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FINITE GROUPS HAVING AN INVOLUTION CENTRALIZER WITH A 2-COMPONENT OF TYPE PSL (3, 3)

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A finite group L is said to be quasisimple if $L=L'$ and $L/Z(L)$ is simple and is said to be 2-quasisimple if $L=L'$ and $L/O(L)$ is quasisimple. Let G denote a finite group. Then $E(G)$ is the subgroup of G generated by all subnormal quasisimple subgroups of G and $F^*(G)=E(G)F(G)$ where $F(G)$ is the Fitting subgroup of G. Also a subnormal quasisimple subgroup of G is called a component of G and a subnormal 2-quasisimple subgroup of G is called a 2-component of G.

We can now state the main result of this paper:

THEOREM A. *Let G be a finite group with $F^*(G)$ simple. Assume that G contains an involution t such that $H=C_G(t)$ possesses a 2-component L with $L/O(L) \cong \mathrm{PSL}(3, 3)$ and such that $C_H(L/O(L))$ has cyclic Sylow 2-subgroups. Then $|F^*(G)|_2 \leqq 2^{10}$.*

In order to state an important consequence of Theorem A, we require two more definitions. A subgroup K of a finite group G is said to be tightly embedded (in G) if $|K|$ is even and $|K \cap K^g|$ is odd for every $g \in G - N_G(K)$. A quasisimple subgroup L of a finite group G is said to be standard (in G) if $[L, L^g] \neq 1$ for all $g \in G$, $C_G(L)$ is tightly embedded in G and $N_G(L) = N_G(C_G(L))$.

THEOREM B. *Let G be a finite group with $O(G) = 1$ and containing a standard subgroup L with $L \cong \mathrm{PSL}(3, 3)$. Then either $L \trianglelefteq G$ or $L \neq \langle L^G \rangle = F^*(G)$ and one of the following five conditions hold:*

(a) $F^*(G) \cong \mathrm{PSL}(3, 9)$;

(b) $F^*(G) \cong \mathrm{PSL}(4, 3)$;

(c) $F^*(G) \cong \mathrm{PSL}(5, 3)$;

(d) $F^*(G) \cong \mathrm{PSp}(6, 3)$;

(e) $F^*(G) = H_1 \times H_2$ *with* $H_1 \cong H_2 \cong L$ *and* $C_G(L) = \langle t \rangle$ *where* t *is an involution such that* $H_1^t = H_2$ *and* $L = \langle h_1 h_1^t \mid h_1 \in H_1 \rangle$.

Note that Theorem B is a step toward the verification of Hypothesis θ^* of [13] and is therefore of import for completing a proof of the Unbalanced Group Conjecture and the $B(G)$-Conjecture and for completing an inductive characterization of all Chevalley groups over finite fields of characteristic 3 (cf. [13, §1]). Also by applying

[13, Lemma 2.9], [3, Theorem], [1, Corollary II], [8, Theorem 5.4.10 (ii)], [3, Table 1] and [6, Tables 3 and 4], it suffices, in proving Theorem B, to assume, in addition to $O(G) = 1$, that $L \neq F^*(G) = \langle L^G \rangle$, $F^*(G)$ is simple and that $C_G(L)$ has cyclic Sylow 2-subgroups. But then Theorem A and the classification of all finite simple groups whose Sylow 2-subgroups have order dividing 2^{10} (cf. [4] and [7]) yield Theorem B. Consequently Theorem B is a consequence of Theorem A.

The remainder of this paper is devoted to demonstrating that the analysis of [12] and [14] can be applied to prove Theorem A.

All groups in this paper are finite. Our notation is standard and tends to follow the notation of [8], [12] and [14]. In particular, if X is a (finite) group, then $S(X)$ denotes the solvable radical of X, $O^2(X)$ is the subgroup of X generated by all elements of X of odd order and is consequently the intersection of all normal subgroups Y of X such that X/Y is a 2-group and $\mathscr{E}(X)$ denotes the set of elementary abelian 2-subgroups of X. Also, if n is a positive integer, then $\mathscr{E}_n(X)$ denotes the set of elementary abelian 2-subgroups of order n of X. Finally $m_2(X)$ denotes the maximal rank of the elements of $\mathscr{E}(X)$, $r_2(X)$ denotes the minimal integer k such that every 2-subgroup of X can be generated by k elements and if $Y \subseteq X$, then $\mathscr{I}(Y)$ denotes the set of involutions contained in Y.

Clearly, if X is a group, then $m_2(X) \leqq r_2(X)$ and $r_2(X) \leqq r_2(Y) + r_2(X/Y)$ for every normal subgroup Y of X.

2. **A proof of Theorem A.** Throughout the remainder of this paper, we shall let G, t, H and L be as in the hypotheses of Theorem A and we shall assume that $|F^*(G)|_2 > 2^{10}$.

Then [9, Main Theorem], [15, Four Generator Theorem], [3, Table 1], [6, Tables 3 and 4] and [2] imply that $4 < r_2(F^*(G)) \leqq r_2(G)$ and that Sylow 2-subgroups of G and $F^*(G)$ contain normal elementary abelian subgroups of order 8.

Clearly $C_H(L/O(L))$ has a normal 2-complement by [8, Theorem 7.6.1], every 2-component K of H with $K \neq L$ lies in $C_H(L/O(L))$ and $O(H) \leqq C_H(L/O(L))$ (cf. [10, § 2]). Thus L is the unique 2-component of H, L char H, $S(H) \cap L = O(L)$ and $S(H) = C_H(L/O(L))$ by [10, Lemma 2.3].

Since $H/S(H)$ is isomorphic to a subgroup of Aut (PSL (3, 3)) with $(LS(H))/S(H)$ corresponding to $\mathscr{I}nn$(PSL (3, 3)) and since $|\text{Aut (PSL (3, 3))}/\mathscr{I}nn\text{(PSL (3, 3))}| = 2$, we have $|H/(S(H)L)| \leqq 2$ and $H^{(\infty)} = L$.

Let $S \in \text{Syl}_2(H)$ and $T = S \cap L$. Then $T \lhd S$, $T \in \text{Syl}_2(L)$, $|T| = 2^4$, T is semidihedral and $T = \langle \lambda, y \mid |y| = 8,\ |y| = 2 \text{ and } \lambda^y = \lambda^3 \rangle$ for suitable elements λ, y of T. Also $\Phi(T) = T' = \langle \lambda^2 \rangle \cong Z_4$ and $\Omega_1(T') =$

$Z(T) = \langle z \rangle$ for an involution z of T. Also $D = \langle \lambda^2, y \rangle \cong D_8$, $Q = \langle \lambda^2, \lambda y \rangle \cong Q_8$ and $\langle \lambda \rangle \cong Z_8$ are the three distinct maximal subgroups of T. Let $P = S \cap S(H)$. Then $P \trianglelefteq S$, P is cyclic, $P \cap T = 1$ and $\Omega_1(P) = \langle t \rangle$. Also $\mathscr{I}(L) = z^L$, $C_{L/O(L)}(z) \cong \mathrm{GL}(2, 3)$ and $S(H) = O(H)P$.

Since $r_2(S) \leqq 1 + r_2(S/P) \leqq 2 + r_2(T) = 4$, we have $S \notin \mathrm{Syl}_2(G)$.

LEMMA 2.1. *The following four conditions hold:*

(a) $|H/(S(H)L)| = 2$ *and* $H/S(H) \cong \mathrm{Aut}\,(\mathrm{PSL}\,(3, 3))$;

(b) *there is an involution* $u \in S - (P \times T)$ *such that* $D = C_T(u) \in \mathrm{Syl}_2(C_L(u))$, $L\langle u \rangle/O(L) \cong \mathrm{Aut}\,(\mathrm{PSL}(3, 3))$, $\mathscr{I}(uL) = u^L$, $C_{L/O(L)}(u) = (O(L)C_L(u))/O(L)$, $O(C_L(u)) = O(L) \cap C_L(u)$, $C_L(u)/O(C_L(u)) \cong \mathrm{PGL}\,(2, 3)$, $O^2(C_G(\langle t, u \rangle))/O(C_G(\langle t, u \rangle)) \cong \mathrm{PSL}\,(2, 3)$, $S = (P \times T)\langle u \rangle$, $\lambda^u = \lambda z$ *and* $C_{T\langle u \rangle}(\langle z, y, u \rangle) = \langle z, y, u \rangle$;

(c) $Z(S) = \langle t, z \rangle$, $P\langle u \rangle$ *is dihedral or semidihedral and* $S \in \mathrm{Syl}_2(C_G(t, z))$; *and*

(d) $Q = \langle \lambda^2, \lambda y \rangle \in \mathrm{Syl}_2(O^2(C_G(\langle t, z \rangle)))$, $C_{O(H)}(z) = O(C_G(\langle t, z \rangle)) = O(O^2(C_G(\langle t, z \rangle)))$ *and* $O^2(C_G(\langle t, z \rangle))/O(C_G(\langle t, z \rangle)) \cong \mathrm{SL}\,(2, 3)$.

Proof. Assume that $H = S(H)L$. Then $S = P \times T$ and $Z(S) = P \times \langle z \rangle$. Since $S \notin \mathrm{Syl}_2(G)$, we have $P = \langle t \rangle$. Then $\langle t, y, z \rangle \in \mathrm{Syl}_2(C_G\langle t, y, z \rangle)$ and [11, Theorem 2] implies that $r_2(G) \leqq 4$. This contradiction implies that (a) holds. For the proofs of (b) and (c) of this lemma, it clearly suffices to assume that $O(H) = 1$. Then $P = O_2(H) = C_H(L)$, $H/P \cong \mathrm{Aut}\,(\mathrm{PSL}\,(3, 3))$ and there is an element $v \in S - (P \times T)$ such that $v^2 \in P$, $C_T(v) = D$ and $C_L(v) \cong \sum 4$ by [6, Table 4]. Thus $S = (P \times T)\langle v \rangle$. Suppose that $\Omega_1(S) \leqq P \times T$. Then $\Omega_1(S) = \langle t \rangle \times D$ char S, $C_S(\Omega_1(S)) = (P \times \langle z \rangle)\langle v \rangle$ char S and $\langle t \rangle$ char S. Since this is impossible, there is an involution $w \in S - (P \times T)$. Then $L\langle w \rangle \cong \mathrm{Aut}\,(\mathrm{PSL}(3, 3))$ since $(T\langle w \rangle) \cap P = 1$ and $T\langle w \rangle \in \mathrm{Syl}_2(L\langle w \rangle)$. Then, as is well known $\mathscr{I}(wL) = w^L$ and there is an involution $u \in Tw$ such that $C_T(u) = D \in \mathrm{Syl}_2(C_L(u))$, $C_L(u) \cong \sum 4$, $S = (P \times T)\langle u \rangle$ and $C_{T\langle u \rangle}(\langle z, y, u \rangle) = \langle z, y, u \rangle$. Also $u \in N_G(\langle \lambda \rangle)$ and $C_{\langle \lambda \rangle}(u) = \langle \lambda^2 \rangle$. Thus $\lambda^u = \lambda z$ and (b) holds. Hence $Z(T\langle u \rangle) = \langle z \rangle$, $\langle t, z \rangle \leqq Z(S) = C_P(u) \times \langle z \rangle$ and (c) holds since $\langle t \rangle$ is not characteristic in S. For (d) observe that $C_G(\langle t, z \rangle) = C_H(z)$ and set $\bar{H} = H/O(H)$. Then $C_{\bar{H}}(\bar{z}) = \overline{C_H(z)}$ and $\bar{z} \in O^{2'}(\bar{H}) = \bar{L} \cong \mathrm{PSL}\,(3, 3)$. But $O^2(C_{\bar{H}}(\bar{z})) = O^2(C_{\bar{L}}(\bar{z})) \cong \mathrm{SL}\,(2, 3)$, $\bar{Q} \in \mathrm{Syl}_2(O^2(C_{\bar{H}}(\bar{z})))$ and $O^2(C_{\bar{H}}(\bar{z})) = O^2(\overline{C_H(z)}) = \overline{O^2(C_H(z))} \cong \mathrm{SL}\,(2, 3)$. Hence $O(H)Q \leqq O(H)O^2(C_H(z))$,

$$Q \leqq C_{O(H)}(z)O^2(C_H(z)) = O^2(C_H(z))\ ,$$

(d) holds and we are done.

LEMMA 2.2. $P = \langle t \rangle$, *t is not a square in* G, $S = \langle t \rangle \times (T\langle u \rangle)$,

$|S| = 2^6$, $S' = \langle \lambda^2 \rangle$, $\langle z \rangle \trianglelefteq N_G(S)$ *and* $t \not\sim z$ *in* G.

Proof. Assume that $P \neq \langle t \rangle$ and let $w \in \mathscr{I}(S - Z(S))$. Suppose that $w \in P \times T$. Then w is conjugate in $P \times T$ to an element of $y\langle t \rangle$. Since $C_S(y) = C_S(yt) = (P \times \langle z, y \rangle)\langle u \rangle$, we have $\Omega_1(C_S(w)') = \langle t \rangle$. Suppose that $w \notin P \times T$. Then $C_P(w) = \langle t \rangle$, $C_S(w) = \langle t \rangle \times C_T(w) \times \langle w \rangle$ and $\Omega_1(C_S(w)') \leqq \langle z \rangle$. Since $Z(S) = \langle t, z \rangle$, we have $Q \trianglelefteq N_G(S)$ by Lemma 2.1 (d), $\langle z \rangle \trianglelefteq N_G(S)$ and $t^{N_G(S)} = t\langle z \rangle$. However $\langle z \rangle \trianglelefteq N_G(S)$ implies $\langle t \rangle \trianglelefteq N_G(S)$ and we have a contradiction. Thus $P = \langle t \rangle$ and the lemma is clear.

Since $\mathscr{I}(uL) = u^L$, we immediately conclude:

COROLLARY 2.3. $\{t, z, tz, u, tu\}$ *is a complete set of representatives for the H-conjugacy classes of involutions in H. Also* $u\mathscr{I}(D) \subseteq u^H$.

Note that $T\langle u \rangle = \langle \lambda, yu, u \mid |yu| = |u| = 2, [yu, u] = 1, |\lambda| = 2^3, \lambda^{yu} = \lambda^{-1}$ and $\lambda^u = \lambda z$ where $z = \lambda^4 \rangle$ and hence [12, Lemma 2.1] lists various facts about $T\langle u \rangle$.

Let $x = \lambda^2 y$. Then $\mathscr{I}(T) = \mathscr{I}(D) = \{z\} \cup y\langle z \rangle \cup x\langle z \rangle$ and $y\langle z \rangle \cup x\langle z \rangle = y^T$. Also $C_S(y) = \langle t, u \rangle \times \langle z, y \rangle$, $C_S(x) = \langle t, u \rangle \times \langle z, x \rangle$, $m_2(\langle t \rangle \times T) = 3$ and $\mathscr{E}_8(\langle t \rangle \times T) = \{\langle t, z, y \rangle, \langle t, z, x \rangle\}$. Hence $m_2(S) = 4$ and $\mathscr{E}_{16}(S) = \{\langle t, u, z, y \rangle, \langle t, u, z, x \rangle\}$. Note also that $u^S = u^T = u\langle z \rangle$ and $\exp(S) = 8$.

Set $A = \langle t, u, z, y \rangle$ and $B = \langle t, u, z, x \rangle$. Then $\mathscr{E}_{16}(S) = \{A, B\}$, $A \sim B$ via T, $\langle A, B \rangle = \langle t, u \rangle \times D$ char S, $N_S(A) = N_S(B) = \langle t, u \rangle \times D$, $C_G(A) = O(C_G(A)) \times A$, $C_G(B) = O(C_G(B)) \times B$ and $N_G(S) = S(N_G(S) \cap N_G(A) \cap N_G(B))$.

Let $X = \langle t, u, z \rangle$. Clearly $C_S(X) = \langle t, u \rangle \times D$.

LEMMA 2.4. *X is the unique element Y of $\mathscr{E}(S)$ such that $Y \trianglelefteq S$ and $|Y| > 4$.*

Proof. Let $Y \in \mathscr{E}(S)$ satisfy $Y \trianglelefteq S$ and $|Y| > 4$. Then we may assume that $Z(S) = \langle t, z \rangle \leqq Y$ and $|Y| = 2^3$. Then $E_4 \cong Y \cap (T\langle u \rangle) = \langle z, \tau \rangle$ where $\tau \in \mathscr{I}(T\langle u \rangle)$ and $[\langle \lambda \rangle, \tau] \leqq \langle z \rangle$. This forces $Y \cap (T\langle u \rangle) = \langle z, u \rangle$ and we are done.

Set $M = N_G(A)$ and $\bar{M} = M/O(M)$. Clearly $C_G(A) = O(M) \times A$ and, interchanging u and uz if necessary, there is a 3-element $\rho \in C_H(u) \cap N_L(A)$ such that x inverts ρ, $C_A(\rho) = \langle t, u \rangle$, $[A, \rho] = \langle z, y \rangle$ and $\rho^3 \in O(M)$. Also $C_{\bar{M}}(\bar{t}) = \overline{C_M(t)} = \bar{A}\langle \bar{\rho}, \bar{x} \rangle = \langle \bar{t}, \bar{u} \rangle \times \langle \bar{y}, \bar{z}, \bar{\rho}, \bar{x} \rangle$ with $\langle \bar{y}, \bar{z}, \bar{\rho}, \bar{x} \rangle \cong \sum 4$, $C_{\bar{M}}(\bar{A}) = \bar{A}$ and $\bar{M}/\bar{A} \hookrightarrow \mathrm{Aut}(A) \cong GL(4, 2) \cong A_8$. Moreover, it is clear that $O^2(C_G(\langle t, u \rangle)) = O(C_G(\langle t, u \rangle))\langle y, z, \rho \rangle$, $\langle y, z \rangle \in Syl_2(O^2(C_G(\langle t, u \rangle)))$ and $O^2(C_G(\langle t, u \rangle))/O(C_G(\langle t, u \rangle)) \cong PSL(2, 3)$.

LEMMA 2.5. *$M = N_G(A)$ controls the G-fusion of elements in $t^G \cap A$.*

Proof. Assume that $t^g \in A$ for $g \in G$. Let $A < S_1 \in \mathrm{Syl}_2(C_G(t^g))$. Since $S^g \in \mathrm{Syl}_2(C_G(t^g))$, we may assume that $S^g = S_1$. If $A^g = A$, then $g \in M$. Suppose that $A^g \neq A$. Then $\mathscr{E}_{16}(S_1) = \{A, A^g\}$ and there is an element $h \in S_1$ such that $A^{gh} = A$. Then $gh \in M$, $t^g = t^{gh}$ and the lemma holds.

Let $S \leqq \mathscr{S} \in \mathrm{Syl}_2(G)$. Then $S \neq \mathscr{S}$, $|\mathscr{S}| > 2^{10}$ and $S < N_{\mathscr{S}}(S)$. Since $Z(S) \trianglelefteq N_G(S)$ and $\langle z \rangle \trianglelefteq N_G(S)$, we have $|N_{\mathscr{S}}(S)/S| = 2$, $t^{N_{\mathscr{S}}(S)} = t\langle z \rangle$ and $Z(N_{\mathscr{S}}(S)) = \langle z \rangle = Z(\mathscr{S})$.

Clearly $O(C_G(S)) = O(N_G(S)) \times \langle t, z \rangle$ and if π is an element of odd order of $N_G(S)$, then $\pi \in C_G(\langle t, z \rangle)$, $\pi \in C_G(X)$, $\pi \in C_G(\langle t, u \rangle \times D)$ and hence $\pi \in O(N_G(S))$. Thus $N_G(S) = O(N_G(S))N_{\mathscr{S}}(S)$.

As in [12, § 4], we have $SCN_5(\mathscr{S}) = \phi$ and there is an element $E \in \mathscr{E}_8(\mathscr{S})$ such that $E \trianglelefteq \mathscr{S}$. Clearly $z \in E$, $|C_E(t)| \geqq 4$ and $z \in C_E(t) \trianglelefteq S = C_{\mathscr{S}}(t)$. Suppose that $\tau \in t^G \cap E$. Then $|\mathscr{S}| = |\tau^{\mathscr{S}}|\,|C_{\mathscr{S}}(\tau)| \leqq 2^2 \cdot |S| = 2^8$. Thus $t^G \cap E = \phi$, $t \notin C_E(t)$, $|C_E(t)| = 4$, $\langle t, C_E(t) \rangle = X = \langle t, y, z \rangle$, $[S, E] \leqq E \cap S = C_E(t)$, $N_{\mathscr{S}}(S) = SE$ and $t^E = t\langle z \rangle$. Interchanging u and tu if necessary, it follows that we may assume that $C_E(t) = \langle u, z \rangle$.

Set $F = \langle y, z \rangle$. Then $A = F \cup tF \cup uF \cup tuF$, $tF \subseteq t^G \cap A$, $t^G \cap (F \cup uF) = \phi$ and $tF \subseteq t^G \cap A \subseteq tF \cup tuF$. Consequently:

COROLLARY 2.6. *Either $t^M = t^G \cap A = tF$ and $|\bar{M}/\bar{A}| = 24$ or $t^M = t^G \cap A = tF \cup tuF$ and $|\bar{M}/\bar{A}| = 48$.*

Now the analyses of [12, § 5-11], with the obvious slight changes, shows that $|O^2(G)|_2 \leqq 2^{10}$. Since $F^*(G) \leqq O^2(G)$, our proof of Theorem A is complete.

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