PACIFIC JOURNAL OF MATHEMATICS
Vol. 87, No. 1, 1980

ON SOME FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS

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We give some fixed point theorems for multivalued nonexpansive mappings or generalized contractions with noncompact domains in Banach spaces. First, we give a fixed point theorem for nonexpansive mappings that generalizes the results of Lami-Dozo, Assad-Kirk and Ko. Furthermore we give similar theorems for nonexpansive mappings or generalized contractions with nonconvex domains.

In 1976, Caristi [4] obtained fixed point theorems for weakly inward singlevalued mappings. The essential part of his proof is based on the following useful existence theorem.

THEOREM (*Browder* [2], *Caristi-Kirk* [3], *Caristi* [4], *Kirk* [9], *Siegel* [18] *and Wong* [19]). *Let X be a complete metric space and $f: X \to X$ an arbitrary mapping. Suppose there exists a lower semicontinuous mapping ψ of X into the nonnegative real numbers such that for each $x \in X$,*

$$d(x, f(x)) \leqq \psi(x) - \psi(f(x)) .$$

Then f has a fixed point in X.

Fixed point theorems for multivalued nonexpansive mappings are obtained by Assad-Kirk [1], Downing-Kirk [5], Itoh-Takahashi [8], Ko [10], Lami-Dozo [11], Lim [12, 13], Reich [15, 16, 17] and the other. Recently Downing-Kirk and Reich obtained some existence theorems containing the results of Lim by using the above theorem essentially. In this paper we shall give extensions of results of Lami-Dozo, Assad-Kirk and Ko by using similar method to Downing-Kirk and Reich. Furthermore we shall obtain similar results in the case of nonconvex domain. Now we shall introduce some necessary notations and definitions. Let X be a Banach space and K be a nonempty convex subset of X. If $x \in K$, we define the inward set of x relative to K, denoted $I_K(x)$ as follows:

$$I_K(x) = \{x + \alpha(y - x) \mid y \in K, \alpha \geqq 1\} .$$

We say that a mapping $f: K \to X$ is *weakly inward* if $f(x)$ belongs to the closure of $I_K(x)$ for each $x \in K$. We denote by $\mathscr{CB}(X)$ the family of nonempty bounded closed subsets of X and denote by $\mathscr{K}(X)$ the family of nonempty compact subsets of X. For $A \in$

$\mathscr{CB}(X)$, we define $d(x, A) = \inf\{\|x - y\| \mid y \in A\}$. If $K \subset X$, $\mathrm{cl}(K)$, $\mathrm{int}(K)$ and ∂K will stand for the closure, interior and boundary of K, respectively. We write $x_n \rightharpoonup x$ to indicate that the sequence of vectors $\{x_n\}$ converges weakly to x; as usual $x_n \to x$ will symbolize (strong) convergence.

DEFINITION 1. Let D be the Hausdorff metric on $\mathscr{CB}(X)$ induced by the norm of X and let $K \in \mathscr{CB}(X)$. $T: K \in \mathscr{CB}(X)$ is said to be *nonexpansive* if $D(T(x), T(y)) \leqq \|x - y\|$ for every $x, y \in K$. $T: K \to \mathscr{CB}(X)$ is said to be a *contraction* if for every $x, y \in K$, $D(T(x), T(y)) \leqq k\|x - y\|$, where $0 \leqq k < 1$. $T: K \to \mathscr{CB}(X)$ is said to be a *generalized contraction* if for each $x \in K$ there is a number $\alpha(x) < 1$ such that $D(T(x), T(y)) \leqq \alpha(x)\|x - y\|$ for each $y \in K$.

DEFINITION 2. A Banach space X is said to satisfy *Opial's condition* if the following holds: If a sequence $\{x_n\}$ is weakly convergent to x in X and $x \neq y$, then

$$(*) \qquad \liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| .$$

A Banach space X is said to satisfy *weak Opial's condition* if the following holds: If a sequence $\{x_n\}$ is weakly convergent to x in X, then for every y in X,

$$(**) \qquad \liminf_{n\to\infty} \|x_n - x\| \leqq \liminf_{n\to\infty} \|x_n - y\| .$$

We remark that (*) and (**) are equivalent to (*)′ and (**)′, respectively (cf. [11]):

$$(*)' \qquad \limsup_{n\leftarrow\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\| ,$$

$$(**)' \qquad \limsup_{n\to\infty} \|x_n - x\| \leqq \limsup_{n\to\infty} \|x_n - y\| .$$

Hilbert spaces and $l^p(1 \leqq p < \infty)$ satisfy Opial's condition and Banach spaces with weakly continuous duality mappings satisfy weak Opial's condition (cf. [14]).

DEFINITION 3. Let K be a convex set in X. $T: K \to \mathscr{CB}(X)$ is said to be *demiclosed* on K if $x_n \rightharpoonup x$, $y_n \to y$ and $y_n \in T(x_n)$ imply $y \in T(x)$. $T: K \to \mathscr{CB}(X)$ is said to be *semiconvex* on K if for any $x, y \in K$, $z = \lambda x + (1 - \lambda)y$, where $0 \leqq \lambda \leqq 1$, and any $x_1 \in T(x)$, $y_1 \in T(y)$, there exists $z_1 \in T(z)$ such that $\|z_1\| \leqq \max\{\|x_1\|, \|y_1\|\}$.

PROPOSITION 1 (Ko [10]). *Let K be a convex set in X and let $T: K \to \mathscr{CB}(X)$. If $I - T$ is semiconvex on K, then for any $x, y \in K$*

and $z = \lambda x + (1 - \lambda)y$, *where* $0 \leqq \lambda \leqq 1$, *we have* $d(z, T(z)) \leqq \max\{d(x, T(x)), d(y, T(y))\}$.

PROPOSITION 2 (*Ko* [10], *Downing-Kirk* [5]). *Let K be a set in X. If $T: K \to \mathscr{CB}(X)$ is upper semicontinuous, then $d(x, T(x))$ is a lower semicontinuous mapping of K into the nonnegative real numbers.*

Before we obtain main theorems, we shall state the following result related to multivalued contractions.

PROPOSITION 3 (*Downing-Kirk* [5], *Reich* [17]). *Let K be a nonempty closed convex subset of X and let $T: K \to \mathscr{K}(X)$ be a contraction. If $T(x) \subset \mathrm{cl}\,(I_K(x))$ for each $x \in K$, then T has a fixed point.*

We shall obtain the first theorem.

THEOREM 1. *Let K be a nonempty weakly compact convex subset of a Banach space X and let $T: K \to \mathscr{K}(X)$ be nonexpansive such that $T(x) \subset \mathrm{cl}\,(I_K(x))$ for each $x \in K$. If $I - T$ is demiclosed or semiconvex on K, then T has a fixed point.*

Proof. Choose a point x_0 in K and a sequence $\{k_n\}$, $0 < k_n < 1$, that converges to 0. By Proposition 3, the mapping $T_n: K \to \mathscr{K}(X)$ defined by $T_n(x) = k_n x_0 + (1 - k_n)T(x)$ for all $x \in K$ has a fixed point x_n. Consequently there exists $y_n \in T(x_n)$ such that $x_n = k_n x_0 + (1 - k_n)y_n$. Suppose $I - T$ is demiclosed on K. Since K is weakly compact, there is a sequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup z \in K$. Also

$$||x_{n_i} - y_{n_i}|| = \frac{k_{n_i}}{1 - k_{n_i}} ||x_0 - x_{n_i}|| \longrightarrow 0 \ .$$

Therefore $0 \in (I - T)(z)$, i.e., $z \in T(z)$. Suppose $I - T$ is semiconvex on K. We have $\inf\{d(x, T(x)) \mid x \in K\} = 0$ because

$$d(x_n, T(x_n)) \leqq ||x_n = y_n|| = \frac{k_n}{1 - k_n} ||x_0 - x_n|| \longrightarrow 0 \ .$$

Let $r > 0$, define $H_r = \{x \in K \mid d(x, T(x)) \leqq r\}$. Since Proposition 1 and Proposition 2 imply that H_r are closed convex, H_r are weakly closed for every $r > 0$. The family $\{H_r \mid r > 0\}$ has the finite intersection property. Therefore, by the weak compactness of K, we have $\cap\{H_r \mid r > 0\} \neq \varnothing$. It is clear that any point in $\cap\{H_r \mid r > 0\}$ is a fixed point of T. □

We obtain the following

COROLLARY 1. *Let K be a nonempty weakly compact convex subset of a Banach space X which satisfies Opial's condition (or weak Opial's condition). If $T: K \to \mathscr{K}(X)$ is nonexpansive (or a generalized contraction) such that $T(x) \subset \mathrm{cl}\,(I_K(x))$ for each $x \in K$, then T has a fixed point.*

Proof. If X satisfies Opial's condition and T is nonexpansive, then $I - T$ is demiclosed on K by the result of Lami-Dozo. Therefore we show that $I - T$ is demiclosed on K if X satisfies weak Opial's condition and T is a generalized contraction. Suppose that $x_n \rightharpoonup x$, $y_n \to y$ and $y_n \in (I - T)(x_n)$. Hence there exists $u_n \in T(x_n)$ such that $y_n = x_n - u_n$. Since $T(x)$ is compact, there exists $v_n \in T(x)$ such that

$$\|v_n - u_n\| \leqq D(T(x), T(x_n)) \leqq \alpha(x)\|x - x_n\| .$$

Also there is a sequence $\{v_{n_i}\}$ of $\{v_n\}$ such that $v_{n_i} \to v \in T(x)$. We have the following relation,

$$\begin{aligned}
\alpha(x) \limsup_{i\to\infty} \|x_{n_i} - x\| &\geqq \limsup_{i\to\infty} \|u_{n_i} - v_{n_i}\| \\
&= \limsup_{i\to\infty} \|x_{n_i} - y_{n_i} - v_{n_i}\| \\
&= \limsup_{i\to\infty} \|x_{n_i} - y - v + y - y_{n_i} + v - v_{n_i}\| \\
&\geqq \limsup_{i\to\infty} \{\|x_{n_i} - y - v\| - \|y_{n_i} - y\| - \|v_{n_i} - v\|\} \\
&\geqq \limsup_{i\to\infty} \|x_{n_i} - y - v\| - \limsup_{i\to\infty} \|y_{n_i} - y\| - \limsup_{i\to\infty} \|v_{n_i} - v\| \\
&= \limsup_{i\to\infty} \|x_{n_i} - y - v\| .
\end{aligned}$$

Since $x_{n_i} \rightharpoonup x$ and X satisfies weak Opial's condition, we have $\limsup_{i\to\infty} \|x_{n_i} - x\| = 0$. Hence $x_{n_i} \to x$ and $x_{n_i} \to y + v$. Therefore $y = x - v \in (I - T)(x)$. □

If K is compact in Theorem 1, we obtain the following

COROLLARY 2. *Let K be a nonempty compact convex subset of a Banach space X and let $T: K \to \mathscr{K}(X)$ be nonexpansive such that $T(x) \subset \mathrm{cl}\,(I_K(x))$ for each $x \in K$. Then T has a fixed point.*

We shall obtain fixed point theorems for nonexpansive mappings or generalized contractions on starshaped subsets of Banach spaces.

DEFINITION 4. A subset K of a Banach space is called *starshaped* if there exists an element $x_0 \in K$ such that for $x \in K$ and $k (0 < k < 1)$, $kx_0 + (1 - k)x \in K$.

DEFINITION 5. For a subset K of a Banach space X and a bounded sequence $\{x_n\}$ in X, we define

$$AR(K, \{x_n\}) = \inf \left\{\limsup_{n\to\infty} \|y - x_n\| \,|\, y \in K\right\}$$

and

$$A(K, \{x_n\}) = \left\{z \in K \,|\, \limsup_{n\to\infty} \|z - x_n\| = AR(K, \{x_n\})\right\} .$$

The set $A(K, \{x_n\})$ and the number $AR(K, \{x_n\})$ are called, respectively, the *asymptotic center* and the *asymptotic radius* of $\{x_n\}$ relative to K.

PROPOSITION 4. *The following hold*:

(1) *If K is convex, then $A(K, \{x_n\})$ is convex*;

(2) *if K is closed, then $A(K, \{x_n\})$ is closed*;

(3) *if K is weakly compact, then $A(K, \{x_n\})$ is nonempty*;

(4) *if X is uniformly convex and K is bounded closed convex, then $A(K, \{x_n\})$ consists of exactly one point*;

(5) $A(K, \{x_n\}) \subset \partial K \cup A(X, \{x_n\})$;

(6) *There exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $AR(K,\{x_{n_{i_j}}\}) = AR(K, \{x_{n_i}\})$ and $A(K, \{x_{n_i}\}) \subset A(K, \{x_{n_{i_j}}\})$ for any subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$.*

Proof. (1), (2), (3) and (4) are clear (cf. [6]). We prove at first (5). Suppose that $A(K, \{x_n\}) \not\subset \partial K \cup A(X, \{x_n\})$. Then there exists $x \in \operatorname{int}(K)$ such that $x \in A(K, \{x_n\})$ and $x \notin A(X, \{x_n\})$. We have

$$\begin{aligned}\inf \left\{\limsup_{n\to\infty} \|y - x_n\| \,|\, y \in X\right\} &< \limsup_{n\to\infty} \|x - x_n\| \\ &= \inf \left\{\limsup_{n\to\infty} \|y - x_n\| \,|\, y \in K\right\} .\end{aligned}$$

Hence there is $v \in X$ such that

$$\limsup_{n\to\infty} \|v - x_n\| < \inf \left\{\limsup_{n\to\infty} \|y - x_n\| \,|\, y \in K\right\} .$$

Since $x \in \operatorname{int}(K)$, there exists $\lambda \in (0, 1)$ such that $\lambda x + (1 - \lambda)v \in K$. Hence

$$\begin{aligned}\inf \left\{\limsup_{n\to\infty} \|y - x_n\| \,|\, y \in K\right\} &\leqq \limsup_{n\to\infty} \|\lambda x + (1 - \lambda)v - x_n\| \\ &\leqq \lambda \limsup_{n\to\infty} \|x - x_n\| + (1 - \lambda) \limsup_{n\to\infty} \|v - x_n\| .\end{aligned}$$

Therefore $\limsup_{n\to\infty} \|x - x_n\| \leqq \limsup_{n\to\infty} \|v - x_n\|$. This is a contradiction. Next we show (6). By Lim [13, Proposition 1], there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $AR(K, \{x_{n_{i_j}}\}) = AR(K, \{x_{n_i}\})$ for any subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$. Let $x \in A(K, \{x_{n_i}\})$. For any subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$,

$$\limsup_{j\to\infty} \|x_{n_{i_j}} - x\| \leqq \limsup_{i\to\infty} \|x_{n_i} - x\| = AR(K, \{x_{n_i}\})$$
$$= AR(K, \{x_{n_{i_j}}\}) \leqq \limsup_{j\to\infty} \|x_{n_{i_j}} - x\| .$$

Hence $\limsup_{j\to\infty} \|x_{n_i} - x_j\| = AR(K, \{x_{n_{i_j}}\})$. Therefore $x \in A(K, \{x_{n_{i_j}}\})$. □

We shall obtain the following theorem for nonexpansive mappings.

THEOREM 2. *Let K be a nonempty weakly compact starshaped subset of a uniformly convex Banach space X and let $T: K \to \mathscr{K}(X)$ be nonexpansive. If for each $x \in \partial K$, $T(x) \subset K$ and $\lambda x + (1-\lambda)T(x) \subset K$ for some $\lambda \in (0, 1)$ or $T(x) \subset \operatorname{int}(K)$, then T has a fixed point.*

Proof. Let x_0 be a starcenter and choose a sequence $\{k_n\}$, $0 < k_n < 1$, that converges to 0. By Assad-Kirk [1], the mapping T_n: $K \to \mathscr{K}(X)$ defined by $T_n(x) = k_n x_0 + (1-k_n)\ T(x)$ for all $x \in K$, has a fixed point x_n. Consequently there exists $y_n \in T(x_n)$ such that $x_n = k_n x_0 + (1-k_n) y_n$. Since $\{x_n\}$ is bounded, we can take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ as (6) in Proposition 4. We rewrite $\{x_{n_i}\}$ to $\{x_n\}$. Let $z \in A(K, \{x_n\})$. Since $T(z)$ is compact, there exists $z_n \in T(z)$ such that $\|z_n - y_n\| \leqq D(T(z), T(x_n)) \leqq \|z - x_n\|$, and there exists a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $z_{n_i} \to \bar{z} \in T(z)$. By (6) in Proposition 4, $A(K, \{x_n\}) \subset A(K, \{x_{n_i}\})$. Hence $z \in A(K, \{x_{n_i}\})$. Since

$$\|x_{n_i} - y_{n_i}\| = \frac{k_{n_i}}{1 - k_{n_i}} \|x_0 - x_{n_i}\| \longrightarrow 0 ,$$

we have

$$\limsup_{i\to\infty} \|\bar{z} - x_{n_i}\|$$
$$\leqq \limsup_{i\to\infty} \|\bar{z} - z_{n_i}\| + \limsup_{i\to\infty} \|z_{n_i} - y_{n_i}\| + \limsup_{i\to\infty} \|y_{n_i} - x_{n_i}\|$$
$$= \limsup_{i\to\infty} \|z_{n_i} - y_{n_i}\|$$
$$\leqq \limsup_{i\to\infty} \|z - x_{n_i}\| = \inf \left\{ \limsup_{i\to\infty} \|y - x_{n_i}\| \,\middle|\, y \in K \right\} .$$

If $z \in \partial K$, then $w = \lambda z + (1-\lambda)\bar{z} \in K$ for some $\lambda \in (0, 1)$ by hypothesis. Suppose that $z \neq \bar{z}$. By uniform convexity of X, we have for some $\delta \in (0, 1)$,

$$\limsup_{i\to\infty} \|w - x_{n_i}\| \leqq (1-\delta) \inf \left\{\limsup_{i\to\infty} \|y - x_{n_i}\| \mid y \in K\right\} .$$

This contradicts the choice of w. If $z \in A(X, \{x_{n_i}\})$, we have

$$\begin{aligned} AR(X, \{x_{n_i}\}) &\leqq \limsup_{i\to\infty} \|\bar{z} - x_{n_i}\| \\ &\leqq \limsup_{i\to\infty} \|\bar{z} - z_{n_i}\| + \limsup_{i\to\infty} \|z_{n_i} - y_{n_i}\| + \limsup_{i\to\infty} \|y_{n_i} - x_{n_i}\| \\ &= \limsup_{i\to\infty} \|z_{n_i} - y_{n_i}\| \leqq \limsup_{i\to\infty} \|z - x_{n_i}\| = AR(X, \{x_{n_i}\}) . \end{aligned}$$

Hence $\bar{z} \in A(X, \{x_{n_i}\})$. By uniform convexity of X, we obtain $z = \bar{z} \in T(z)$. □

The following theorem for generalized contractions is obtained.

THEOREM 3. *Let K be a nonempty weakly compact starshaped subset of a Banach space X and $T: K \to \mathscr{K}(X)$ be a generalized contraction. If for each $x \in \partial K$, $T(x) \subset K$, then T has a fixed point.*

Proof. As in Theorem 2, we obtain $x_n \in K$ such that $x_n \in T_n(x_n)$. Consequently, there exists $y_n \in T(x_n)$ such that $x_n = k_n x_0 + (1 - k_n) y_n$. Since $\{x_n\}$ is bounded, we can take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ as (6) in Proposition 4. We rewrite $\{x_{n_i}\}$ to $\{x_n\}$. Let $z \in A(K, \{x_n\})$. Since $T(z)$ is compact, there exists $z_n \in T(z)$ such that

$$\|z_n - y_n\| \leqq D(T(z), T(x_n)) \leqq \alpha(z) \|z - x_n\| ,$$

and there exists a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $z_{n_i} \to \bar{z} \in T(z)$. Since $A(K, \{x_n\}) \subset A(K, \{x_{n_i}\})$, $z \in A(K, \{x_{n_i}\})$. Also

$$\|x_{n_i} - y_{n_i}\| = \frac{k_{n_i}}{1 - k_{n_i}} \|x_0 - x_{n_i}\| \longrightarrow 0 .$$

If $z \in \partial K$, then $\bar{z} \in K$ by hypothesis. Hence

$$\begin{aligned} AR(K, \{x_{n_i}\}) &\leqq \limsup_{i\to\infty} \|\bar{z} - x_{n_i}\| \\ &\leqq \limsup_{i\to\infty} \|\bar{z} - z_{n_i}\| + \limsup_{i\to\infty} \|z_{n_i} - y_{n_i}\| + \limsup_{i\to\infty} \|y_{n_i} - x_{n_i}\| \\ &= \limsup_{i\to\infty} \|z_{n_i} - y_{n_i}\| \leqq \limsup_{i\to\infty} \alpha(z) \|z - x_{n_i}\| \\ &= \alpha(z) AR(K, \{x_{n_i}\}) . \end{aligned}$$

Since $1 - \alpha(z) > 0$, $AR(K, \{x_{n_i}\}) = 0$, which implies that $x_{n_i} \to \bar{z}$ and $x_{n_i} \to z$. Therefore $z = \bar{z} \in T(z)$. If $z \in A(X, \{x_{n_i}\})$, we have

$$\begin{aligned} AR(X, \{x_{n_i}\}) &\leqq \limsup_{i\to\infty} \|\bar{z} - x_{n_i}\| \\ &\leqq \limsup_{i\to\infty} \|\bar{z} - z_{n_i}\| + \limsup_{i\to\infty} \|z_{n_i} - y_{n_i}\| + \limsup_{i\to\infty} \|y_{n_i} - x_{n_i}\| \end{aligned}$$

$$= \limsup_{i\to\infty} ||z_{n_i} - y_{n_i}|| \leqq \limsup_{i\to\infty} \alpha(z)||z - x_{n_i}||$$
$$= \alpha(z)AR(X, \{x_{n_i}\}) .$$

Since $1 - \alpha(z) > 0$, $AR(X, \{x_{n_i}\}) = 0$, which implies that $x_{n_i} \to \bar{z}$ and $x_{n_i} \to z$. Therefore $z = \bar{z} \in T(z)$. □

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Received October 18, 1978.

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