PACIFIC JOURNAL OF MATHEMATICS
Vol. 85, No. 2, 1979

ON THE COMPLETENESS OF SEQUENCES OF PERTURBED POLYNOMIAL VALUES

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If S is an arbitrary sequence of positive integers, define $P(S)$ to be the set of all integers which are representable as a sum of distinct terms of S. Call a sequence S *complete* if $P(S)$ contains all sufficiently large integers, and *subcomplete* if $P(S)$ contains an infinite arithmetic progression. We will prove the following theorem: Let nth term of the integer sequence S have the form $f(n) + O(n^\alpha)$, where f is a polynomial and where $0 \leqq \alpha < 1$; then S is subcomplete. We further show that S is complete if, in addition, for every prime p there are infinitely many terms of S not divisible by p. (We call any sequence satisfying this last property an *R-sequence*.) We will then extend these results to considerably more general sequences.

It can be shown in various ways ([3], [4]) that if f is a polynomial which maps positive integers to positive integers, then the sequence $S = \{f(1), f(2), \cdots\}$ is subcomplete, and if in addition S is an R-sequence, S is complete. In this work we use results of Folkmann's fine paper [2] to generalize these results to perturbed polynomial sequences $f(1) + t(1)$, $f(2) + t(2)$, $\cdots$, where t is a function with sufficiently slow growth. We first state two results of [2].

THEOREM A (Folkman). *Let $A = \{a_n\}$ be a nondecreasing sequence of positive integers satisfying $a_n = O(n^\alpha)$ for some $0 \leqq \alpha < 1$. Then A is subcomplete.*

THEOREM B (Folkman). *Let $A = \{a_n\}$ be a nondecreasing sequence of positive integers with disjoint subsequences $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$. Suppose that*

$$(1) \qquad \lim_{n\to\infty} \frac{1}{b_{n+m}} \sum_{i=1}^{n} b_i = \infty \quad \textit{for each} \quad m > 0 ,$$

that $c_n > d_n$ for each n, and that the sequence $\{c_n - d_n\}$ is subcomplete. Then A is subcomplete.

We now state

THEOREM 1. *Let $S = \{s_1, s_2, \cdots\}$ be asequence of positive integers of the form $s_n = f(n) + O(n^\alpha)$ where f is a polynomial of degree $\geqq 1$ and $0 \leqq \alpha < 1$. Then S is subcomplete.*

Before proving this theorem we first state the case $k = 1$ of it as a lemma. The author is grateful to Carl Pomerance of the University of Georgia for the lemma in its present form. The autho's verion of this lemma required $\alpha < 1/2$, and Theorems 1, 3, and 4 were correspondingly weaker.

LEMMA 1 (Pomerance). *Let* $S = \{s_1, s_2, \cdots\}$ *be a sequence of integers of the form* $s_n = an + O(n^\alpha)$, *where* $a > 0$ *and* $0 \leqq \alpha < 1$. *Then* S *is subcomplete.*

Proof. Let t_n be the sequence S arranged in nondecreasing order. If $t_n = s_m$, it is clear that $|m - n| = O(n^\alpha)$, so that

$$t_n = am + O(m^\alpha) = an + O(n^\alpha) .$$

Hence we may assume without loss of generality that S is monotone nondecreasing. Write $s(n)$ for s_n and form three disjoint subsequences of S given by

$$\begin{aligned} b_n &= s(3n + 2) , \\ c_n &= s(3[n + Mn^\alpha] + 1) , \\ d_n &= s(3n) , \end{aligned}$$

where M is large enough that $c_n > d_n$ for all n. Then $0 < c_n - d_n = O(n^\alpha)$ for all n. Let $\{e_n\}$ be the sequence $\{c_n - d_n\}$ in nondecreasing order. Then

$$e_n \leqq \max_{1 \leqq i \leqq n} (c_i - d_i) = O(n^\alpha) ,$$

and by Theorem A, $\{e_n\}$, and hence $\{c_n - d_n\}$, is subcomplete. Hence, by Theorem B, S is subcomplete. This completes the proof.

Proof of Theorem 1. The case $k = 1$ is just Lemma 1, so we assume the theorem to have been proved for some degree $k \geqq 1$. Let S satisfy the hypotheses with f having degree $k + 1$. Without loss of generality we may assume that S is strictly increasing. Form three disjoint subsequences of S given by $b_n = s_{3n}$, $c_n = s_{3n-1}$, $d_n = s_{3n-2}$. Then

$$\lim_{n \to \infty} \frac{1}{b_{n+m}} \sum_{i=1}^{n} b_i = \infty$$

for any m, and $c_n - d_n = f_0(n) + O(n^\alpha)$, where f_0 is a polynomial of degree k. Thus $\{c_n - d_n\}$ is subcomplete by the induction hypothesis, and hence S subcomplete by Theorem B. This completes the proof.

Note that Theorem 1 does not require f to be integer-valued, or even to have rational coefficients. We will see later that Theorem

1 can be made considerably more general than this. We also remark that Theorem 1 can be proved for bounded perturbations by means of Theorem B alone. To get the full result we must use the powerful Theorem A.

We will prove a theorem which enables us to conclude that an R-sequence satisfying the hypotheses of Theorem 1 is complete. Some preliminary results are necessary. We first state two further theorems taken from [2] and [3] respectively.

THEOREM C (Folkman). *Let* $B = \{b_1, b_2, \cdots\}$ *be an increasing sequence satisfying* (1). *Then for each integer* $r > 0$, *there is an integer* $q(r)$ *such that for any* $k \geqq 0$, *at least one of the numbers*

$$(k + 1)r\ , \qquad (k + 2)r, \cdots, (k + q(r))r$$

is in $P(B)$.

THEOREM D (Graham). *Let* A *be an* R*-sequence. Then for any integer* m, $P(A)$ *contains a complete system of residues modulo* m.

We next prove three simple lemmas.

LEMMA 2. *Let* S *be a sequence with disjoint subsequences* A *and* B. *If* A *is an* R*-sequence and* B *is subcomplete, then* S *is complete.*

Proof. Since B is subcomplete, $P(B)$ contains an infinite arithmetic progression $\{r + u, 2r + u, \cdots\}$. By Theorem D, $P(A)$ contains a complete system of residues modulo r, say $k_1 < k_2 < \cdots < k_r$. Let n be any number $\geqq r + u + k_r$. For some k_i we have $k_i \equiv n - u (\bmod r)$. Then $(n - u - k_i)/r$ is an integer $j \geqq 1$. Thus $n = (jr + u) + k_i$. Since $k_i \in P(A)$ and $jr + u \in P(B)$, $n \in P(S)$. Thus S is complete.

LEMMA 3. *Let the increasing sequence* $B = \{b_n\}$ *satisfy* (1). *Let* $B' = \{b'_n\} = \{b_{i_n}\}$ *be a subsequence of* B *with* $i_{n+1} \leqq i_n + 2$. *Then* B' *satisfies* (1).

Proof. Let $b'_n = b_j$. Then

$$\frac{1}{b'_{n+m}} \sum_{i=1}^{n} b'_i \geqq \frac{1}{b_{j+2m}} (b_j + b_{j-2} + \cdots)$$
$$\geqq 1/2 \frac{1}{b_{j+2m}} \sum_{i=1}^{j} b_i\ .$$

But the last expression $\rightarrow \infty$ as $j \rightarrow \infty$ for any m; so B' satisfies (1).

LEMMA 4. *Let A be a subcomplete sequence, and let B be an increasing sequence satisfying* (1). *Then it is possible to form a subcomplete sequence B' by adjoining to B a finite number of terms of A.*

Proof. Let $P(A)$ contain the infinite arithmetic progression $\{r + u, 2r + u, \cdots\}$. By Theorem C there is a q such that for any $k \geqq 0$, at least one of $(k + 1)r, \cdots, (k + q)r$ is in $P(B)$. It is clear that there is a finite subsequence A_0 of A such that $P(A_0)$ contains all the numbers $r + u, 2r + u, \cdots, qr + u$. Let $j \geqq q + 1$, and choose i among $j - q, \cdots, j - 1$ so that ir is in $P(B)$. Then $jr + u = ir + (j - i)r + u$. But $(j - i)r + u \in P(A_0)$. Thus any number $jr + u$ with $j \geqq q + 1$ is a sum of a number in $P(A_0)$ and a number in $P(B)$. Therefore if we form B' by adjoining the terms of A_0 to B, we see that B' is subcomplete.

We are now in a position to prove

THEOREM 2. *Let S be an R-sequence which is increasing, with disjoint subsequences $A = \{a_n\}$ and $B = \{b_n\}$. If A is subcomplete and B satisfies* (1), *then S is complete.*

Proof. Let $Q = \{q_1, q_2, \cdots\}$ be the set of all primes q with the property that there are infinitely many terms of B which are not divisible by q. We must partition B into two subsequences B_0 and B_1, where for each $q \in Q$, B_0 has infinitely many terms not divisible by q, and where B_1 satisfies (1). This can be done in the following manner. First put into B_0 a term b_i not divisible by q_1. Next put into B_0 a term b_{i+j}, $j \geqq 2$, not divisible by q_2. Continue to place terms b_i into B_0, where successively the terms are not divisible by $q_1, q_2, q_1, q_2, q_3, q_1, q_2, q_3, q_4, \cdots$; this can be done so that each term chosen has an index at least two greater than the previous one chosen. This defines B_0. But by construction B_1, formed by the terms remaining, satisfies the hypothesis of Lemma 3. Thus we have accomplished the desired partition.

We now apply Lemma 4 to the sequences A and B_1 to form a subcomplete sequence B_2 consisting of the terms B_1 and a finite number of terms of A. Now form a sequence A_1 consisting of all terms of S not in B_2. Then A_1 is an R-sequence, since S is an R-sequence and since any prime q which is a non-divisor of infinitely many terms of B_2 also is a nondivisor of infinitely many terms of B_0, and hence of A_1. Thus S has the disjoint subsequences A_1 and B_2, with A_1 an R-sequence and B_2 subcomplete. Therefore, by Lemma 4, S is complete.

We may now derive our desired result on perturbed polynomials as an easy corollary to Theorem 2.

THEOREM 3. *Let S satisfy the conditions of Theorem* 1, *and let S be an R-sequence. Then S is complete.*

Proof. Let $S_1 = \{s_1, s_3, \cdots\}$ and $S_2 = \{s_2, s_4, \cdots\}$. Then s_1 is subcomplete since it satisfies the conditions of Theorem 1, and S_2 clearly satisfies (1), and may be assumed without loss of generality to be increasing. Hence S is complete by Theorem 2, and the result is proved.

It is possible to extend Theorems 1 and 3 to considerably more general sequences, namely ones in which f is a "polynomial" with nonintegral exponents. Specifically, we have

THEOREM 4. *Let $a_1, a_2, \cdots, a_r$ and $\gamma_1 > \gamma_2 > \cdots > \gamma_r$ be real numbers, where $a_1 > 0$ and $\gamma_1 \geqq 1$. Let $f(n) = a_1 n^{\gamma_1} + a_2 n^{\gamma_2} + \cdots + a_r n^{\gamma_r}$. Let $S = \{s_1, s_2, \cdots\}$ be a sequence of positive integers of the form $s_n = f(n) + O(n^\alpha)$. Then S is subcomplete. If in addition, S is an R-sequence, S is complete.*

Proof. The proof is very similar to that of Theorems 1 and 3, so we will not carry out all the details. The proof for $1 \leqq \gamma_1 < 2$ is the same as for Lemma 1, except that an is replaced by $f(n)$ and α is replaced by

$$\max\,(\alpha, \gamma_1 - 1, \max_{\gamma_i < 1} \gamma_i)\ .$$

Now assume the theorem true for $k \leqq \gamma_1 < k + 1$, where k is an integer $\geqq 1$. If S satisfies the hypotheses with $k + 1 \leqq \gamma_1 < k + 2$, the construction of Theorem 1 can be applied. The only additional detail is that terms like $n^\gamma - (n - 1)^\gamma$ produce infinite series. However, this causes no difficulty, since all but a finite number of terms grow more slowly than n^α and can be included in the perturbation term. Thus S is seen to be subcomplete.

Finally, if S is an R-sequence, Theorem 2 may be applied to show that S is complete. This completes the proof.

We conclude with a few remarks on possible extensions of the results given. One obvious possibility is to extend the allowable functions f in Theorem 4. This can certainly be done since it is not hard to see that f may be permitted to be an absolutely convergent infinite series with terms of the form $a_i n^{\gamma_i}$. More interesting would be an extension to functions satisfying some smoothness condition. Another possibility would be to weaken the condition on the perturbation term. A result of [1] shows that Theorem 1 is false with $\alpha > 1$. It seems possible that the theorem holds for $\alpha = 1$. It would be interesting to weaken the conditions of Theorem 2. Thus, in [2] it is shown that for a sequence of Theorem A to be complete, it suffices that $P(A)$ contain a complete system of residues with respect to every modulus. It seems unlikely that such a weak condition would suffice in the present case, but the author knows no counterexample.

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Received July 12, 1977 and in revised form May 11, 1979.

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