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ON UNIVERSAL EXTENSIONS OF DIFFERENTIAL FIELDS

E. R. KOLCHIN

Dedicated to Gerhard Hochschild on the occasion of his 65th birthday

The main result of this paper is the following:

THEOREM: Let $\mathscr{U}$ be a universal extension of the differential field $\mathscr{F}$ of characteristic zero and let $\mathscr{G}$ be a strongly normal extension of $\mathscr{F}$ in $\mathscr{U}$. Then $\mathscr{U}$ is a universal extension of $\mathscr{G}$.

Introduction. We deal with differential fields, always of characteristic zero, relative to a nonempty finite set of commuting derivation operators. By an *extension* of a differential field, we always mean a differential field extension. An extension $\mathscr{F}'$ of a differential field $\mathscr{F}$ is said to be *finitely generated* if $\mathscr{F}'$ has a finite subset Φ such that $\mathscr{F}' = \mathscr{F}\langle\Phi\rangle =$ the smallest extension of $\mathscr{F}$ in $\mathscr{F}'$ that contains Φ.

Let $\mathscr{F}$ be a differential field. Recall that an extension $\mathscr{U}$ of $\mathscr{F}$ is called *universal* if, for any finitely generated extension $\mathscr{F}_1$ of $\mathscr{F}$ in $\mathscr{U}$ and any finitely generated extension $\mathscr{G}$ of $\mathscr{F}_1$ not necessarily in $\mathscr{U}$, $\mathscr{G}$ can be embedded in $\mathscr{U}$ over $\mathscr{F}_1$, i.e., there exists an extension of $\mathscr{F}_1$ in $\mathscr{U}$ that is isomorphic (in the sense of differential fields) to $\mathscr{G}$ over $\mathscr{F}_1$. Such a universal extension of $\mathscr{F}$ always exists ([2] p. 132, Th. 2). It is not unique, but if $\mathscr{U}$ and $\mathscr{V}$ are two universal extensions of $\mathscr{F}$, then there exist universal extensions $\mathscr{U}'$ and $\mathscr{V}'$ of $\mathscr{F}$, lying in $\mathscr{U}$ and $\mathscr{V}$, respectively, such that $\mathscr{U}'$ is isomorphic to $\mathscr{V}'$ over $\mathscr{F}$ ([2] p. 135, Exerc. 7).

Let $\mathscr{U}$ be a universal extension of the differential field $\mathscr{F}$ and let $\mathscr{G}$ be an extension of $\mathscr{F}$ in $\mathscr{U}$. Under favorable conditions, $\mathscr{U}$ is then a universal extension of $\mathscr{G}$, too. For example, this is the case when $\mathscr{G}$ is finitely generated over $\mathscr{F}$ ([2] p. 133, Prop. 4), and also when $\mathscr{G}$ is algebraic over $\mathscr{F}$ ([2] p. 134, Exerc. 1). The main purpose of the present note is to point out another such favorable condition. We shall show (§1) that when $\mathscr{G}$ is a strongly normal extension of $\mathscr{F}$, in the general sense of Kovacic [4] (i.e., not necessarily finitely generated), then $\mathscr{U}$ is universal over $\mathscr{G}$. This result shows that, in the study of strongly normal extensions, it is not necessary to replace $\mathscr{U}$ by a larger universal extension of $\mathscr{F}$ (see Kovacic [4] p. 518).

Every strongly normal extension of $\mathscr{F}$ in $\mathscr{U}$ is embeddable over $\mathscr{F}$ in a constrained closure of $\mathscr{F}$ in $\mathscr{U}$ ([3] p. 162, Th. 3 or Blum [1] p. 42 (15)) and hence, in particular, is constrained over $\mathscr{F}$

([3] p. 148, Th. 1). It is tempting to conjecture that the above result generalizes to constrained extensions of $\mathscr{F}$ in $\mathscr{U}$. We shall show (§2) by a counterexample that $\mathscr{U}$ can fail to be universal over a constrained closure of $\mathscr{F}$ in $\mathscr{U}$.

1. Strongly normal extensions. Recall ([2] p. 393), for a finitely generated extension $\mathscr{G}$ of $\mathscr{F}$ in a given universal extension $\mathscr{U}$ of $\mathscr{F}$, that $\mathscr{G}$ is called strongly normal over $\mathscr{F}$ if every isomorphism σ over $\mathscr{F}$ of $\mathscr{G}$ onto an extension of $\mathscr{F}$ in $\mathscr{U}$ is strong, i.e., has the property that $\sigma c = c$ for every constant c in $\mathscr{G}$ and $\mathscr{G}\mathscr{K} = \sigma\mathscr{G} \cdot \mathscr{K}$, where $\mathscr{K}$ denotes the field of constants of $\mathscr{U}$. This definition is apparently a relative one, depending on the universal extension $\mathscr{U}$ of $\mathscr{F}$ in which $\mathscr{G}$ is embedded. It is eary to see, however, that if $\mathscr{G}$ is strongly normal over $\mathscr{F}$ relative to one $\mathscr{U}$, then $\mathscr{G}$ is strongly normal over $\mathscr{F}$ relative to every $\mathscr{U}$, so that the notion of strongly normal finitely generated extension is an absolute one. When $\mathscr{G}$ is not necessarily finitely generated over $\mathscr{F}$, $\mathscr{G}$ is said, following Kovacic [4] p. 518, to be strongly normal over $\mathscr{F}$ if $\mathscr{G}$ is the union of strongly normal finitely generated extensions. Hence, also this more general notion is absolute.

It follows from [2] pp. 402-403, Th. 5, and the definition that if $\mathscr{G}$ is any strongly normal extension of $\mathscr{F}$ and $\mathscr{E}$ is any extension of $\mathscr{F}$, both contained in an extension of $\mathscr{F}$ having the same field of constants as $\mathscr{F}$, then $\mathscr{G}\mathscr{E}$ is a strongly normal extension of $\mathscr{E}$, and $\mathscr{G}$ and $\mathscr{E}$ are linearly disjoint over $\mathscr{G} \cap \mathscr{E}$.

We now prove the main theorem of this paper which was stated in the opening paragraph.

Proof. (a) We must show that if $\mathscr{G}_1$ is a finitely generated extension of $\mathscr{G}$ in $\mathscr{U}$ and $\mathscr{H}$ is any finitely generated extension of $\mathscr{G}_1$ not necessarily in $\mathscr{U}$, then there exists an embedding $\mathscr{H} \rightarrow \mathscr{U}$ over $\mathscr{G}_1$. As before, denote the field of constants of $\mathscr{U}$ by $\mathscr{K}$, and put $\mathscr{C} = \mathscr{F} \cap \mathscr{K}$, $\mathscr{C}_1 = \mathscr{G}_1 \cap \mathscr{K}$. Then $\mathscr{C} = \mathscr{G} \cap \mathscr{K}$ ([2] p. 393, Prop. 9), $\mathscr{C}_1$ is a finitely generated field extension of $\mathscr{C}$ ([2] p. 113, Cor. 1 to Prop. 14), $\mathscr{U}$ is a universal extension of $\mathscr{F}\mathscr{C}_1$, and $\mathscr{G}\mathscr{C}_1$ is a strongly normal extension of $\mathscr{F}\mathscr{C}_1$ ([2] p. 396, Th. 2). Thus, we may replace $(\mathscr{F}, \mathscr{G}, \mathscr{G}_1, \mathscr{H})$ by $(\mathscr{F}\mathscr{C}_1, \mathscr{G}\mathscr{C}_1, \mathscr{G}_1, \mathscr{H})$, i.e., we may suppose that $\mathscr{F}$, $\mathscr{G}$, $\mathscr{G}_1$ have the same field of constants $\mathscr{C}$.

(b) That being the case, fix a finite family β of generators of $\mathscr{G}_1$ over $\mathscr{G}$. Then $\mathscr{U}$ is a universal extension of $\mathscr{F}\langle\beta\rangle$ and $\mathscr{G}_1 = \mathscr{G}\mathscr{F}\langle\beta\rangle$ is a strongly normal extension of $\mathscr{F}\langle\beta\rangle$. Thus, we may replace $(\mathscr{F}, \mathscr{G}, \mathscr{G}_1, \mathscr{H})$ by $(\mathscr{F}\langle\beta\rangle, \mathscr{G}_1, \mathscr{G}_1, \mathscr{H})$, i.e., we may suppose that $\mathscr{G}_1 = \mathscr{G}$.

(c) That being the case, let $\mathscr{D}$ denote the field of constants of $\mathscr{H}$. Then $\mathscr{D}$ is a finitely generated field extension of $\mathscr{C}$, so that there exists an isomorphism $\mathscr{D} \approx \mathscr{D}'$ over $\mathscr{C}$ with $\mathscr{D}'$ a field extension of $\mathscr{C}$ in $\mathscr{K}$. Because $\mathscr{G}$ and $\mathscr{D}$ are linearly disjoint over $\mathscr{C}$ ([2] p. 87, Cor. 1 to Th. 1), and likewise $\mathscr{G}$ and $\mathscr{D}'$, this can be extended to an isomorphism $\mathscr{G}\mathscr{D} \approx \mathscr{G}\mathscr{D}'$ over $\mathscr{G}$. This can in turn be extended to an isomorphism $\mathscr{H} \approx \mathscr{H}'$, where $\mathscr{H}'$ is a finitely generated extension of $\mathscr{G}\mathscr{D}'$ not necessarily in $\mathscr{U}$. Now, $\mathscr{U}$ is a universal extension of $\mathscr{F}\mathscr{D}'$, $\mathscr{G}\mathscr{D}'$ is a strongly normal extension of $\mathscr{F}\mathscr{D}'$ in $\mathscr{U}$, and $\mathscr{H}'$ is a finitely generated extension of $\mathscr{G}\mathscr{D}'$ with field of constants $\mathscr{D}'$. An embedding $\mathscr{H}' \to \mathscr{U}$ over $\mathscr{G}\mathscr{D}'$ would, when composed with the isomorphism $\mathscr{H} \approx \mathscr{H}'$ over $\mathscr{G}$, yield an embedding $\mathscr{H} \to \mathscr{U}$ over $\mathscr{G}$. Thus, we may replace $(\mathscr{F}, \mathscr{G}, \mathscr{H})$ by $(\mathscr{F}\mathscr{D}', \mathscr{G}\mathscr{D}', \mathscr{H}')$, i.e., we may suppose that the field of constants of $\mathscr{H}$ is $\mathscr{C}$.

(d) That being the case, fix a finite family α of generators of the extension $\mathscr{H}$ of $\mathscr{G}$, and put $\mathscr{E} = \mathscr{F}\langle\alpha\rangle$. Then $\mathscr{G} \cap \mathscr{E}$ is a finitely generated extension of $\mathscr{F}$ ([2] p. 112, Prop. 14), so that $\mathscr{U}$ is universal over $\mathscr{G} \cap \mathscr{E}$. Thus, we may replace $(\mathscr{F}, \mathscr{G}, \mathscr{H}, \mathscr{E})$ by $(\mathscr{G} \cap \mathscr{E}, \mathscr{G}, \mathscr{H}, \mathscr{E})$, i.e., we may suppose that $\mathscr{G} \cap \mathscr{E} = \mathscr{F}$. Since $\mathscr{G}$ is strongly normal over $\mathscr{F}$, then the differential field $\mathscr{H} = \mathscr{G}\mathscr{E}$ is strongly normal over $\mathscr{E}$ and $\mathscr{G}$ and $\mathscr{E}$ are linearly disjoint over $\mathscr{F}$.

(e) Because $\mathscr{U}$ is universal over $\mathscr{F}$, there exists an isomorphism $\mathscr{E} \approx \mathscr{E}_0$ over $\mathscr{F}$ with $\mathscr{E}_0$ an extension of $\mathscr{F}$ in $\mathscr{U}$, and this isomorphism can be extended to an isomorphism σ: $\mathscr{H} \approx \mathscr{H}_0$, where $\mathscr{H}_0$ is an extension of $\mathscr{F}$ (and of $\mathscr{E}_0$) not necessarily in $\mathscr{U}$. Put $\mathscr{G}_0 = \sigma\mathscr{G}$. Then $\mathscr{H}_0 = \mathscr{G}_0\mathscr{E}_0$, this differential field is a strongly normal extension of $\mathscr{E}_0$, and $\mathscr{G}_0$ and $\mathscr{E}_0$ are linearly disjoint over $\mathscr{F}$. Evidently $\mathscr{U}$ is universal over $\mathscr{E}_0$ (because $\mathscr{E}_0$ is finitely generated over $\mathscr{F}$), and hence the strongly normal extension $\mathscr{G}_0\mathscr{E}_0$ of $\mathscr{E}_0$ can be embedded in $\mathscr{U}$ over $\mathscr{E}_0$, i.e., there exists an isomorphism σ_0: $\mathscr{G}_0\mathscr{E}_0 \approx \mathscr{G}_2\mathscr{E}_0$ over $\mathscr{E}_0$ with $\sigma_0\mathscr{G}_0 = \mathscr{G}_2 \subset \mathscr{U}$. The field of constants of $\mathscr{G}_2\mathscr{E}_0$, like those of $\mathscr{H}_0 = \mathscr{G}_0\mathscr{E}_0$ and $\mathscr{H} = \mathscr{G}\mathscr{E}$, is $\mathscr{C}$, and hence $\mathscr{G}_2\mathscr{E}_0$ and $\mathscr{K}$ are linearly disjoint cover $\mathscr{C}$. Therefore $\mathscr{G}_2\mathscr{E}_0$ and $\mathscr{G}_2\cdot\mathscr{K}$ are linearly disjoint over $\mathscr{G}_2$. But by (d), $\mathscr{E}$ and $\mathscr{G}$ are linearly disjoint over $\mathscr{F}$, so that $\mathscr{E}_0$ and $\mathscr{G}_0$ are, too, and hence also $\mathscr{E}_0$ and $\mathscr{G}_2$. Therefore $\mathscr{E}_0$ and $\mathscr{G}_2\cdot\mathscr{K}$ are linearly disjoint over $\mathscr{F}$. But $\mathscr{G}$ is strongly normal over $\mathscr{F}$, so that $\mathscr{G} \subset \sigma_0\sigma\mathscr{G} \cdot \mathscr{K} = \mathscr{G}_2\cdot\mathscr{K}$. Hence $\mathscr{E}_0$ and $\mathscr{G}$ are linearly disjoint over $\mathscr{F}$. Therefore, $id_{\mathscr{E}_0}$ and the isomorphism $\mathscr{G}_2 \approx \mathscr{G}$ (restriction of $(\sigma_0 \circ \sigma)^{-1}$) extend to an isomorphism τ: $\mathscr{G}_2\mathscr{E}_0 \approx \mathscr{G}\mathscr{E}_0$. The composite isomorphism $\tau \circ \sigma_0 \circ \sigma$ is an embedding of $\mathscr{H}$ into $\mathscr{U}$ over $\mathscr{G}$.

2. A counterexample for constrained extensions. Recall that an extension $\mathscr{G}$ of a differential field is said to be *constrained* ([3] p. 144) if every finite family of elements of $\mathscr{G}$ is constrained over $\mathscr{F}$ in the sense of [2] p. 142, that a differential field is said to be *constrainedly closed* ([3] p. 145) if it has no constrained extension other than itself, and that $\mathscr{G}$ is said to be a *constrained closure* of $\mathscr{F}$ ([3] p. 147) if $\mathscr{G}$ is constrainedly closed and is embeddable over closed $\mathscr{F}$ in every constrainedly extension of $\mathscr{F}$. A constrained closure of $\mathscr{F}$ always exists, and it is a constrained extension of $\mathscr{F}$.

We are going to exhibit an ordinary differential field $\mathscr{F}$, a universal extension $\mathscr{U}$ of $\mathscr{F}$, and an extension $\mathscr{G}$ of $\mathscr{F}$ in $\mathscr{U}$ such that $\mathscr{G}$ is a constrained closure of $\mathscr{F}$ and $\mathscr{U}$ is not universal over $\mathscr{G}$.

Let $\mathscr{C}$ be any denumerable field of characteristic zero and put $\mathscr{F} = \mathscr{C}(x) =$ the field of rational fractions over $\mathscr{C}$ in an indeterminate x; $\mathscr{F}$ has a unique structure of ordinary differential field with field of constants $\mathscr{C}$ in which the derivative of x is 1. By [3] p. 149, Prop. 4, we may fix a denumerable universal extension $\mathscr{U}$ of $\mathscr{F}$. By [3] p. 146, Cor. 1 to Prop. 3, $\mathscr{U}$ is constrainedly closed.

The set of solutions in $\mathscr{U}$ different from 0 and 1 of the differential equation

$$y' = y^3 - y^2$$

is denumerable and hence can be arranged in a sequence

$$\eta_0, \eta_1, \eta_2, \cdots .$$

By [3] §8, this set is infinite and is an independent set of conjugates over $\mathscr{F}$, and $\mathscr{F}\langle\eta_0, \eta_1, \eta_2, \cdots\rangle$ is constrained over $\mathscr{F}$ (see [3] p. 144, Prop. 1). Because $\mathscr{U}$ is constrainedly closed, $\mathscr{F}\langle\eta_0, \eta_1, \eta_2, \cdots\rangle$ has a constrained closure $\mathscr{G}$ in $\mathscr{U}$. The differential ideal $[y' - y^3 + y^2]$ of the differential polynomial algebra $\mathscr{G}\{y\}$ is evidently prime and does not have a generic zero in $\mathscr{U}$ (because all its zeros in $\mathscr{U}$ are in $\mathscr{G}$). Therefore, $\mathscr{U}$ is not universal over $\mathscr{G}$. (The same argument shows that $\mathscr{U}$ is even not universal over $\mathscr{F}\langle\eta_0, \eta_1, \eta_2, \cdots\rangle$.) We are going to show that $\mathscr{G}$ is a constrained closure of $\mathscr{F}$.

By [3] p. 144, Prop. 2(a), $\mathscr{G}$ is constrained over $\mathscr{F}$. Let $\mathscr{H}$ be any denumerable constrained closure of $\mathscr{F}$ (e.g., any constrained closure of $\mathscr{F}$ in $\mathscr{U}$). The set of solutions in $\mathscr{H}$ of the above differential equation can be arranged in a sequence

$$\zeta_0, \zeta_1, \zeta_2, \cdots .$$

As before, this set is infinite and is an independent set of conjugates over $\mathscr{F}$. Therefore, there exists an isomorphism

$$\varphi\colon \mathscr{F}\langle \eta_0, \eta_1, \eta_2, \cdots\rangle \approx \mathscr{F}\langle \zeta_0, \zeta_1, \zeta_2, \cdots\rangle .$$

Now, $\mathscr{F}\langle \zeta_0, \zeta_1, \zeta_2, \cdots\rangle$ is normal over $\mathscr{F}$ in $\mathscr{H}$ (see [3] §6 p. 153). Hence, by [3] p. 159, Cor. 1 to Th. 2, $\mathscr{H}$ is a constrained closure of $\mathscr{F}\langle \zeta_0, \zeta_1, \zeta_2, \cdots\rangle$. Therefore, by [3] p. 158, Th. 2(b), φ can be extended to an isomorphism $\mathscr{G} \approx \mathscr{H}$, so that $\mathscr{G}$ is a constrained closure of $\mathscr{F}$.

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COLUMBIA UNIVERSITY
NEW YORK, NY 10027