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THE SECOND LIE ALGEBRA COHOMOLOGY GROUP AND WEYL MODULES

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Dedicated to Gerhard Hochschild on the occasion of his 65th birthday

The Lie algebra 1-cohomology of classical Lie algebras in characteristic p is nonzero at some modules. In fact, Hochschild showed that the restricted 1-cohomology is nonzero. Here I will systematically produce Weyl modules at which the (unrestricted) 2-cohomology is nonzero. This will provide examples of nonsplit abelian extensions of Lie algebras by Weyl modules in characteristic p, where the Lie algebras are the reductions modulo p of integral Chevalley forms of complex semisimple Lie algebras.

The method requires passages among cohomology groups for Lie algebras defined over the integers, rationals, and the prime field of characteristic p. In proving Theorem 1, we show that the 2-cohomology of an integral Lie algebra has nonzero p-torsion at a module whenever the reduced (modulo p) algebra has nonzero 1-cohomology at the reduced module and the extension of the integral Lie algebra to the rationals has zero for its first and second cohomology at the extension of the module by the rationals. The tensor product of the integral cohomology group with the prime field is a nonzero piece of the cohomology group of the reduced Lie algebra. This will give the examples that we seek provided that we produce modules which satisfy the conditions given above for the groups of the reduced and rational algebras. The second condition is satisfied in the classical case since the rational Lie algebra will be semisimple. As for the first condition, in [4] I gave irreducible modules in characteristic p at which the 1-cohomology is nonzero. I will show that the cohomology is also nonzero at a Weyl module associated to the irreducible module. The module that we seek for the integral Lie algebra is one that reduces modulo p to the Weyl module. I have carried out this enterprise of showing that the cohomology is nonzero at the Weyl module only for the Lie algebra of the special linear group although I believe that it can be carried out generally.

In §3, I take a particular one cocycle g (constructed in [4]) and, working with an integral cochain which reduces to g, apply the coboundary operator to obtain explicitly a p-torsion element of the integral cohomology group.

I wish to thank Professor Hochschild for his kindness and generosity over the years.

NOTATION. Let L be a Lie algebra over the ring Z of integers, let V be a Z-module, and let F be any commutative ring with unit.

$V_F = V \otimes_Z F$. F_p, the field with p element $V_p = V_{F_p}$.

$L_F = L \otimes_Z F$, an F-Lie algebra.

If V is an L-module, then V_F is an L_F-module.

$V_F^{L_F} = \{v \in V_F \mid kv = 0$ for all $k \in L_F\}$, the space of invariants.

$C^i(L_F, V_F) =$ space of i-cochains for the Lie algebra cohomology of L_F with values in V_F

$= \mathrm{Hom}_F(\wedge^i L_F, V_F)$, where $\wedge^i L_F$ is the ith exterior power of L_F.

$Z^i(L_F, V_F) =$ space of i-cocycles.

$B^i(L_F, V_F) =$ space of i-coboundaries.

$H^i(L_F, V_F) = i$th cohomology group.

$\delta^i = i$th coboundary operator.

Let $\mathrm{sl}_{n+1,F}$ be the Lie algebra of $n+1$ by $n+1$ trace 0 matrices with entries in F. Let E_{ij} be the trace zero matrix whose only nonzero entry is a 1 in the ijth position. The $\{E_{ij}\}$ belong to the standard Chevalley basis for $\mathrm{sl}_{n+1,F}$.

$Y_1 = E_{21}$.

$h_F =$ space of diagonal matrices in $\mathrm{sl}_{n+1,F}$.

$b_F =$ space of upper triangular matrices in $\mathrm{sl}_{n+1,F}$.

$b =$ space of upper triangular matrices in $\mathrm{sl}_{n+1,Z}$.

α_i is the element in the dual space of h_F whose value at the matrix with diagonal entries t_j is $t_i - t_{i+1}$.

1. Take a Lie algebra L over the ring Z of integers which is a free Z-module of finite rank. Let V be any L-module which is a free Z-module of finite rank. Let Q be the rational numbers.

Suppose that L_Q is a semisimple Lie algebra and that the space of L_Q-invariants in V_Q and the space of L_p-invariants in V_p have the same dimension, say m. Then we have the following theorem.

THEOREM. *If $H^1(L_p, V_p)$ is nonzero, then $H^2(L_p, V_p)$ is nonzero.*

Proof. The proof is a comparison of the ranks and dimensions of various spaces coming from cochain complexes. The canonical map $C^2(L, V) \otimes_Z F_p \to C^2(L_p, V_p)$ is a linear isomorphism since L and V are free Z-modules. Furthermore, the image of $Z^2(L, V) \otimes_Z F_p$ lies in $Z^2(L_p, V_p)$. Hence, the dimension of $Z^2(L_p, V_p)$ is at least as large as the rank of the free Z-module $Z^2(L, V)$, which equals the dimension of $Z^2(L_Q, V_Q)$.

The spaces $Z^i(L_Q, V_Q)$ and $B^i(L_Q, V_Q)$ coincide for $i = 1, 2$ since $H^i(L_Q, V_Q) = (0)$ for $i = 1, 2$ for a semisimple Lie algebra [2]. Therefore, $\dim(Z^2(L_Q, V_Q))$ equals $\dim(C^1(L_Q, V_Q)) - \dim(V_Q) + m$.

The dimension of $B^2(L_p, V_p)$, which equals $\dim(C^1(L_p, V_p)) - \dim(Z^1(L_p, V_p))$, is less than $\dim(C^1(L_p, V_p)) - \dim(B^1(L_p, V_p))$ since $H^1(L_p, V_p)$ is nonzero by hypothesis. Therefore, the dimension of $B^2(L_p, V_p)$ is less than $\dim(C^1(L_p, V_p)) - \dim(V_p) + m$, which equals $\dim(C^1(L_Q, V_Q)) - \dim(V_Q) + m$. By the conclusions of the previous two paragraphs, the dimension of $Z^2(L_p, V_p)$ exceeds the dimension of $B^2(L_p, V_p)$ and $H^2(L_p, V_p)$ is nonzero.

2. We will turn our attention specifically to the Lie algebra sl_{n+1,F_p} of the special linear group SL_{n+1,F_p} over the prime field F_p. Let λ_i be the character on the diagonal subgroup of SL_{n+1,F_p} whose value at the diagonal matrix with diagonal entries $\{t_j\}_j$ is $t_1 \cdots t_i$. Let W be the irreducible SL_{n+1,F_p}-module whose highest weight is $(p-2)\lambda_1 + \lambda_2$. W will also stand for the space W with the differential sl_{n+1,F_p}-module structure. For any SL_{n+1,F_p}-module W', we will give the weight of a line stable under the diagonal subgroup by a character on the diagonal subgroup, even though it may be the differential sl_{n+1,F_p}-module structure of W' that is under consideration.

In [4], we showed that the one-cohomology of sl_{n+1,F_p} at W is nonzero, except possibly for sl_{3,F_3}. Below, we associate to W a Weyl module V_p and show that sl_{n+1,F_p} has nonzero two-cohomology at V_p, except possibly when $p = n + 1$. Let V_Q be the irreducible $\mathrm{SL}_{n+1,Q}$-module whose highest weight is $(p-2)\lambda_1 + \lambda_2$. We construct a Z-form V for V_Q which is stable under the Kostant Z-form U_Z for the universal enveloping algebra of $\mathrm{sl}_{n+1,Q}$ (relative to the standard basis for $\mathrm{sl}_{n+1,Q}$) and whose reduction modulo p, V_p, contains W as a submodule for its SL_{n+1,F_p}-module structure.

The standard Z-form of V_Q is the U_Z-submodule that is generated from a highest weight vector [1, A]. The Z-form V is the variant that is produced as follows. Take a lowest weight vector v^* from the $\mathrm{SL}_{n+1,Q}$-module V_Q^* that is contragradient to V_Q. The weight of v^*, reduced modulo p, equals the lowest weight of W^*. The U_Z-module $U_Z v^*$ is a Z-form for V_Q^* for which the SL_{n+1,F_p}-module $U_Z v^* \otimes_Z F_p$ contains a unique maximal proper submodule. The quotient module is isomorphic to W^* since the lowest weight of the quotient module is the weight of v^*, reduced modulo p. The U_Z-module $V = \mathrm{Hom}(U_Z v^*, Z)$ is a Z-form for V_Q and the SL_{n+1,F_p}-module $V_p = V \otimes_Z F_p$, which is contragradient to $U_Z v^* \otimes_Z F_p$, has a unique irreducible submodule. This submodule is isomorphic to W.

The following lemma and its use in proving the proposition below were suggested by John Ballard. The proof has been consigned to the last section.

LEMMA. *If $p \neq n + 1$, then V_p contains no weight $p\lambda$ where λ*

is dominant.

PROPOSITION. *If $p \neq n+1$, then $H^1(\mathrm{sl}_{n+1,F_p}, V_p)$ is nonzero.*

Proof. From the long exact sequence of cohomology associated with the exact sequence $0 \to W \to V_p \to V_p/W \to 0$, take the exact sequence $(V_p/W)^{\mathrm{sl}_{n+1},F_p} \to H^1(\mathrm{sl}_{n+1,F_p}, W) \to H^1(\mathrm{sl}_{n+1,F_p}, V_p)$. It will be enough to show that the first term is (0) since $H^1(\mathrm{sl}_{n+1,F_p}, W)$ is nonzero. $(V_p/W)^{\mathrm{sl}_{n+1},F_p}$ is an SL_{n+1,F_p}-submodule of V_p/W, which is trivial as an sl_{n+1,F_p}-module. A highest weight vector of this submodule would be dominant weight of the form $p\lambda$, which the lemma excludes from V_p. Therefore, V_p/W has (0) as its space of invariants.

THEOREM. *If $p \neq n+1$, then $H^2(\mathrm{sl}_{n+1,F_p},\_)$ is nonzero at the Weyl module V_p.*

Proof. In the theorem of section one, the hypothesis that $H^1(\mathrm{sl}_{n+1,F_p}, V_p)$ be nonzero is satisfied by the proposition above. The hypothesis that the spaces of invariants of V_Q and V_p have the same dimension is satisfied. In fact, $V_p^{\mathrm{sl}_{n+1},F_p}$ is zero, by the argument used in proving the proposition. Therefore, $V_Q^{\mathrm{sl}_{n+1},Q}$ is zero as well. We may conclude that $H^2(\mathrm{sl}_{n+1,F_p}, V_p)$ is nonzero.

3. We will show the origin of one nonzero cohomology class in $H^2(\mathrm{sl}_{n+1,F_p}, V_p)$.

Denote $\mathrm{sl}_{n+1,Z}$ by L, denote the weight $(p-2)\lambda_1 + \lambda_2$ by β, and let W and V be the modules given in §2. In [4], we constructed a noncobounding one cocycle g for L_p with values in W with the properties that g is zero on b_p and $g(Y_1)$ is a highest weight vector in W. The reduction modulo p of the unique line of weight β in V in the unique line of weight β in V_p. Choose any cochain g_Z in $C^1(L, V)$ for which $g_Z \otimes_Z F_p = g$, $g_Z(b) = (0)$, and $g_Z(Y_1)$ is a vector of weight β. We will show that the 2-cochain $f = ((1/p)\delta^1(g_Z)) \otimes_Z F_p$ represents a nonzero cohomology class for L_p with values in V_p. Since $\delta^1 g = 0$, $\delta^1 g_Z$ is divisible by p in $C^2(L, V)$ and we may form the cochain $(1/p)(\delta^1 g_Z)$ in $C^2(L, V)$. This is a cocycle since $\delta^2((1/p)(\delta^1(g_Z))) = (1/p)(\delta^2\delta^1 g_Z) = 0$ and it maps $b \wedge b$ to 0 since $g_Z(b) = (0)$. Therefore, $f = ((1/p)(\delta^1 g_Z)) \otimes_Z F_p$ is a two cocycle for L_p with values in V_p, which maps $b_p \wedge b_p$ to 0. We show that f is not a coboundary.

Suppose that f were the coboundary $\delta^1 u$ for some cochain $u: L_p \to V_p$. The restriction of u to b_p is a cocycle since $f(b_p \wedge b_p) = (0)$. By the lemma of §3.1 of [4], the canonical image of $H_b^1(b_p, V_p)$ in $H^1(h_p, V_p)$ is contained in the image of $H^1(h_p, V_p^{b_p})$ in $H^1(h_p, V_p)$;

here, since $V_p^{b_p} = (0)$, (0) is the image of $H^1(b_p, V_p)$ in $H^1(h_p, V_p)$. Therefore, there is an element v of V_p such that $u - \delta^0 v$ is zero on h_p. Now use the symbol u for the 'normalized' cochain $u - \delta^0 v$. We show that f cannot equal $\delta^1 u$ by showing that $f(Y_1 \wedge H_1)$ has a nonzero component of weight β, while $\delta^1 u(Y_1 \wedge H_1)$ has zero for its component of weight β. $\delta^1 u(Y_1 \wedge H_1) = (-2 - H_1)u(Y_1)$ has zero as its component of weight β since $-2 - H_1$ annihilates vectors of weight β. By the construction of g_Z, $g_Z(Y_1)$ has as its component of weight β a vector x which is nonzero modulo p. Write $g_Z(Y_1) = x + y$. Then $(1/p)\delta^1 g_Z(Y_1 \wedge H_1) = (1/p)(-2-H_1)g_Z(Y_1) = x + (1/p)(-2-H_1)y$, reduced modulo p, has a nonzero component of weight β.

Thus, we see that $(1/p)\delta^1 g_Z$ represents a cohomology class in $H^2(\mathrm{sl}_{n+1,Z}, V)$ whose reduction modulo p gives a nonzero element of $H^2(\mathrm{sl}_{n+1,F_p}, V_p)$. We wish to remark that the cohomology class of f does not depend on the choice of lifting of g to an integral cochain. In fact, if g'_Z is another cochain such that $g'_Z \otimes_Z F_p = g$, then $(1/p)\delta^1 g_Z - (1/p)\delta^1 g'_Z$ is the two coboundary $\delta^1((1/p)(g_Z - g'_Z))$, and $(1/p)\delta^1 g_Z$ and $(1/p)\delta^1 g'_Z$ represent the same class in $H^2(\mathrm{sl}_{n+1,Z}, V)$.

It would be good to know if $H^2(\mathrm{sl}_{n+1,F_p}, W)$ is also nonzero. In particular, does the cohomology class of the element f lie in the canonical image of $H^2(\mathrm{sl}_{n+1,F_p}, W)$ in $H^2(\mathrm{sl}_{n+1,F_p}, V_p)$?

4. Here we give a proof of the lemma of §2. The weights of the $\mathrm{SL}_{n+1,Q}$-module V_Q have the form $\beta - \sum a_i\alpha_i$, where $\beta = (p-2)\lambda_1 + \lambda_2$ and the a_i are integers in the range $0 \leqq a_1, a_n \leqq p-1$, and $0 \leqq a_i \leqq p$ for $i \neq 1, n$, c.f. [4, 3.3]. The weights of the SL_{n+1,F_p}-module V_p inherit the same property. The roots α_i are expressed in terms of the fundamental dominant weights λ_i as follows: $\alpha_1 = 2\lambda_1 - \lambda_2$, $\alpha_n = 2\lambda_n - \lambda_{n-1}$, and $\alpha_i = 2\lambda_i - \lambda_{i-1} - \lambda_{i+1}$ for $i \neq 1, n$.

Suppose that $p\lambda = \beta - \sum a_i\alpha_i$ is a weight of V_p, where λ is a dominant weight $\sum b_i\lambda_i$, $b_i \geqq 0$. Let j be the highest index such that a_j is nonzero. We will show that j must be n and that $p = n+1$. If $n > j > 2$, then the coefficient of λ_{j+1} in $\beta - \sum a_i\alpha_i = p\lambda$ is $pb_{j+1} = a_j$, and $a_j = p$ since $0 < a_j \leqq p$. The coefficient of λ_j is $pb_j = -2a_j + a_{j-1} = -2p + a_{j-1}$. This is not possible since b_j is nonnegative and $a_{j-1} \leqq p$. A similar argument shows that it is not possible to have $n > j$ when $j = 1$ or 2. Hence, j must be n.

The coefficient of λ_n in $p\lambda = \beta - \sum a_i\alpha_i$ is $pb_n = -2a_n + a_{n-1}$. Since b_n is nonnegative, a_n is positive, and $a_{n-1} \leqq p$, a_{n-1} must equal $2a_n$. More generally, we will check that the formula $a_i = (n+1-i)a_n$ holds for any $i \geqq 2$. Suppose that it holds for all $i \geqq k$, where k is an index in the range $n-1 > k > 2$. The coefficient of λ_k is $pb_k = a_{k-1} + a_{k+1} - 2a_k = a_{k-1} - (n+2-k)a_n$. As above for $k = n-1$, we conclude that $a_{k-1} = (n+2-k)a_n$. By the formula, the coefficient of λ_2 is $pb_2 = 1 + a_1 + a_3 - 2a_2 = 1 + a_1 - na_n$ and so, $a_1 = na_n - 1$. The coefficient of λ_1 is $pb_1 = p - 2 + a_2 - 2a_1 = p - (n+1)a_n$, and so, $p = (n+1)a_n$. We may conclude that $p = n+1$, $a_n = 1$, $\beta = (n-1)\alpha_1 + \sum_{i=2}^{n}(n+1-i)\alpha_i$, and that only in characteristic $p = n+1$ can V_p have a dominant weight of the form $p\lambda$.

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