/\partial r^m(r, s) > 0(<0)$ for $(r, s) \in W_2$ such that $r < \phi(s)(r > \phi(s))$. It follows from (23) that there is a unique zero curve $C_{r^{m-1}}$ of $\partial^{m-1}t^1/\partial r^{m-1}$ in W_2, which is of the first kind through R_0, with R_0 as a bending maximum point. Let $s = \tilde{\phi}(r)$ describe $C_{r^{m-1}}$. We also find that $\partial^{m-1}t^1/\partial r^{m-1}(r, s) < 0(>0)$ for $(r, s) \in W_2$ such that $s > \tilde{\phi}(r)(s < \tilde{\phi}(r))$. Applying similar argument repeatedly we can derive that C_r is of the first kind through R_0 with the other properties listed in (ii).

By (i) and (21) we have that (iii) holds.

For Case 3, given $R_0 \in (A_s \cap C_0 \cap \Omega_2) - A_r$, by (20) and applying the result of Case 1 we find that there is a unique zero curve C_s of t_s as shown in Fig. 3, which is of the first kind through R_0, with R_0 as a bending maximum point and that R_0 is an isolated point of $\partial N \cap C_0$.

For Case 4, given $R_0 \in (A_s \cap C_0 \cap \{r = -a\}) - (A_r \cup V)$, by (20) and applying Lemma 12 (i), (iii) we find that there is a deleted neighborhood of R_0 in Ω_2 in which $t_s > 0$ and $t_r < 0$ and that R_0 is an isolated point of $\partial N \cap C_0$. By (18) $t_{2s}^1(-a, s) = t_s(-a, s)(-a \leqq s \leqq a)$ while by (19) $t_{2s}^1(r, s) = -t_r^1(-s, -r)$. Applying Lemma 12 (ii) we find that there is a unique zero curve C_s of t_{2s}^1, which is of the first kind through R_0, with R_0 as a bending minimum point. Suppose $r = r(s)$ describes C_s. We also have that $t^1_{2s}(r, s) > 0$ (<0) for $r < r(s)$ ($r > r(s)$) and $|s - s_0|$ small.

For Case 5 we prove

LEMMA 13. *Given $R_0 = (r_0, s_0) \in A_r \cap A_s \cap C_0 \cap \mathring{\Omega}_2$, the following hold.*

(i) *There is a convex neighborhood W of R_0 in $\mathring{\Omega}_2$ such that $t_s > t_r$ in $W - \{R_0\}$.*

(ii) *There is a unique zero curve $C_r(C_s)$ of $t_r(t_s)$, which is of the first kind through R_0, with R_0 as a bending minimum (maximum) point.*

(iii) *Suppose $s = s(r)$ ($r = r(s)$) describes $C_r(C_s)$. The function $s = s(r)$ ($r = r(s)$) is smooth everywhere except perhaps at $r = r_0$ ($s = s_0$). For $s < s(r)(s > s(r))$, $|r - r_0|$ and $|s - s_0|$ small, $t_r(r, s) < 0$ (>0). For $r > r(s)$ ($r < r, (s)$), $|r - r_0|$ and $|s - s_0|$ small, $t_s(r, s) > 0$ (<0).*

(iv) *C_s lies to the left of C_r as shown in* Fig. 4.

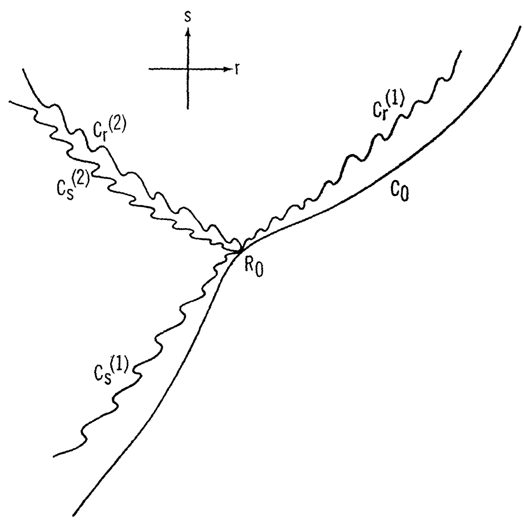

FIGURE 4

(v) *The point R_0 is an isolated point of $\partial N \cap C_0$.*

Proof. By Lemma 10 (i), (iii) and Lemma 9 (i)

$$(24)\quad t_r(r, s_0) < 0 \quad \text{for} \quad r_0 < r \leqq s_0 \text{ and } t_s(r_0, s) > 0 \quad \text{for} \quad r_0 \leqq s < s_0\,.$$

Since $t(r, s)$ is analytic, there are positive integers m and n such that

$$(25)\quad \begin{aligned} &\partial^i t_r/\partial r^i(R_0) = 0 (0 \leqq i \leqq m-1),\ \alpha_0 = \partial^m t_r/\partial r^m(R_0) \neq 0\,, \\ &\partial^j t_s/\partial s^j(R_0) = 0 (0 \leqq j \leqq n-1),\ \beta_0 = \partial^n t_s/\partial s^n(R_0) \neq 0\,. \end{aligned}$$

Using (4) and (25) we can show by induction that

$$\partial^{i+j}t/\partial r^i\partial s^j(R_0) = 0 \ (1 \leqq i \leqq m; 1 \leqq j \leqq n) . \tag{26}$$

Expanding $t(r, s)$ into power series about R_0 and applying (26) give

$$\begin{aligned} t(r, s) = T_0 + (\alpha_0/(m+1)!)(r - r_0)^{m+1} + (\beta_0/(n+1)!)(s - s_0)^{n+1} \\ + o(|r - r_0|^{m+1}) + o(|s - s_0|^{n+1}) . \end{aligned} \tag{27}$$

Differentiating (4) and using (25) and (26) we obtain

$$\begin{aligned} \alpha_1 &= \partial^{m+1}t_s/\partial r^{m+1}(R_0) = \rho(r_0 - s_0)\alpha_0 \neq 0 , \\ \beta_1 &= \partial^{n+1}t_r/\partial s^{n+1}(R_0) = -\rho(r_0 - s_0)\beta_0 \neq 0 . \end{aligned} \tag{28}$$

Expanding t_r and t_s into power series about R_0 and applying (25), (26), and (28) give

$$\begin{aligned} t_r(r, s) &= (\alpha_0/m!)(r - r_0)^m + (\beta_1/(n+1)!)(s - s_0)^{n+1} + o(|r - r_0|^m) \\ &\quad + o(|s - s_0|^{n+1}) , \\ t_s(r, s) &= (\beta_0/n!)(s - s_0)^n + (\alpha_1/(m+1)!)(r - r_0)^{m+1} \\ &\quad + o(|s - s_0|^n) + o(|r - r_0|^{m+1}) . \end{aligned} \tag{29}$$

Recall that $t(r_0(s), s) \equiv T_0$, where $r = r_0(s)$ describes C_0. By the continuity of t_r and by Lemma 11, given any $\eta > 0$, there is a point (r_i, s_i) lying to the left of C_0, $0 < (-1)^i(r_0 - r_i) < \eta$ and $0 < (-1)^i (s_0 - s_i) < \eta$, such that the line segments $\{(r, s_i): r_i \leqq r \leqq r_0(s_i)\}$ lie in N $(i = 1, 2)$. Then by Lemma 9(i)

$$t(r_i, s_i) > T_0 (i = 1, 2) . \tag{30}$$

By (24), (29), (27), and (30) we can show that m and n are even numbers. It follows from (24) and (29) that

$$\alpha_0 < 0 \quad \text{and} \quad \beta_0 > 0 . \tag{31}$$

By (29) and (31) we can derive (i).

Differentiating (4) and using (7), (25), (26), (31) we obtain

$$\begin{aligned} (-1)^{j+1}\alpha_j = (-1)^{j+1}\partial^{j+m+1}t/\partial r^{m+1}\partial s^j(R_0) > 0 \ (1 \leqq j \leqq n) , \\ \beta_i = \partial^{i+n+1}t/\partial r^i\partial s^{n+1}(R_0) > 0 (1 \leqq i \leqq m) . \end{aligned} \tag{32}$$

Expanding $\partial^m t_s/\partial r^m$ into power series about R_0 gives

$$\begin{aligned} \partial^m t_s/\partial r^m(r, s) = \alpha_1(r - r_0) + (\beta_m/n!)(s - s_0)^n \\ + o(|r - r_0|) + o(|s - s_0|^n) . \end{aligned}$$

It follows from (32) that $\partial^m t_s/\partial r^m(r, s) > 0$ for small positive $r - r_0$ and $|s - s_0|$. This and (31) imply that for small nonnegative $s - s_0$ there is a unique zero curve C_{rm} of $\partial^m t/\partial r^m$ through R_0, which is increasing. Let $r = \phi(s)$, $s \geqq s_0$, describe C_{r^m}. By (31)

$$\partial^m t/\partial r^m(r, s) > 0 \quad \text{for} \quad r < \phi(s), |r - r_0| \quad \text{and} \quad s - s_0 \quad \text{small},$$

and

$$(33) \quad \partial^m t/\partial r^m(r, s) < 0 \quad \text{for} \quad r > \phi(s), r - r_0 \quad \text{and} \quad s - s_0 \quad \text{small}.$$

Expanding $\partial^{m-1}t_s/\partial r^{m-1}$ into power series about R_0 and applying (32) give that $\partial^{m-1}t_s/\partial r^{m-1} > 0$ in a deleted neighborhood of R_0. This and (33) imply that for small nonnegative $s - s_0$ there is a unique zero curve $C_{r^{m-1}}$ of $\partial^{m-1}t/\partial r^{m-1}$, which is of the first kind through R_0, with R_0 as a bending minimum point. Let $s = \tilde{\phi}(r)$ describe $C_{r^{m-1}}$. By (33) we have that $\partial^{m-1}t/\partial r^{m-1}(r, s) < 0$ (>0) for $s_0 \leqq s < \tilde{\phi}(r)(s > \tilde{\phi}(r))$, $|r - r_0|$ and $s - s_0$ small. Applying similar argument repeatedly we can derive that there is a unique zero curve C_r of t_r, which is of the first kind through R_0, described by a function $s = s(r)$ with R_0 as a bending minimum point and that $t_r(r, s) < 0$ (>0) for $s_0 \leqq s < s(r)$ $(s > s(r))$, $|r - r_0|$ and $s - s_0$ small. By (29), (31), and (32) we also have that $t_r(r, s) < 0$ for $s < s_0$, $|r - r_0|$ and $s_0 - s$ small. This derives the first parts of (ii) and (iii). Similarly we can drive the remaining parts of (ii) and (iii).

The part (iv) follows from (i), (ii), and (iii), and the part (v) follows from (ii) and (21).

LEMMA 14. *The set $\partial N \cap C_0$ is finite.*

Proof. According to the discussions of Cases 1, 3, and 4 and Lemma 12 (iii) for Case 2 and Lemma 13 (v) for Case 5, we find that both the sets $A_r \cap C_0$ and $A_s \cap C_0$ contain only isolated points of $\partial N \cap C_0$. It follows from (21) that $\partial N \cap C_0$ contains only isolated points. This and the Bolzano-Weierstrass theorem imply that $\partial N \cap C_0$ is finite.

We now drop the condition (7) $(i > 0)$ and assume only (7) $(i = 0)$. Lemmas 7, 8, 9, 10, and 11 still hold. The results of Cases 1 and 3 also hold.

Presently for Case 2 we are unable to exclude the possibility that the zero curve C_r of t_r^1 in Lemma (12) (ii) contains countably infinitely many bending maximum points in any neighborhood of R_0. Likewise for case 4 the zero curve C_s of t_{2s}^1 may contain countably infinitely many bending minimum points in any neighborhood of R_0.

Recall that $r = r_0(s)$, $s^* \leqq s \leqq a$, describes C_0. By Lemma 11 any subset of the set $\{s: s^* \leqq s \leqq a;\ (r_0(s), s) \in \partial N \cap C_0\}$ is not a perfect set. For Case 5 there may be three cases. Case 5a. The point R_0 is an isolated point of $\partial N \cap C_0$. Case 5b. There is a (infinite) sequence of isolated points $\{R_k\}$ of $\partial N \cap C_0$ convergent to R_0. Moreover, the set $\{R_k\}$ is the only subset of $\partial N \cap C_0$ with the above property in a neighborhood of R_0 and $\{R_k\}$ contains no subsequence convergent to a point other than R_0. Case 5c. There is a sequence of points $\{R_k\}$ of $\partial N \cap C_0$ convergent to R_0 such that for each k the point R_k is of Case 5b. Suppose $\{R_{k_n}\}$ is the sequence of isolated points of $\partial N \cap C_0$ convergent to R_k.

For Case 5a, Lemma 13(i), (iii), (iv), (v) still hold. The proof of Lemma 13(i) is exactly the same as before. Lemma 13(ii) is modified to:

There is a unique zero curve $C_r(C_s)$ of $t_r(t_s)$ through R_0 which lies above $\{s = s_0\}$ (to the left of $\{r = r_0\}$).

To verify this statement and Lemma 13(iii), (iv) we use the results obtained in proving Lemma 13(i). By (28), (7) ($i = 0$), and (31)

$$\alpha_1 > 0 \quad \text{and} \quad \beta_1 > 0 . \tag{34}$$

By (29), (31), and (34) we can choose the convex neighborhood W of R_0 in Lemma 13(i) sufficiently small such that

$$\begin{aligned} t_r(r, s) < 0 \ \text{ for } \ (r, s) \in \{(r, s) \in W: s \leqq s_0\} - \{R_0\} , \\ t_r(r_0, s) > 0 \ \text{ for } \ (r_0, s) \in \{(r, s) \in W: s > s_0\} , \\ t_s(r, s) > 0 \ \text{ for } \ (r, s) \in \{(r, s) \in W: r \geqq r_0\} - \{R_0\} , \\ t_s(r, s_0) < 0 \ \text{ for } \ (r, s_0) \in \{(r, s) \in W: r < r_0\} . \end{aligned} \tag{35}$$

Now R_0 is assumed to be an isolated point of $\partial N \cap C_0$. By Lemma 10(iii) there exists an arc γ on C_0 containing R_0 such that $\gamma - \{R_0\} \subset N$. We choose W such that $W \cap C_0 \subset \gamma$. By Lemma 9(i) $t_r < 0$ and $t_s > 0$ along $(W \cap C_0) - \{R_0\}$. Let

$$\begin{aligned} W_1 &= \{(r, s) \in W: r > r_0; s > s_0; (r, s) \text{ lies to the left of } C_0\} , \\ W_2 &= \{(r, s) \in W: r < r_0, s > s_0\} , \\ W_3 &= \{(r, s) \in W: r < r_0; s < s_0; (r, s) \text{ lies to the left of } C_0\} . \end{aligned}$$

According to (35) t_r vanishes somewhere in W_1 and in W_2; t_s vanishes somewhere in W_2 and in W_3. By (4), (7) ($i = 0$), and Lemma 13(i),

$$t_{rs} > 0 \quad \text{in} \quad W - \{R_0\} . \tag{36}$$

It follows from the implicit function theorem that there is a unique $C_r = C_r^{(1)} \cup C_r^{(2)}$, which is described by a function $s = s(r)$ smooth everywhere except perhaps at $r = r_0$, through R_0 such that $C_r^{(1)} - \{R_0\} \subset W_1$ and $C_r^{(2)} - \{R_0\} \subset W_2$. There is also a unique $C_s = C_s^{(1)} \cup C_s^{(2)}$, which is described by a function $r = r(s)$ smooth everywhere except perhaps at $s = s_0$, through R_0 such that $C_s^{(1)} - \{R_0\} \subset W_3$ and $C_s^{(2)} - \{R_0\} \subset W_2$. This derives the above modified version of Lemma 13 (ii) and the first part of Lemma 13(iii). From (36) the second part of

Lemma 13 (iii) follows. This and Lemma 13(i) imply Lemma 13(iv).

There are two subcases of Case 5a. (i) Both C_r and C_s are of the first kind through R_0. (ii) C_r (or C_s) contains countably infinitely many bending minimum (or maximum) points in any neighbourhood of R_0.

For Case 5b (Case 5c), for each k (for each k and for each n) the point $R_k(R_{k_n})$ is of one of the following three cases: (i) $R_k(R_{k_n}) \in A_r - A_s$. This is Case 1. (ii) $R_k(R_{k_n}) \in A_s - A_r$. This is Case 3. (iii) $R_k(R_{k_n}) \in A_r \cap A_s$. This is Case 5a.

4. The characterizations of the breakdown points.

THEOREM 1. *Suppose $X(R)$ is a solution of* (3) *over Ω. Given a point $R_0 = (r_0, s_0)$ in Ω, $-a < r_0 < a$ and $-a \leqq s_0 < a$, assume that there is a zero curve C_r of t_r, which is of the first kind through R_0, described by a function $s = s(r)$ with R_0 as a bending minimum point. Assume further that $s(r)$ is smooth everywhere in a neighborhood of r_0 except perhaps at r_0. Suppose W is a convex neighborhood of R_0 in Ω. Let $W_1 = \{(r, s) \in W: s \geqq s(r)\}$ and $W_2 = \{(r, s) \in W: s \geqq s_0\}$. Suppose the following hold.*

(i) $t_r < 0$ *in* $W_2 - W_1$,

(ii) $t_s > 0$ *in* $W_2 - \{R_0\}$,

(iii) $t_s > t_r$ (*or, by* (3), $-x_s > x_r$) *in* $W_2 - \{R_0\}$.

Then there is a unique 1-shock originating at $P_0 = X(R_0)$. Moreover, if $t_s(R_0) > 0$, then this shock is the only shock curve originating at P_0.

By a method similar to that used in proving Theorem 4.3 of [2] we can prove Theorem 1.

DEFINITION 2. Let $I_B = \{(x, T_0): 0 \leqq x \leqq 1\}$, $T_0 > 0$, be the line segment at which the solution U breaks down. Suppose γ is a segment of I_B. Then γ is strongly regular (s.r.) if it contains no b.d.p.; γ is weakly regular (w.r.) if it contains only finitely many b.d.p.'s, each of which is of one of the kinds I and II as stated in § 1; γ is regular (r.) if it is either s.r. or w.r.

THEOREM 2. *Under* (5), (6), (7), *and* (8) *the segment $I_B \cap D_2^0$ is r.*

Proof. It suffices to assume that the solution of U breaks down in D_2^0 and to prove that $I_B \cap D_2^0$ is w.r.

From Lemma 14, $I_B \cap D_2^0$ contains only finitely many b.d.p.'s. To show that each of them is of one of the kinds I and II we consider the five possible cases in § 3. Let $P_0 = X_2^0(R_0)$.

For Case 1 by Theorem 1 there is a unique shock curve originating at P_0, which is an 1-shock. Then P_0 is of the kind I. For Case 3 by (20) and using the result of Case 1 we find that there is a unique shock curve originating at P_0, which is a 2-shock. Thus P_0 is of the kind I.

For Case 2, by Lemma 12 (ii) and Lemma 6 (iii) and applying the same methods as those used in proving Theorems 3.1 and 4.3 of [2], we can prove that there is a unique shock curve originating at P_0, which is an 1-shock. Then P_0 is of the kind I.

For Case 4, by (19) and applying the above arguments for Case 2 we find that there is a unique shock curve originating at P_0, which is a 2-shock. Thus P_0 is of the kind I.

For Case 5, by Lemma 13(ii), (iii), (i) the conditions (i) and (iii) of Theorem 1 hold. We can weaken the condition (ii) of Theorem 1 to: $t_s > 0$ in $\{(r, s) \in W_2: r \geqq r_0, \text{ or } r > \theta(s) \text{ for } r < r_0\} - \{R_0\}$, where $\theta(s)$ is a decreasing function describing a C_s in W_2 through R_0, and derive the first conclusion of Theorem 1. For Case 5 by Lemma 13 (ii), (iii), (iv) this modified condition holds. Thus there is a unique 1-shock originating at P_0. By Lemma 13 (ii), (iii) and (20), and applying the above modified version of Theorem 1 we can also prove that there is a unique 2-shock originating at P_0. Thus P_0 is of the kind II.

THEOREM 3. *Under* (5), (6), (7) ($i = 0$), *and* (8) *the segment* $\mathfrak{l}_B \cap D_2^0$ *is either* r. *or with countably many* b.d.p.'s, *each of which is of one of the four kinds as stated in* § 1.

Proof. We assume that the solution U breaks down in D_2^0 and consider the five possible cases in § 3. Let $P_0 = X_2^0(R_0)$.

As we observed at the end of § 3 the results of Cases 1 and 3 still hold. For each of these two cases, from the proof of Theorem 2 we can verify that P_0 is of the kind I.

For Case 2 since the zero curve C_r of t_r^1 either is of the first kind through R_0 or contains countably infinitely many bending maximum points in any neighborhood of R_0, from the proof of Theorem 2 we can verify that P_0 is of one of the kinds I and III. Similarly for Case 4, P_0 is of one of the kinds I and III.

For Case 5a(i), by the proof of Theorem 2, P_0 is of the kind II. For Case 5a(ii), by the proof of Theorem 2 we can show that P_0 is of the kind III. For Case 5b (5c) from the above discussions of Cases 1, 3, and 5a we can see that P_0 is of one of the kinds III and IV (of the kind IV).

DEFINITION 3. For $k \geqq 0$ and $i = 1, 2$, the initial signs (i.s.) of t_{ir}^{2k} and $t_{is}^{2k}(t_r^{2k+1}$ and $t_s^{2k+1})$ are their own signs if $D_i^{2k}(D^{2k+1})$ lies strictly below $\mathfrak{l}_B$, or their original signs before the occurence of the changes of signs if $\mathfrak{l}_B$ intersects $D_i^{2k}(D^{2k+1})$; the initial signs distribution (i.s.d.) of a region $D_i^{2k}(D^{2k+1})$ is $(+, -)$ if the i.s.'s of t_{ir}^{2k} and t_{is}^{2k} $(t_r^{2k+1}$ and $t_s^{2k+1})$ are $+$ and $-$ respectively. We define the i.s.d.'s $(-, +)$, $(-, -)$, and $(+, +)$ similarly.

By Lemma 6(ii), (iii) we have

LEMMA 15. *For $k \geqq 0$ if the region $D_1^{4k}(D_2^{4k}, D^{4k+1}, D_1^{4k+2}, D_2^{4k+2}, D^{4k+3}$ respectively) either lies strictly below $\mathfrak{l}_B$ or intersects $\mathfrak{l}_B$, then its* i.s.d. *is* $(+, -)$ $((-, +), (-, -), (-, +), (+, -), (+, +)$ *respectively).*

THEOREM 4. *Under* (5), (6), (7), *and* (8) *the segment* $\mathfrak{l}_B$ *is* w.r.

Proof. We consider first the following four cases: Case 1. $\mathfrak{l}_B \subset D_1^0 \cup D^1 \cup D_2^0$, Case 2. $\mathfrak{l}_B \subset D_1^2 \cup D^1 \cup D_2^2$, Case 3. $\mathfrak{l}_B \subset D_1^2 \cup D^3 \cup D_2^2$, Case 4. $\mathfrak{l}_B \subset D_1^4 \cup D^3 \cup D_2^4$.

Case 1. We have observed in § 3 that $\mathfrak{l}_B \cap D_1^0$ is s.r.. By Theorem 2, $\mathfrak{l}_B \cap D_2^0$ is r.

By Lemma 15 the i.s.d. of D^1 is $(-, -)$. As we construct the set N in § 3 we construct the set N^1 which is the largest subset of Ω satisfying (i) N^1 is open and connected, (ii) N^1 contains the vertex (a, a), (iii) $t_r^1 < 0$ and $t_s^1 < 0$ in N^1. Also as we construct the curve C_0 in § 3 we construct the curve C_0^1 in Ω such that $X^1(C_0^1) = (\mathfrak{l}_B \cap D^1) - (D_1^0 \cup D_2^0)$. The curve C_0^1 satisfies: (i) C_0^1 is decreasing. (ii) The interior of the region to the right of C_0^1 in Ω is contained in N^1. Suppose the solution U breaks down at $(\mathfrak{l}_B \cap D^1) - (D_1^0 \cup D_2^0)$. Then $N^1 \subsetneqq \Omega$; if $R \in \partial N^1$, $t_r^1(R) = 0$ or $t_s^1(R) = 0$; $\partial N^1 \cap C_0^1 \neq \varnothing$. Let $A_r^1 = \{R \in \Omega; t_r^1(R) = 0\}$. The condition (7) $(i = 0)$ implies that $q^*(r - s) > 0$ for $(r, s) \in \Omega$. By (11) we can derive that $t_s^1 < 0$ along C_0 and so that $\partial N^1 \cap C_0^1 = A_r^1 \cap C_0^1$. Given any point $R_0 \in A_r^1 \cap C_0^1$, by (4) and the implicit function theorem we derive the same conclusion of Lemma 12(ii). Applying the same methods as those used in proving Theorems 3.1 and 4.3 of [2] we can prove that at $X^1(R_0)$ there originates a unique shock curve, which is an 1-shock. Thus the b.d.p.'s on $(\mathfrak{l}_B \cap D^1) - (D_1^0 \cup D_2^0)$ are of the kind I. By the method used in proving Lemma 14 we can prove that $\partial N^1 \cap C_0^1$ is finite. That is, the segment $(\mathfrak{l}_B \cap D^1) - (D_1^0 \cup D_2^0)$ contains only finitely many b.d.p.'s. We recall that we assumed that U breaks down at $(\mathfrak{l}_B \cap D^1) - (D_1^0 \cup D_2^0)$. It may not break down there. Thus $(\mathfrak{l}_B \cap D^1) -$

$(D_1^0 \cup D_2^0)$ is r. It follows that I_B is w.r.

Case 2. By the same argument used in Case 1 we can show that $(I_B \cap D^1) - (D_1^2 \cup D_2^2)$ is r.

By Lemma 15 the i.s.d. of D_2^2 is $(+, -)$. We construct the set N_2^2 which is the largest subset of Ω satisfying (i) N_2^2 is open and connected, (ii) N_2^2 contains the vertex $(-a, a)$, (iii) $t_{2r}^2 > 0$ and $t_{2s}^2 < 0$ in N_2^2. We also construct the curve $C_{2,0}^2$ in Ω such that $\{(x, t) \in X_2^2(C_{2,0}^2): x \leqq 1\} = I_B \cap D_2^2$. The curve $C_{2,0}^2$ satisfies: (i) $C_{2,0}^2$ is increasing. (ii) The interior of the region to the left of $C_{2,0}^2$ in Ω is contained in N_2^2. By (19) the curve $C_{2,0}^2$ is symmetric with respect to the diagonal $r = -s$. Let $R_1 = (r_0, a)$ and $R_2 = (-a, -r_0)$, $-a < r_0 < a$, be the points of intersection of $C_{2,0}^2$ with $\{s = a\}$ and $\{r = -a\}$. Now R_2 is one of the end points of the curve C_0^1 constructed in Case 1 over which $t_s^1 < 0$. It follows from (18) that $t_{2s}^2(R_2) = t_s^1(R_2) < 0$. By (19) $t_{2r}^2(R_1) = -t_{2s}^2(R_2) > 0$. Combining this with the fact that the interior of the region to the left of $C_{2,0}^2$ in Ω is contained in N_2^2 we find that $t_{2r}^2(r, a) > 0$ for $-a \leqq r \leqq r_0$ and $t_{2s}^2(-a, s) < 0$ for $-r_0 \leqq s \leqq a$. Let $\hat{\Omega}$ be the square with vertices $(-a, a)$, R_1, $(r_0, -r_0)$ and R_2. Integrating (10) and (11) with the sides $\{(r, a): -a \leqq r \leqq r_0\}$ and $\{(-a, s): -r_0 \leqq s \leqq a\}$ as initial curves and applying successive approximations we can prove that $t_{2r}^2 > 0$ and $t_{2s}^2 < 0$ in $\hat{\Omega}$. Now the curve $C_{2,0}^2$ is contained in $\hat{\Omega}$. It follows from Lemmas 2 and 3 that $I_B \cap D_2^2$ is s.r.

By Lemma 15 the i.s.d. of D_1^2 is $(-, +)$ which is the same as that of D_2^1. By a method similar to that used in proving Theorem 2 we can prove that $I_B \cap D_1^2$ is r. It follows that I_B is w.r.

Case 3. As in Case 2, $I_B \cap D_2^2$ is s.r. and $I_B \cap D_1^2$ is r.

By Lemma 15 the i.s.d. of D^3 is $(+, +)$. We construct the set N^3 which is the largest subset of Ω satisfying (i) N^3 is open and connected, (ii) N^3 contains the vertex $(-a, -a)$, (iii) $t_r^3 > 0$ and $t_s^3 > 0$ in N^3. We also construct the curve C_0^3 in Ω such that $X^3(C_0^3) = (I_B \cap D^3) - (D_1^2 \cup D_2^2)$. The curve C_0^3 satisfies (i) C_0^3 is decreasing. (ii) The interior of the region to the left of C_0^3 in Ω is contained in N^3. Suppose U breaks down at $(I_B \cap D^3) - (D_1^2 \cup D_2^2)$. Then $N^3 \subsetneqq \Omega$; if $R \in \partial N^3$, $t_r^3(R) = 0$ or $t_s^3(R) = 0$; $\partial N^3 \cap C_0^3 \neq \emptyset$. Let $A_s^3 = \{R \in \Omega: t_s^3(R) = 0\}$. By (10) we can derive that $t_r^3 > 0$ along C_0^3 so that $\partial N^3 \cap C_0^3 = A_s^3 \cap C_0^3$. Given any point $R_0 = (r_0, s_0) \in A_s^3 \cap C_0^3$, by (4) and the implicit function theorem we can derive that there is a unique zero curve C_s of t_s^3, which is of the first kind through R_0, with R_0 as a bending minimum point. Suppose $r = r(s)$ describes C_s. We also have that $t_s^3(r, s) > 0 (<0)$ for $r < r(s)$ $(r > r(s))$ and $|s - s_0|$ small.

Applying methods similar to those used in proving Theorems 3.1 and 4.3 of [2] we can prove that at $X^3(R_0)$ there originates a unique shock curve, which is a 2-shock. Thus the b.d.p.'s on $(l_B \cap D^3) - (D_1^2 \cup D_2^2)$ are of the kind I. By the method used in proving Lemma 14 we can prove that $\partial N^3 \cap C_0^3$ is finite. Now U may not break down at $(l_B \cap D^3) - (D_1^2 \cup D_2^2)$. Thus this segment is r. It follows that l_B is w.r.

Case 4. As in Case 3, $(l_B \cap D^3) - (D_1^4 \cup D_2^4)$ is r.

By Lemma 15 the i.s.d.'s of D_1^4 and D_2^4 are $(+, -)$ and $(-, +)$ respectively. By a method similar to that used in showing that $l_B \cap D_2^2$ is s.r. in Case 2 we can show that $l_B \cap D_1^4$ is s.r. By the same method as that used in showing Theorem 2 we can show that $l_B \cap D_2^4$ is r. Thus l_B is w.r.

Suppose in general l_B is contained in $D_1^{2n} \cup D^{2n+1} \cup D_2^{2n}(D_1^{2n+2} \cup D^{2n+1} \cup D_2^{2n+2})$, $n \geqq 2$. By Lemma 15 the i.s.d.'s of D_i^{4k+j} and D^{4k+l} are the same as those of D_i^j and D^l respectively, $i = 1, 2; k \geqq 1;$ $j = 0, 2; l = 1, 3; 4k + j \leqq 2n(4k + j \leqq 2n + 2); 4k + l \leqq 2n + 1$. We can apply the arguments used in the above four cases to derive that l_B is w.r.

THEOREM 5. *Under* (5), (6), (7) $(i = 0)$, *and* (8) *the segment* l_B *is either* w.r. *or with countably many* b.d.p.'s, *each of which is of one of the four kinds as stated in* § 1.

Applying the arguments used in the proofs of Theorems 3 and 4 we can prove Theorem 5.

REMARK 1. The methods used in this paper are applicable to the initial value problem of (1) with initial conditions $u(x, 0) = g(x)$, $v(x, 0) = f(x)$, $-\infty < x < \infty$, where $f(g)$ is an odd (even) periodic analytic function with period 2; $f'(x) + Q(g(x))g'(x)$ and $f'(x) - Q(g(x))g'(x)$ vanish on only two finite sets contained in $[0, 1]$. The existence of a local solution and certain recursion formulas similar to Lemma 5 can be derived by the methods of MacCamy and Mizel [10] (see Remark 3 of [1] about the existence of a local solution). We observe from the proof of Theorem 4 that the i.s.d. plays the main role in characterizing the b.d.p.'s. There are four possible distributions $(+, -)$, $(-, +)$, $(-, -)$, and $(+, +)$, which all appeared in the proof of Theorem 4. For the general problem using the recursion formulas similar to Lemma 5 we can derive a lemma similar to Lemma 6, which determines the i.s.d.'s similar to those in Lemma 15. Then the methods used in §§ 3 and 4 can be applied to derive the conclusion of Theorem 4 under (7) and that of Theorem 5 under (7) ($i = 0$).

REMARK 2. Equations (1) govern the motion of an isentropic gas, where u is the specific volume, v the velocity, and $-\int Q^2(u)du$ the pressure. For a polytropic gas $Q(u) = c^2 u^{(-\gamma-1)/2}$, $\gamma > 1$, so that the condition (7) holds.

REMARK 3. We may replace the condition (7) by

$$(37)\quad \partial^i \rho/\partial s^i(r-s) = (-1)^i \partial^i \rho/\partial r^i(r-s) > 0 \quad \text{for} \quad i \geqq 0 \text{ and } (r, s) \in \Omega \,.$$

By applying the same methods as those used in §§ 3 and 4 we can derive the conclusion of Theorem 4 under the further conditions imposed on the initial functions as in Remark 1. The second order quasilinear wave equation $y_{tt} - (1 + \varepsilon y_x)^\alpha y_{xx} = 0$ (see [12]), where α and ε are positive constants, is equivalent to (1) if we set $u = y_x$, $v = y_t$, and $Q(u) = (1 + \varepsilon u)^{\alpha/2}$. For this system the condition (37) holds.

REMARK 4. The conditions of analyticity on Q and f assumed in (5) and (6) were used in the proofs of Lemmas 7 and 11 to exclude the possibility that C_r or C_s contains horizontal or vertical segments. They were also used in the proofs of Lemmas 12 and 13 to initiate the Taylor's series arguments. Now we weaken these conditions to those of C^2. Assume that U breaks down in D_2^0. From (21), $\partial N \cap C_0 = (A_r \cap C_0) \cup (A_s \cap C_0)$. Suppose $\partial N \cap C_0$ contains a horizontal segment. Since $t_2^0(r, s) \equiv T_0$ along C_0, $t_{2r}^0 \equiv 0$ along this segment. It follows from (3) that the X_2^0 image of this segment is a point. Similarly the X_2^0 image of a vertical segment on $\partial N \cap C_0$ is a point. These indicate that the method used in showing the existence of shock curves originating at b.d.p.'s on $I_B \cap D_2^0$ may be applied to this problem. Under the further conditions (7) ($i = 0$) and (8) we can derive the conclusion of Theorem 5. The derivation depends heavily on the use of (10) and (11), which replaces that of the Taylor's series arguments. As for the problem considered in Remark 1, weakening the conditions of analyticity on Q, f, and g to those of C^2 and assuming (7) ($i = 0$) we can also derive the conclusion of Theorem 5. For brevity we do not pursue these problems here.

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