PACIFIC JOURNAL OF MATHEMATICS
Vol. 86, No. 2, 1980

EXISTENCE OF EIGENVALUES FOR SECOND-ORDER DIFFERENTIAL SYSTEMS

S. C. TEFTELLER

The paper is concerned with establishing the existence of eigenvalues for the second order differential system $y' = k(x, \lambda)z$, $z' = g(x, \lambda)y$, together with boundary conditions $y(a) = A(\lambda)y(b) + B(\lambda)z(b)$, $z(a) = C(\lambda)y(b) + D(\lambda)z(b)$. A general theorem is obtained establishing the existence of eigenvalues for both self-adjoint and nonself-adjoint boundary problems. This result is then simplified for nonself-adjoint problems, extending the previous work of H. J. Ettlinger and E. Kamke.

1. Introduction. Second-order differential systems involving a parameter, together with boundary conditions at two points, have played a fundamental role in many physical and mechanical processes. The mathematical study of these boundary value problems "began" with the fundamental work of Sturm and Liouville and has flourished ever since.

In this paper, the differential system

$$\begin{aligned} y' &= k(x, \lambda)z\ , \\ z' &= g(x, \lambda)y\ , \end{aligned} \tag{1}$$

is considered, where $k(x, \lambda)$ and $g(x, \lambda)$ are real-valued functions on X: $-\infty < a \leqq x \leqq b < \infty$, L: $\lambda_{\#} - \eta < \lambda < \lambda_{\#} + \eta$, $0 < \eta \leqq \infty$. The system (1) is studied together with the boundary conditions

$$\alpha_1(\lambda)y(a, \lambda) - \beta_1(\lambda)z(a, \lambda) = \gamma_1(\lambda)y(b, \lambda) - \delta_1(\lambda)z(b, \lambda)\ , \tag{2a}$$

$$\alpha_2(\lambda)y(a, \lambda) - \beta_2(\lambda)z(a, \lambda) = \gamma_2(\lambda)y(b, \lambda) - \delta_2(\lambda)z(b, \lambda)\ . \tag{2b}$$

The problem (1, 2a, 2b) has been studied H. J. Ettlinger [3, 4, 5] and E. Kamke [6, 7]. G. D. Birkhoff [1] also studied this problem, but he considered a second-order differential equation rather than a system. However, his equation may be written as a system and the results remain intact.

Under the hypothesis that the boundary problem be self-adjoint and that the coefficient of the differential equation satisfy a monotoniety condition with respect to the parameter, Birkhoff established the existence of an infinite sequence of characteristic values for the boundary problem and determined the oscillatory behavior of the associated solutions.

Later, H. J. Ettlinger, in a series of papers, considered both the self-adjoint and nonself-adjoint boundary problems. By assuming monotone behavior of coefficients in the boundary conditions and that the coefficients of the system satisfy a limiting condition with respect to the parameter, Ettlinger established results analogous to those of Birkhoff for the self-adjoint problem. For the nonself-adjoint problem, he established the existence of at least one characteristic value under the hypothesis that determinants of the boundary condition coefficients satisfy certain inequalities.

Generally speaking, the techniques used by Birkhoff and Ettlinger are similar. Contrasting these techniques are those of E. Kamke, who also studied both the self-adjoint and nonself-adjoint versions of (1, 2a, 2b). By means of the polar coordinate transformation, he was able to obtain results complementing those of Birkhoff and Ettlinger. The results of this paper will be obtained by techniques similar to those of Kamke.

In [6], Kamke considers the matrices of coefficients for the boundary conditions. Using the rank of these matrices, he shows that the problem falls into one of four categories. The first category is where only one point is involved and the second category contains the usual Sturmian boundary conditions. The fourth category consists of conditions studied by W. M. Whyburn [11, 12] and G. J. Etgen and the author [2]. Kamke's third category is the problem investigated in this paper.

Specifically, we suppose that $\alpha_2\beta_1 - \alpha_1\beta_2 \neq 0$ and $\gamma_2\delta_1 - \gamma_1\delta_2 \neq 0$ and moreover, that we can normalize to obtain $\alpha_2\beta_1 - \alpha_1\beta_2 \equiv 1$ on L. Since premultiplication of the coefficient matrices,

$$\begin{bmatrix} \alpha_1(\lambda) - \beta_1(\lambda) \\ \alpha_2(\lambda) - \beta_2(\lambda) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \gamma_1(\lambda) - \delta_1(\lambda) \\ \gamma_2(\lambda) - \delta_2(\lambda) \end{bmatrix}$$

by any 2×2 matrix with determinant equal to 1 will not alter the set of eigenvalues and corresponding eigenfunctions, we can write the boundary conditions as

$$y(a) = A(\lambda)y(b) + B(\lambda)z(b)\ , \tag{3a}$$

$$z(a) = C(\lambda)y(b) + D(\lambda)z(b)\ . \tag{3b}$$

It will be in this latter form that we consider the boundary conditions.

This work will generalize that of Kamke, Birkhoff, and Ettlinger by establishing the existence of sets of characteristic numbers for the boundary problem and determining the oscillatory behavior of the associated solutions for both the self-adjoint and nonself-adjoint problems. The techiques used here are applicable to both problems simultaneously. Since the distinction between problem-types does not need to be made, this approach may be considered to be more general. In addition, an interesting application is made to a class of nonself-adjoint problems. For such problems the results of this paper indeed extend the work of the forementioned authors.

2. Preliminary definitions and results. The following hypotheses on the coefficients involved in the boundary problem (1, 3a, 3b) will be assumed throughout:

(H-0) All involved functions in (1) and the boundary conditions are assumed to be real-valued.

(H-1) For each $x \in X$, each of $k(x, \lambda)$ and $g(x, \lambda)$ is continuous on L.

(H-2) For each $\lambda \in L$, each of $k(x, \lambda)$ and $g(x, \lambda)$ is measurable on X.

(H-3) There exists a Lebesgue measurable function $M(x)$ on X such that $|k(x, \lambda)| \leqq M(x)$ and $|g(x, \lambda)| \leqq M(x)$ on $X \times L$.

(H-4) $k(x, \lambda) > 0$ on $X \times L$.

(H-5) Each of the functions $A(\lambda)$, $B(\lambda)$, $C(\lambda)$, $D(\lambda)$ is continuous on L.

(H-6) $A^2(\lambda) + B^2(\lambda) > 0$, $C^2(\lambda) + D^2(\lambda) > 0$, and $A(\lambda)D(\lambda) - B(\lambda)C(\lambda) \neq 0$ on L.

(H-7) Let $\sigma(\lambda)$ be a function defined on L by

$$\sigma(\lambda) = (AD - BC)/[(A^2 + C^2)(B^2 + D^2)]^{1/2} .$$

From (H-6) we have that $\sigma(\lambda) \neq 0$ on L. We will assume that $0 < \arcsin \sigma(\lambda) < \pi$ if $AD - BC > 0$ and $-\pi < \arcsin \sigma(\lambda) < 0$ if $AD - BC < 0$.

Hypotheses (H-0)-(H-3) are the familiar Carathéodory conditions, which allow the application of fundamental existence theorems for differential systems to obtain the existence of two unique solution pairs $\{u_i(x, \lambda), v_i(x, \lambda)\}$, $i = 1, 2$, of (1) satisfying the initial conditions

$$\begin{aligned} u_1(a, \lambda) &\equiv 1; u_2(a, \lambda) \equiv 0 ; \\ v_1(a, \lambda) &\equiv 0; v_2(a, \lambda) \equiv 1 ; \end{aligned} \tag{4}$$

on L. It follows that these pairs are linearly independent and any solution pair $\{y, z\}$ of (1) must be of the form

$$y = c_1u_1 + c_2u_2 , \quad z = c_1v_1 + c_2v_2 , \tag{5}$$

where c_1 and c_2 are real numbers. Hence $\{y, z\}$ will be a solution of (1) satisfying (3a, 3b), if and only if

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{bmatrix} Au_1(b) + Bv_1(b) & Au_2(b) + Bv_2(b) \\ Cu_1(b) + Dv_1(b) & Cu_2(b) + Dv_2(b) \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} .$$

Therefore, $\{y, z\}$ is a nontrivial solution pair (i.e., $y^2+g^2>0$ on $X\times L$) if and only if

$$(6)\qquad \begin{aligned} &Au_1(b) + Bv_1(b) + Cu_2(b) + Dv_2(b) \\ &\qquad = 1 + (AD - BC)[u_1(b)v_2(b) - u_2(b)v_1(b)] \, . \end{aligned}$$

In particular, $\{y, z\}$ is a nontrivial solution if not both c_1 and c_2 are zero, and if

$$(7)\qquad \begin{aligned} Cu_1(b) + Dv_1(b) &= Au_2(b) + Bv_2(b) = 0 \, , \\ Cu_2(b) + Dv_2(b) &= Au_1(b) + Bv_1(b) = 1 \, . \end{aligned} \quad \text{and}$$

It is not clear that either (6) or (7) can generally be satisfied for any $\lambda \in L$. We seek to establish the existence of such values of $\lambda \in L$. The values of λ for which (6), but not (7), holds are called *simple eigenvalues*, and for these values, there corresponds exactly one solution pair of (1). The existence of simple eigenvalues will be our primary concern.

Let us define functions ϕ_i and ψ_i, $i = 1, 2$, on $X \times L$ as follows:

$$(8)\qquad \begin{aligned} \phi_i(x, \lambda) &= A(\lambda)u_i(x, \lambda) + B(\lambda)v_i(x, \lambda) \, , \\ \psi_i(x, \lambda) &= C(\lambda)u_i(x, \lambda) + D(\lambda)v_i(x, \lambda) \, ; \quad i = 1, 2 \, . \end{aligned}$$

W. M. Whyburn [10], has shown that the pairs $\{\phi_i, \psi_i\}$ each satisfy the differential system

$$(9)\qquad \begin{aligned} \phi_i' &= [(ACk - BDg)\phi_i + (B^2g - A^2k)\psi_i]\Delta^{-1} \, , \\ \psi_i' &= [(C^2k - D^2g)\phi_i + (BDg - ACk)\psi_i]\Delta^{-1} \, ; \end{aligned}$$

where $\Delta = AD - BC$. (Note that the existence of Δ^{-1} is guaranteed by (H-6).) Define the functions Φ, Ψ, K, G, and ω by

$$(10)\qquad \begin{aligned} &\Phi(x, \lambda) = \phi(x, \lambda)e^{-\omega(x,\lambda)} \, , \quad \Psi(x, \lambda) = \psi(x, \lambda)e^{\omega(x,\lambda)} \, , \\ &K = (B^2g - A^2k)\Delta^{-1}e^{-2\omega} \, , \quad G = (C^2k - D^2g)\Delta^{-1}e^{2\omega} \, , \\ &\omega(x, \lambda) = \int_a^x (ACg - BDk)\Delta^{-1}dt \, . \end{aligned}$$

In terms of these functions, the system (9) is equivalent to the system

$$(11)\qquad \begin{aligned} \Phi' &= K(x, \lambda)\Psi \, , \\ \Psi' &= G(x, \lambda)\Phi \, . \end{aligned}$$

Let $\{\Phi_i, \Psi_i\}$ be the solution pairs of (11) corresponding to the solution pairs $\{\phi_i, \psi_i\}$, $i = 1, 2$, of (9). We apply the polar coordinate transformation to these pairs to obtain

$$\Phi_i(x, \lambda) = \rho_i(x, \lambda) \sin \theta_i(x, \lambda) \ , \qquad (12)$$
$$\Psi_i(x, \lambda) = \rho_i(x, \lambda) \cos \theta_i(x, \lambda) \ ,$$

where ρ_i and θ_i are solutions of

$$\rho_i' = \rho_i(K + G) \sin \theta_i \cos \theta_i \ , \qquad (13)$$
$$\theta_i' = K \cos^2 \theta_i - G \sin^2 \theta_i \ , \quad i = 1, 2 \ ;$$

satisfying the initial conditions

$$\rho_1(a, \lambda) = [A^2(\lambda) + C^2(\lambda)]^{1/2} \ , \quad \rho_2(a, \lambda) = [B^2(\lambda) + D^2(\lambda)]^{1/2} \ ,$$
$$\theta_1(a, \lambda) = \arctan A(\lambda)/C(\lambda) \ , \quad \theta_2(a, \lambda) = \arctan B(\lambda)/D(\lambda) \ ,$$
$$-\pi < \theta_1(a, \lambda) \ , \quad \theta_2(a, \lambda) < \pi \ .$$

Hypothesis (H-6) insures that $\rho_i(a, \lambda) > 0$ on L, $i = 1, 2$.

LEMMA 1. *The following inequality holds on* $X \times L$:

$$0 < \theta_1(x, \lambda) - \theta_2(x, \lambda) < \pi \quad \textit{if} \quad AD - BC > 0 \quad \textit{on} \quad L \ ,$$

or

$$-\pi < \theta_1(x, \lambda) - \theta_2(x, \lambda) < 0 \quad \textit{if} \quad AD - BC < 0 \quad \textit{on} \quad L \ .$$

Proof. Using equations (8) and (10), we have

$$\phi_1\psi_2 - \phi_2\psi_1 \equiv \Phi_1\Psi_2 - \Psi_1\Phi_2 \equiv (AD - BC)(u_1v_2 - u_2v_1)$$

on $X \times L$. Differentiating and evaluating at $x = a$, we see that $u_1v_2 - u_2v_1 \equiv 1$ on $X \times L$. Hence $\Phi_1\Psi_2 - \Psi_1\Phi_2 \equiv AD - BC$ on $X \times L$. Applying the transformation (12), it follows that $\rho_1(x, \lambda)\rho_2(x, \lambda) \sin [\theta_1(x, \lambda) - \theta_2(x, \lambda)] \equiv AD - BC$ on $X \times L$. Hence $\sin [\theta_1(x, \lambda) - \theta_2(x, \lambda)] > 0$ or $\sin [\theta_1(x, \lambda) - \theta_2(x, \lambda)] < 0$ on $X \times L$, depending on whether $AD - BC$ is positive or negative on $X \times L$, respectively. The desired inequalities follow from (H-7) and the initial conditions for ρ_1 and ρ_2.

It should be noted that (H-7) is made only for convenience. Since $\sin (\theta_1 - \theta_2) \neq 0$ on $X \times L$, there is some integer n so that $n\pi < \theta_1(x, \lambda) - \theta_2(x, \lambda) < (n+1)\pi$ on $X \times L$. As a simplification, (H-7) provides that $n = 1$ if $AD - BC > 0$ and that $n = -1$ if $AD - BC < 0$ on L.

COROLLARY. *For each* $x \in X$, *the zeros of* $\Phi_1(x, \lambda)$ *and* $\Phi_2(x, \lambda)$, *and the zeros of* $\Psi_1(x, \lambda)$ *and* $\Psi_2(x, \lambda)$ *separate each other on* L.

LEMMA 2. *Suppose that either of the following conditions holds on* $X \times L$:

(i) $\int_a^x |K(t, \lambda) + G(t, \lambda)| \, dt < \ln |1/\sigma(\lambda)|$,

(ii) $\int_a^x |K(t, \lambda) + G(t, \lambda)| \, dt < \ln \pi - \ln |2 \arcsin \sigma(\lambda)|$, ($\sigma(\lambda)$ *defined* *by* (H-7).) *Then* $0 < |\theta_1(x, \lambda) - \theta_2(x, \lambda)| < \pi/2$ *or* $\pi/2 < |\theta_1(x, \lambda) - \theta_2(x, \lambda)| < \pi$ *on* $X \times L$.

Proof. Let $\Delta(\lambda) = A(\lambda)D(\lambda) - C(\lambda)D(\lambda)$. If $\Delta(\lambda) > 0$ on L, then from Lemma 1, we know that $0 < \sin[\theta_1(x, \lambda) - \theta_2(x, \lambda)] \equiv \Delta(\lambda)/\rho_1(x, \lambda)\rho_2(x, \lambda)$. In (13), we can solve the first order equation in $\rho_i(x, \lambda)$, $i = 1, 2$, to obtain

$$\rho_1(x)\rho_2(x) = \rho_1(a)\rho_2(a) \exp\left\{\int_a^x (K + G)(\sin\theta_1\cos\theta_1 + \sin\theta_2\cos\theta_2)dt\right\} .$$

Now

$$\Delta/\rho_1(x)\rho_2(x) = \sigma \exp\left\{-\int_a^x (K + G)\sin(\theta_1 + \theta_2)\cos(\theta_2 - \theta_1)dt\right\} .$$

Since $-\int_a^x (K + G)\sin(\theta_1 + \theta_2)\cos(\theta_2 - \theta_1)dt \leqq \int_a^x |K + G|dt$, then if condition (i) holds, $\Delta/\rho_1(x)\rho_2(x) < 1$ and $0 < \sin[\theta_1(x, \lambda) - \theta_2(x, \lambda)] < 1$ on $X \times L$. Hence the desired inequality follows from (H-7).

To apply condition (ii), we again use equation (13) to obtain the equation $(\theta_1 - \theta_2)' = -(K + G)(\sin^2\theta_1 - \sin^2\theta_2)$, which can also be written as $\tau' + H(x, \lambda)\tau = 0$, where $\tau(x, \lambda) = \theta_1(x, \lambda) - \theta_2(x, \lambda) > 0$, and $H = [(K + G)\sin(\theta_1 + \theta_2)\sin(\theta_1 - \theta_2)]/(\theta_1 - \theta_2)$. Solving this first-order equation, one obtains $\tau(x, \lambda) = \tau(a, \lambda)\exp\left\{-\int_a^x H(t, \lambda)dt\right\}$. Since $\exp\left\{-\int_a^x H(t, \lambda)dt\right\} \leqq \exp\int_a^x |K(t, \lambda) + G(t, \lambda)|dt$, and $\tau(a, \lambda) = \arcsin\sigma(\lambda)$, then if (ii) holds, we have $\exp\left\{-\int_a^x H(t, \lambda)dt\right\} < \pi/2\tau(a, \lambda)$. Therefore, $0 < \tau(x, \lambda) < \pi/2$ on $X \times L$.

The case where $\Delta(\lambda) < 0$ is handled exactly as above by noting that $0 < \sin[\theta_2(x, \lambda) - \theta_1(x, \lambda)] \equiv -\Delta(\lambda)/\rho_1(x, \lambda)\rho_2(x, \lambda)$.

COROLLARY. *For each* $x \in X$, *the zeros of* $\Phi_1(x, \lambda)$ *and* $\Psi_2(x, \lambda)$, *and the zeros of* $\Phi_2(x, \lambda)$ *and* $\Psi_1(x, \lambda)$ *separate each other on* L.

Consider now equations (6) and (7). Applying (8) and (10), we have

$$\Phi_1(b, \lambda)e^{\omega(b,\lambda)} + \Psi_2(b, \lambda)e^{-\omega(b,\lambda)} = \xi(\lambda) , \tag{14}$$

or

$$\begin{aligned} \Psi_1(b, \lambda)e^{-\omega(b,\lambda)} &= \Phi_2(b, \lambda)e^{\omega(b,\lambda)} = 0 , \\ \Psi_2(b, \lambda)e^{-\omega(b,\lambda)} &= \Phi_1(b, \lambda)e^{\omega(b,\lambda)} = 1 . \end{aligned} \tag{15}$$

Here we define the function $\xi(\lambda)$ by

$$\xi = 1 + (AD - BC)(u_1v_2 - u_2v_1) \equiv 1 + (AD - BC) . \tag{16}$$

3. **Existence of eigenvalues.** Using the results of the previous section, we are now in a position to be able to specify conditions which will guarantee the existence of eigenvalues for the system (1, 3a, b). Our first theorem is a result which holds for both the self-adjoint and nonself-adjoint problems. This result will then be specialized to consider a class of nonself-adjoint problems only.

THEOREM 1. *Let* $\{y(x, \lambda), z(x, \lambda)\}$ *be a nontrivial solution pair of* (1). *Suppose that either condition* (i) *or* (ii) *of Lemma* 2 *holds, and that* $K(x, \lambda) > 0$ *on* $X \times L$. *In addition to hypotheses* (H-0)-(H-7), *assume conditions so that there exist integers* m *and* n *having the property that* $\inf \theta_1(b, \lambda) < m\pi$ *and* $\sup \theta_1(b, \lambda) > n\pi$. *If* $n \geqq m + 1$, *and if*

$$[\rho_1^2(b, \lambda)\rho_2^2(b, \lambda) - \Delta^2(\lambda)]^{1/2}e^{-\omega(b,\lambda)}/\rho_1(b, \lambda) \geqq |\xi(\lambda)| , \tag{17}$$

then there exist p, $p = n - m$, *nonempty sets of simple eigenvalues* $J_0, J_1, \cdots, J_{p-1}$ *for the system* (1, 3a, 3b). *Moreover, the number of distinct eigenvalues for the system* (1, 3a, 3b) *is at least* $p/2$ *if* p *is even and at least* $(p + 1)/2$ *if* p *is odd.*

Proof. Let $\{y, z\}$ be a solution pair of (1) and let $\{u_i, v_i\}$, $i = 1, 2$, be the solution pairs of (1) defined by (4). As seen above, $\{y, z\}$ is of the form (5), and the eigenvalues of (1, 3a, 3b) will be those values of λ for which (14) or (15) is satisfied. Let Φ_i, Ψ_i, $i = 1, 2$, be defined as above. By assuming that either hypothesis (i) or (ii) of Lemma 2 holds, we see that (15) cannot be satisfied on L. Hence all eigenvalues will be simple.

Supposing that $K(x, \lambda) > 0$ on $X \times L$ insures that $\theta_i'(x, \lambda) > 0$ when $\Phi_i(x, \lambda) = 0$ and hence $\theta_i(x, \lambda) = 0 (\mathrm{mod}\, \pi)$ if and only if $\Phi_i(x, \lambda) = 0$.

Let m and n be integers with the properties described in the hypothesis. Then there are values of λ, say $\bar{\lambda}$ and λ^*, such that $\theta_1(b, \bar{\lambda}) = m\pi$ and $\theta_1(b, \lambda^*) = n\pi$. Clearly, $\bar{\lambda} \neq \lambda^*$, so assume that $\bar{\lambda} < \lambda^*$. Since $K(x, \lambda) > 0$ on $X \times L$, we have that $\theta_1(b, \lambda) > m\pi$ on $(\bar{\lambda}, \lambda^*]$. It follows from the continuity of $\theta_1(b, \lambda)$ on L and the fact that $p = n - m \geqq 1$, that there exist $p + 1$ values of λ, $\bar{\lambda} = \lambda_0 < \lambda_1 < \cdots < \lambda_p = \lambda^*$, on $[\bar{\lambda}, \lambda^*]$ such that $\theta_1(b, \lambda_j) = (m + j)\pi$, $j = 0, 1, \cdots, p$.

Choose any integer j, $0 \leqq j \leqq p$, and without loss of generality, assume that $\cos \theta_1(b, \lambda_j) = +1$ and $\cos \theta_1(b, \lambda_{j+1}) = -1$. Now $\rho_1\rho_2 \sin(\theta_1 - \theta_2) \equiv AD - BC \equiv \Delta$ on $X \times L$ implies that $\sin \theta_2(b, \lambda_j) = -\Delta(\lambda_j)/\rho_1(b, \lambda_j)\rho_2(b, \lambda_j)$ and $\sin \theta_2(b, \lambda_{j+1}) = \Delta(\lambda_{j+1})/\rho_1(b, \lambda_{j+1})\rho_2(b, \lambda_{j+1})$. It is a consequence of Lemma 2 and the usual trigonometric identities that $\cos \theta_2(b, \lambda_j) = [\rho_1^2(b, \lambda_j)\rho_2^2(b, \lambda_j) - \Delta^2(\lambda_j)]^{1/2}/\rho_1(b, \lambda_j)\rho_2(b, \lambda_j)$, and

$\cos\theta_2(b, \lambda_{j+1}) = -[\rho_1^2(b, \lambda_{j+1})\rho_2^2(b, \lambda_{j+1}) - \Delta^2(\lambda_{j+1})]^{1/2}/\rho_1(b, \lambda_{j+1})\rho_2(b, \lambda_{j+1})$. (It may be that the signs in the last two equations may be reversed.) Define the function $s(\lambda)$ on L by

$$(18) \qquad s(\lambda) = \Phi_1(b, \lambda)e^{\omega(b,\lambda)} + \Psi_2(b, \lambda)e^{-\omega(b,\lambda)} .$$

We have from above that $s(\lambda_j) = \rho_2(b, \lambda_j)e^{-\omega(b,\lambda_j)}\cos\theta_2(b, \lambda_j)$, and $s(\lambda_{j+1}) = \rho_2(b, \lambda_{j+1})e^{-\omega(b,\lambda_{j+1})}\cos\theta_2(b, \lambda_{j+1})$. Hence if condition (17) holds then $s(\lambda_j) \geqq \xi(\lambda_j)$ and $s(\lambda_{j+1}) \leqq \xi(\lambda_{j+1})$, or vice versa. Since $s(\lambda)$ and $\xi(\lambda)$ are continuous on L, it follows that there is at least one value of λ on $[\lambda_j, \lambda_{j+1}]$ such that $s(\lambda) = \xi(\lambda)$, and therefore equation (14) is satisfied. Let $J_j = \{\lambda \in [\lambda_j, \lambda_{j+1}] \mid (14) \text{ is satisfied}\}$, $j = 0, 1, \cdots, p-1$.

The work above establishes that the continuous curves $s(\lambda)$ and $\xi(\lambda)$ must intersect at least once on each of the intervals $[\lambda_j, \lambda_{j+1}]$, $j = 0, 1, \cdots, p-1$. It could happen that these curves intersect only at alternate endpoints, $\lambda_1, \lambda_3, \cdots$, with λ_{2j+1} serving as the eigenvalue for both $[\lambda_{2j}, \lambda_{2j+1}]$ and $[\lambda_{2j+1}, \lambda_{2j+2}]$. Therefore there will be at least $p/2$ or $(p+1)/2$ distinct eigenvalues for (1, 3a, 3b) depending on where p is even or odd. This completes the proof of the theorem.

REMARKS. It should be noted that a similar result to Theorem 1 can be established by supposing that $G(x, \lambda) < 0$ on $X \times L$, that m and n are integers such that $\inf\theta_2(b, \lambda) < (2m+1)\pi/2$ and $\sup\theta_2(b, \lambda) > (2n+1)\pi/2$, and that condition (17) be replaced by the condition that

$$(19) \qquad [\rho_1^2(b, \lambda)\rho_2^2(b, \lambda) - \Delta^2(\lambda)]^{1/2}e^{\omega(b,\lambda)}/\rho_2(b, \lambda) \geqq |\xi(\lambda)| .$$

In Theorem 1 we allude to conditions which would insure the existence of integers m and n such that $\theta_1(b, \lambda)$ goes from less than $m\pi$ to more than $n\pi$ on L. Conditions on $K(x, \lambda)$ and $G(x, \lambda)$ (and hence $k(x, \lambda)$, $g(x, \lambda)$, A, B, C, D) which insure this behavior may be found in Ince [8, pp. 231–248] or Ettlinger [3, 4, 5], or Whyburn [10, p. 852]. It should also be noted that if both $K(x, \lambda) > 0$ and $G(x, \lambda) < 0$ on XL, then the eigenvalues are distinct. Furthermore, by assuming that $A(\lambda) \geqq 0$ or $D(\lambda) \geqq 0$ on L insures that the integer $m \geqq 0$, i.e., the polar angle $\Phi_1(x, \lambda)$ is nonnegative on L. As a final remark, we note that either conition (17) or (19) concerns the amplitudes of the pairs $\{\Phi_i, \Psi_i\}$, $i = 1, 2$, and the function $\omega(b, \lambda)$. This is consistent with results obtained for other boundary problems with nonzero right hand sides (see e.g., [2, 11, 12]).

4. A nonself-adjoint problem. We will now specialize the results of the preceding section to a class of nonself-adjoint problems.

Consider the boundary conditions (3a, 3b). It is readily verified that this boundary problem is self-adjoint if and only if $\Delta(\lambda) \equiv A(\lambda)D(\lambda) - B(\lambda)C(\lambda) \equiv 1$ on L. Since we are interested in the non-self-adjoint problem, assume the problem has the property that $\Delta(\lambda) \equiv -1$ on L. Using this specialization, we see that λ will be a simple eigenvalue of (1, 3a, 3b) if and only if

$$s(\lambda) = \Phi_1(b, \lambda)e^{\omega(b,\lambda)} + \Psi_2(b, \lambda)e^{-\omega(b,\lambda)} = 0 \ . \tag{20}$$

Based on this assumption we now have the following result.

THEOREM 2. *Let* $\{y(x, \lambda), z(x, \lambda)\}$ *be a nontrivial solution pair of* (1). *Suppose that either condition* (i) or (ii) *of Lemma* 2 *holds and that* $K(x, \lambda) > 0$ *on* $X \times L$. *In addition to hypotheses* (H-0)-(H-7), *suppose* $A(\lambda) \geqq 0$. *Then* $\theta_1(b, \lambda) > -\pi$ *on* L. *Assume* $m \geqq 0$ *is the least integer such that* $\inf \theta_1(b, \lambda) < m\pi$ *and let* n *be any integer such that* $\sup \theta_1(b, \lambda) > n\pi$. *If* $n \geqq m + 1$ *then there exist* p, $p = n - m$, *nonempty sets of simple eigenvalues* $J_0, J_1, \cdots, J_{p-1}$ *for the nonself-adjoint system* (1, 3a, 3b). *The number of distinct eigenvalues for the system is* $p/2$ *if* p *is even and* $(p + 1)/2$ *if* p *is odd.*

Proof. This proof is a simplification of Theorem 1. Using the statements and notions of that proof, we see that if $A(\lambda) \geqq 0$ on L, then $0 \leqq \theta_1(a, \lambda) < \pi$ on L. Since $K(x, \lambda) > 0$ on $X \times L$, $\theta_1(x, \lambda)$ passes through integer multiples of π in the positive direction only as x increases on X, and therefore $\theta_1(b, \lambda) > -\pi$ on L. Defining $\lambda_0, \lambda_1, \cdots, \lambda_p$ as in Theorem 1, we see that $s(\lambda_j) = \Psi_2(b, \lambda_j)e^{-\omega(b,\lambda_j)}$. From the corollary to Lemma 2, we know that the zeros of $\Phi_1(b, \lambda)$ and $\Psi_2(b, \lambda)$ separate each other on L. Hence $s(\lambda_j) > 0$ and $s(\lambda_{j+1}) < 0$ or vice versa. The continuity of $s(\lambda)$ guarantees the existence of a value of λ on $[\lambda_j, \lambda_{j+1}]$ such that $s(\lambda) = 0$, and therefore equation (20) is satisfied. The remainder of the proof follows as in Theorem 1.

COROLLARY. *Suppose, in addition to the hypotheses of Theorem* 2, *that* $k(x, \lambda) \geqq K(x, \lambda)$ *and* $g(x, \lambda) \leqq G(x, \lambda)$ *on* $X \times L$. *Then there exist* p *nonempty sets of simple eigenvalues* $L_0, L_1, \cdots, L_{p-1}$ *for* (1, 3a, 3b), *such that if* $\sigma_j \in L_j$, *then* $\theta_1(b, \sigma_j) > (m + j)\pi$. *Moreover, if* $j > 2$, *then the corresponding solution* $\{y(x, \sigma_j), z(x, \sigma_j)\}$ *has the property that* $y(x, \sigma_j)$ *has at least* $j - 2$ *zeros on* X.

Proof. From the continuity of $\theta_1(b, \lambda)$ and the fact that $\theta_1(b, \lambda)$ increases from less than $m\pi$ to more than $n\pi$, select λ_j such that for $\lambda > \lambda_j$, $\theta_1(b, \lambda) > (m + j)\pi$, $j = 0, 1, \cdots, p - 1$. Let L_j be the set of all eigenvalues on $[\lambda_j, \lambda_{j+1}]$. From the proof of Theorem 2, each L_j is nonempty.

Suppose $\sigma_j \in L_j$, $j > 2$. Then $\theta_1(b, \sigma_j) > (m + j)\pi \geqq j\pi$ and $\theta_1(b, \sigma_j) - \theta_1(a, \sigma_j) > (j - 1)\pi$, so that $\theta_1(x, \sigma_j) = O(\operatorname{mod} \pi)$ at least $j - 1$ times on X and $\Phi_1(x, \sigma_j)$ has at least $j - 1$ zeros on X. If $k(x, \lambda) \geqq K(x, \lambda)$ and $G(x, \lambda) \geqq g(x, \lambda)$ on $X \times L$, then the Sturm comparison theorem can be applied to see that between two zeros of $\Phi_1(x, \sigma_j)$ there is at least one zero of $y(x, \sigma_j)$.

REFERENCES

1. G. D. Birkhoff, *Existence and oscillation theorem for a certain boundary value problem*, Trans. Amer. Math. Soc., **10** (1909), 259-270.

2. G. J. Etgen and S. C. Tefteller, *A two-point boundary problem for nonlinear second order differential systems*, SIAM J. Math. Anal., **2** (1971), 64-71.

3. H. J. Ettlinger, *Existence theorem for the non-self-adjoint linear system of the second order*, Annals of Math., **21** (1919-1920), 278-290.

4. ———, *Extension of an existence theorem for a non-self-adjoint system*, Bull. Amer. Math. Soc., **27** (1920-1921), 322-325.

5. ———, *Oscillation theorems for the real, self-adjoint linear system of the second order*, Trans. Amer. Math. Soc., **22** (1921), 136-143.

6. E. Kamke, *Über die Existenz von Eigenwerten bei Randwertaufgaben Zweiter Ordnung*, Math. Zeit., **44** (1939), 618-634.

7. ———, *Neue Herleitung der Oszillationssätze für die Linearen Selfstadjungierten Randwertaufgaben Zweiter Ordnung*, Math. Zeit., **44** (1939), 635-658.

8. E. L. Lnce, *Ordinary Differential Equations*, Dover, New York, 1956.

9. W. M. Whyburn, *Second-order differential systems with integral and k-point boundary conditions*, Trans. Amer. Math. Soc., **30** (1928), 630-640.

10. ———, *Existence and oscillation theorems for non-linear systems of the second order*, Trans. Amer. Math. Soc., **30** (1928), 848-854.

11. ———, *A non-linear boundary value problem for second order differential systems*, Pacific J. Math., **5** (1955), 147-160.

12. ———, *A note on a non-self-adjoint differential system of the second order*, J. Elisha Mitchell Sci. Soc., **69** (1953), 116-118.

Received November 23, 1976 and in revised form July 13, 1978.

UNIVERSITY OF ALABAMA IN BIRMINGHAM
BIRMINGHAM, AL 35294
AND
EXXON PRODUCTION RESEARCH COMPANY
HOUSTON, TX 77001