PACIFIC JOURNAL OF MATHEMATICS
Vol. 87, No. 1, 1980

A CHARACTERIZATION OF LC-NON-REMOVABLE IDEALS IN COMMUTATIVE BANACH ALGEBRAS

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Let A be a commutative Banach algebra with an identity e. Our main result states that an ideal $I \subset A$ is contained in a proper ideal I_B of B for every locally convex extension B of A if and only if the ideal I consists of joint topological divisors of zero.

All algebras in this paper are assumed to be commutative, complex and with an identity element denoted by e. All ideals are assumed to be proper, i.e., different from the whole algebra. By a topological algebra we mean a topological linear space together with an associative jointly continuous multiplication. If K is any class of topological linear spaces, then we say that a topological algebra is in K if it is in K as a topological linear space.

If K is any class of topological algebras, then a K-extension of A is an algebra $B \in K$ which contains A under an identity preserving topological isomorphism into. In this case we write $A \subset B$. An ideal $I \subset A \in K$ is called K-removable if there is a K-extension B of A such that I is contained in no ideal of B. If this holds we say that the extension B removes the ideal I. Thus B removes I if and only if there are elements $x_1, x_2, \cdots, x_n \in I$, $b_1, b_2, \cdots, b_n \in B$ such that

$$e = \sum_{i=1}^{n} x_i b_i \ . \tag{1}$$

Otherwise we say that an ideal I is K-nonremovable.

As the class K we shall consider the following classes of commutative algebras with identities: B—the class of Banach algebras, LC—locally convex algebras, M—multiplicatively convex algebras (shortly m—convex algebras), and T—topological algebras. We shall consider only complete algebras, however the ideals are not assumed to be closed.

In this paper we give a characterization of LC-removability of ideals in Banach algebras. It turns out that this removability coincides with T-removability, but we do not know whether it coincides with B-removability. There is no satisfactory characterization of B-nonremovable ideals. Some description of these ideals is given in [2] in terms of a certain B-extension of the algebra in question.

We give now a short description of some concepts and facts we shall use in the sequel.

The topology of a locally convex algebra A is given by means of a family $(||x||_\alpha)$ of seminorms such that for each α there is a β with

$$||xy||_\alpha \leqq ||x||_\beta ||y||_\beta$$

for all $x, y \in A$. If, moreover, A is metrizable, then its topology can be given by means of an increasing sequence

$$||x||_1 \leqq ||x||_2 \leqq \cdots$$

of seminorms such that

$$(2) \qquad ||xy||_i \leqq ||x||_{i+1} ||y||_{i+1}\,, \qquad i = 1, 2, \cdots$$

for all $x, y \in A$. Such algebras will be called shortly B_0-algebras.

A locally convex algebra A is called m-convex if its topology can be given by means of a family of submultiplicative seminorms, i.e.,

$$||xy||_\alpha \leqq ||x||_\alpha ||y||_\alpha$$

for all α and all $x, y \in A$.

We shall need the following extensions of Banach algebras. Consider the algebra of all bounded sequences $\tilde{x} = (x_n) \subset A$ with pointwise algebra operations. The formula

$$(3) \qquad ||\tilde{x}|| = \limsup ||x_n||$$

defines there a submultiplicative seminorm. We set A_∞ for the quotient of this algebra modulo the ideal of zeros of the seminorm (3). It is a Banach algebra with the norm (3). Elements of A_∞ may be regarded as sequences $\tilde{x}$ with two sequence identified when their difference tends to zero. The algebra A_∞ contains A isometrically if we identify elements of A with the constant sequences.

Let $t = (t_1, \cdots, t_n)$ be an n-tuple of indeterminates and consider the algebra of all power series $x(t) = \sum_{|i|=1}^{\infty} x_i t^i$, where $i = (i_1, \cdots, i_n)$, $t^i = t_1^{i_1} \cdots t_n^{i_n}$, $|i| = i_1 + \cdots + i_n$, and $x_i = x_{i_1 \cdots i_n} \in A$, such that

$$(4) \qquad ||x(t)|| = \sum_i ||x_i|| < \infty\ .$$

This algebra will be designated by $A(t)$. It contains A isometrically if we identify elements of A with the constant power series.

Since the class B is contained in the class LC we can consider also LC-extensions of Banach algebras. For our purposes it is sufficient to remark that if a Banach algebra A is algebraically contained in $B \in LC$ and for each $x \in A$ and each index α we have $\|x\|_\alpha = \|x\|$, then the imbedding is topological.

Let A be a Banach algebra. An ideal $I \subset A$ is said to consist of joint topological divisors of zero if there exists a net (z_α) of elements of A, $\|z_\alpha\| = 1$, such that

$$\lim_\alpha \|z_\alpha x\| = 0 \tag{5}$$

for all $x \in I$. In this case we say that the net (z_α) annihilates the ideal I and write $(z_\alpha) \perp I$. Observe that the relation (5) is equivalent to the following: for each n-tuple $(x_1, \cdots, x_n)$ of elements of I we have $\inf\{\sum \|x_i z\| : \|z\| = 1, z \in A\} = 0$. Thus if an ideal $I \subset A$ does not consist of joint topological divisors of zero, then there is an n-tuple $(x_1, \cdots, x_n) \subset I$ such that

$$\sum_{i=1}^{n} \|x_i z\| \geqq \|z\| \tag{6}$$

for all $z \in A$. The family of all ideals in A which consist of joint topological divisors of zero will be denoted by $\ell(A)$ and its members will be called shortly ℓ-ideals. We put also $\mathcal{L}(A) = \ell(A) \cap \mathfrak{M}(A)$, where $\mathfrak{M}(A)$ is the maximal ideal space of A. It is known ([4]) that every ℓ-ideal I is contained in a maximal ideal $M \in \mathcal{L}(A)$.

For details on the above the reader is refered to [4], [5], [7].

The following lemma is a well known fact in the theory of rings (cf. [3]).

LEMMA 1. *Let A be a commutative ring with an identity element and let $t = (t_1, \cdots, t_n)$ be a system of indeterminates. If for two nonzero polynomials $p(t)$ and $q(t)$ with coefficients in A we have $p(t)q(t) = 0$, then there is a nonzero element $x \in A$ such that*

$$xp(t) = 0\ , \tag{7}$$

i.e., *the element x annihilates all coefficients in $p(t)$.*

LEMMA 2. *Let A be commutative Banach algebra with an identity element and let $t = (t_1, \cdots, t_n)$ be a system of n indeterminates. Let $(x_1, \cdots, x_n)$ be an n-tuple of elements of A satisfying relation* (6) *for all $z \in A$, and put*

$$w = \sum_{i=1}^{n} x_i t_i\ . \tag{8}$$

Then there is a sequence (α_k) of real numbers, $\alpha_0 = 1$, $\alpha_k \geqq 1$, such that

$$\alpha_k \|w p_k\| \geqq \|p_k\|\ , \qquad k = 0, 1, \cdots \tag{9}$$

for all homogeneous polynomials $p_k \in A(t)$ *of* kth *degree. The norm in* (9) *is given by the formula* (4).

Proof. For $k = 0$ the relation (9) with $\alpha_0 = 1$ follows immediately from the inequality (6). Suppose that for some $k \geqq 1$ the relation (9) fails. This means that for each integer m we can find a homogeneous polynomial $p_k^{(m)}$ of degree k, with $\|p_k^{(m)}\| = 1$, such that

$$(10) \qquad m\|wp_k^{(m)}\| < \|p_k^{(m)}\| .$$

Thus $\lim_m wp_k^{(m)} = 0$. Denote by $x_i^{(m)}$ the coefficient by t^i for $p_k^{(m)}$. Since $\|x_i^{(m)}\| \leqq 1$ for all m, the sequence $\tilde{x}_i = (x_i^{(m)})$ represents an element in A_∞. The relation (10) implies that in $A_\infty(t)$ we have $w\tilde{p}_k = 0$, where $\tilde{p}_k = \sum_{|i|=k} \tilde{x}_i t^i$. One can easily see that $\tilde{p}_k$ is a nonzero polynomial in $A_\infty(t)$. Applying Lemma 1 we find an element $\tilde{x} \in A_\infty$, $\|\tilde{x}\| = 1$, such that $\|w\tilde{x}\| = 0$. However if $\tilde{x} = (x_i)$, then relation (6) implies $\|wx_i\| \geqq \|x_i\|$, which in turn implies $\|w\tilde{x}\| \geqq \|\tilde{x}\|$ what is a contradiction. Thus, the desired sequence (α_k) exists.

LEMMA 3. *Let* (a_k), $k = 0, 1, 2, \cdots$ *be a sequence of positive real numbers with* $a_0 = 1$. *There exists a sequence* (b_k), $k = 0, 1, \cdots$, $b_0 = 1$, *with* $b_i \geqq a_i$ *and*

$$(11) \qquad a_{m+n} \leqq b_m b_n$$

for all $m, n \geqq 0$.

Proof. Put $b_0 = 1$ and suppose that we already have numbers b_i for $i < k$ which satisfy (11) for $m, n < k$. Put

$$b_k = \max\{a_k, a_{k+1}/b_1, a_{k+2}/b_2, \cdots, a_{2k-1}/b_{k-1}, a_{2k}^{1/2}\} .$$

One can easily see that relation (11) holds now for all $m, n \leqq k$ and $b_k \geqq a_k$. The conclusion follows.

We can prove now our main result.

THEOREM 4. *Let* A *be a commutative Banach algebra with an identity element and let* I *be an ideal in* A. *Then* I *is an* LC-*nonremovable ideal if and only if it consists of joint topological divisors of zero.*

Proof. Let $I \in \mathcal{I}(A)$ and let B be any locally convex extension of A. If I is removed by B there are elements $x_1, \cdots x_n \in I$ and $b_1, \cdots, b_n \in B$ such that relation (1) holds true. Multiplying both sides by a net $(z_\alpha) \perp I$, $\|z_\alpha\| = 1$ we obtain a contradiction. So I is an LC-nonremovable ideal.

Suppose now that I does not consist of joint topological divisors of zero. We can find elements $x_1, \cdots, x_n \in I$ so that relation (6) holds true for all $z \in A$. We shall be done if we construct a locally convex algebra B (it will be in fact a B_0-algebra), and elements $b_1, \cdots, b_n \in B$ such that formula (1) holds true. Taking w given by the formula (8) we find by Lemma 2 a suitable sequence (α_k) satisfying relation (9). Define $a_0^{(1)} = 1$ and $a_m^{(1)} = \alpha_0 \alpha_1 \cdots \alpha_{m-1}$ for $m = 1, 2, \cdots$. Thus, $a_i = a_i^{(1)}$ satisfies the assumptions of Lemma 3. Put $a_i^{(2)} = b_i$, $i = 0, 1, 2, \cdots$, where (b_i) is the sequence in conclusion of Lemma 3 and then proceed by an induction. For a given sequence $a_m^{(k)}$, $m = 0, 1, \cdots$ put $a_m = a_m^{(k)}$ and define $a_m^{(k+1)} = b_m$ according to Lemma 3. The matrix $(a_m^{(k)})$, $k = 1, 2, \cdots$ $m = 0, 1, \cdots$ satisfies the following

$$a_0^{(k)} = 1 \quad \text{for} \quad k = 1, 2, \cdots , \tag{12}$$

$$a_i^{(k)} \leqq a_i^{(k+1)} \quad \text{for} \quad k = 1, 2, \cdots \quad \text{and all} \quad i \geqq 0 , \tag{13}$$

$$a_{i+j}^{(k)} \leqq a_i^{(k+1)} a_j^{(k+1)} \quad \text{for} \quad k \geqq 1 \quad \text{and} \quad i, j \geqq 0 . \tag{14}$$

Let $t = (t_1, \cdots, t_n)$ be a system of indeterminates and consider the locally convex algebra $\widetilde{B}(t)$ consisting of all polynomials $p(t)$ in n variables with coefficients from A. Each such polynomial can be written in the form

$$p(t) = \sum_{k=0}^{m} p_k(t) , \tag{15}$$

where $p_k(t)$ is a homogeneous polynomial of degree k with coefficients in A. The seminorms in $\widetilde{B}(t)$ are defined as follows. For a polynomial p of the form (15) we put

$$\| p(t) \|_i = \sum_{k=0}^{m} a_k^{(i)} \| p_k(t) \| , \qquad i = 1, 2, \cdots , \tag{16}$$

where the norm $\| p_k(t) \|$ is given by the formula (4). Relation (13) shows that for any polynomial $p \in \widetilde{B}(t)$ we have

$$\| p \|_i \leqq \| p \|_{i+1} \quad \text{for} \quad i = 1, 2, \cdots . \tag{17}$$

For any two polynomials p, q of the form (15) we have by (14)

$$\begin{aligned} \| pq \|_i &= \sum_k a_k^{(i)} \| \sum_s p_{k-s} q_s \| \\ &\leqq \sum_{k,s} a_{k-s}^{(i+1)} a_s^{(i+1)}) \| p_{k-s} \| \| q_s \| = \| p \|_{i+1} \| q \|_{i+1} . \end{aligned} \tag{18}$$

Thus the multiplication is jointly continuous in $\widetilde{B}(t)$ and its completion $B(t)$ is a B_0-algebra with the seminorms (16) and formal multiplication of power series. Let us note that for all polynomials of zero degree p_0 we have by (12)

$$(19) \qquad ||p_0||_i = ||p_0|| \,.$$

Thus, $B(t)$ is an extension of A if we identify elements of A with polynomials of degree zero. Let w be the element of $B(t)$ given by (8) and let J be the closed ideal of $B(t)$ generated by $w - e$, i.e., J is the closure in $B(t)$ of the set $(w - e)\tilde{B}(t)$. Put $B = B(t)/J$. We shall show that B is an extension of A under the imbedding $x \to [x] = x + J$. The topology of B is given by means of the sequence of seminorms

$$(20) \qquad ||[p]||_i = \inf_{j \in J} ||p + j||_i \,,$$

and one can easily see that the seminorms (20) also satisfy relations (17) and (18). Relation (20) implies that for each $p \in B(t)$ we have

$$||[p]||_i \leqq ||p||_i$$

and so, by (19)

$$||[x]||_i \leqq ||x||$$

for all $x \in A$. In view of (17) we shall be done if we show

$$(21) \qquad ||x|| \leqq ||[x]||_1$$

for all $x \in A$ since it will imply $||x|| = ||[x]||_i$ for all x and i and our imbedding will be a topological isomorphism into. Since $\tilde{B}(t)$ is dense in $B(t)$ we have

$$||[x]||_1 = \inf ||x + (w - e)\sum_{i=0}^{m} p_i||_1 \,,$$

where p_i is a homogeneous polynomial of degree i and the infimum is taken with respect to all elements $\sum_{i=0}^{m} p_i$ in $\tilde{B}(t)$. Setting $p_{m+1} = 0$, we have by (9) the following estimation

$$\begin{aligned} ||x + (w - e)\sum_{i=0}^{m} p_i||_1 &= ||x - p_0|| + \sum_{i=0}^{m} a_{i+1}^{(1)}||wp_i - p_{i+1}|| \\ &= ||x - p_0|| + \sum_{i=0}^{m} \alpha_0 \cdots \alpha_i ||wp_i - p_i||_1 \\ &\geqq ||x|| - ||p_0|| + \sum_{i=0}^{m} \alpha_0 \cdots \alpha_i(||wp_i|| - ||p_{i+1}||) \\ &= ||x|| + (||wp_0|| - ||p_0||) + \sum_{i=0}^{m} \alpha_0 \cdots \alpha_{i-1}(\alpha_i||wp_i|| - ||p_i||) \geqq ||x|| \,, \end{aligned}$$

which establishes relation (21) and we are done.

COROLLARY 5. *An ideal of a Banach algebra is* LC-*nonremovable if and only if it is T-nonremovable.*

As we mentioned earlier we do not know what is characterization of nonremovable (i.e., B-nonremovable) ideals in Banach algebras. From the above result it follows that either this characterization is the same as in Theorem 4, or has a relative character: there are ideals which are nonremovable through Banach algebra extensions but are removable through locally convex extensions.

Let K be a class of topological algebras and let $A \in K$. A family (I_α) of K-remorable ideals of A is called a K-removable family if there exists a single extension $B \in K$ of the algebra A which removes all ideals I_α. In [1] Arens asked whether a finite family of removable (i.e., B-removable) ideals of a Banach algebra is a removable family. In [8] we showed that for $A \in K$ the following are equivalent

(i) Every finite family of K-removable ideals is K-removable and

(ii) Every maximal K-nonremovable ideal is prime.

Here by a maximal K-nonremovable ideal we mean an ideal $I \subset A$ such that for any ideal $J \supset I$ we have either $I = J$, or J is K-removable. Since for a Banach algebra A the class of ŁC-nonremovable ideals coincides with $\mathcal{ł}(A)$ and every ideal $I \in \mathcal{ł}(A)$ is contained in a maximal ideal $M \in \mathcal{Ł}(A)$, we have the following result

THEOREM 6. *Let A be a commutative Banach algebra with an identity element. Then every finite family of* LC-*removable ideals is an* ŁC-*removable family.*

In [9] we reduced the problem of the characterization of M-nonremovable ideals of an algebra $A \in M$ to that of the characterization of B-nonremovable ideals in Banach algebras. Unfortunately, this result gives no information about a characterization of LC-nonremovable ideals in m-convex algebras.

Let us remark that M-removability of ideals in Banach algebras is the same as B-removability, and that the M-removability of ideals in m-convex algebras has a relative character. By a result in [6] there is an ideal $I \subset A \in M$ which is M-nonremovable and LC-removable.

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Received July 12, 1979. This paper was written during the author's stay in the University of Kansas, Lawrence, during the second term of 1978/79.

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