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INCREASING SEQUENCES OF BETTI NUMBERS

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We study the sequence of Betti numbers $\{\beta_i(M)\}_{i\geq 1}$ of an arbitrary finitely generated nonfree module M over a commutative noetherian local ring R and show that for a certain class of rings this sequence is always nondecreasing, while for a certain subclass of rings, the subsequence $\{\beta_i(M)\}_{i\geq 2}$ is strictly increasing.

In [3], a class of commutative noetherian local rings $(R, \mathfrak{m})$ called BNSI rings was introduced. These rings have the property that for every finitely generated nonfree module M, the sequence of Betti numbers $\{\beta_i(M)\}_{i\geq 1}$ is strictly increasing. Recall that $\beta_i(M)$ is the dimension of the $R/\mathfrak{m}$-vector space $\mathrm{Tor}_i^R(M, R/\mathfrak{m})$; equivalently, it is the rank of the free module F_i where

$$\cdots \longrightarrow F_i \longrightarrow F_{i-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

is a minimal R-free resolution of M. A class of BNSI rings was given in [3, Theorem 3.2A]: Let $(S, \mathfrak{n})$ be a noetherian local ring and let J be an ideal which is not contained in any prime ideal of grade 1. If S is a domain, then $S/\mathfrak{n}J$ is a BNSI ring.

In this note, using a result of G. Levin [2] we prove:

THEOREM 1.1. *Let $(S, \mathfrak{n})$ be a noetherian local ring of Krull dimension $d \geqq 2$. Then for n sufficiently large, the local ring $(R, \mathfrak{m}) = (S/\mathfrak{n}^n, \mathfrak{n}/\mathfrak{n}^n)$ has the property that for all finitely generated nonfree R-modules M, the sequence $\{\beta_i(M)\}_{i\geq 2}$ is strictly increasing. In fact, for all $i \geqq 2$, $\beta_{i+1}(M) - \beta_i(M) \geqq d - 1$.*

Thus R is nearly a BNSI ring, except that our proof gives no estimate for $\beta_2(M) - \beta_1(M)$. Another drawback is that we can not estimate how large n must be, since it comes, indirectly, from the Artin-Rees Lemma. To fill these gaps (at least partially) we offer the weaker, but more general:

COROLLARY 2.2. *Let $(S, \mathfrak{n})$ be a noetherian local ring of Krull dimension $\geqq 1$, and let $R = S/\mathfrak{n}^n$, with $n \geqq 1$. Then for all finitely generated R-modules M, the sequence $\{\beta_i(M)\}_{i\geq 1}$ is nondecreasing.*

It should be pointed out that if S is assumed to be a domain and grade $\mathfrak{n} \geqq 2$, then by the theorem from [3] cited above, $S/\mathfrak{n}^n$ is a BNSI ring for all $n \geqq 2$.

1. We begin with:

Proof of Theorem 1.1. Let $0 \to K \to R^{n_0} \to M \to 0$ be exact, with $K \subset \mathfrak{m}R^{n_0}$, and let

$$\cdots \longrightarrow R^{n_i} \longrightarrow \cdots \longrightarrow R^{n_1} \longrightarrow K \longrightarrow 0$$

be a minimal R-free resolution of K. Then $n_{i+1} = \beta_i(K) = \beta_{i+1}(M)$, and since $K \subset \mathfrak{m}R^{n_0}$, $\mathrm{ann}(\mathfrak{m}) \cdot K = 0$. Similarly, all the higher syzygies of M are annihilated by $\mathrm{ann}(\mathfrak{m})$. Thus it suffices to prove that for any finitely generated R-module N which is annihilated by $\mathrm{ann}(\mathfrak{m})$, $\beta_2(N) - \beta_1(N) \geqq d - 1$.

By [2, Formula (8), p. 9], for n sufficiently large we have

$$(*) \qquad P_R^N(t) = P_S^N(t)/1 - t(P_S^R(t) - 1)$$

where for any noetherian local ring Q and finitely generated Q-module X, $P_Q^X(t)$ is the Poincaré series $\sum_{i=0}^{\infty} \beta_i(X)t^i$. Now $P_S^R(t) = 1 + b_2 t + \cdots$. Since

$$S^{b_2} \longrightarrow S \longrightarrow R \longrightarrow 0$$

is part of a minimal S-resolution of R, $b_2 =$ the minimal number of generators of $\mathfrak{n}^n$. But $\mathfrak{n}^n$ is $\mathfrak{n}$-primary, and so by Krull's Generalized Ideal Theorem [1, Theorem 152], $b_2 \geqq$ height $\mathfrak{n} = d$. Now

$$1 - t(P_S^R(t) - 1) = 1 - b_2 t^2 - \cdots$$

and so if $(1 - b_2 t^2 - \cdots)^{-1} = \sum_{i=0} c_i t^i$, then $c_0 = 1$, $c_1 = 0$, and $c_2 = c_0 b_2 = b_2$.

Now let $P_S^N(t) = \sum_{j=0}^{\infty} p_j t^j$. Thus

$$S^{p_2} \longrightarrow S^{p_1} \longrightarrow S^{p_0} \longrightarrow N \longrightarrow 0$$

is part of a minimal S-free resolution of N. We claim that $p_2 + p_0 \geqq p_1$. To see this, localize at a minimal prime of S to obtain an artin ring T. Then the sequence

$$T^{p_2} \xrightarrow{f} T^{p_1} \xrightarrow{g} T^{p_0}$$

is exact, so $l(T^{p_1}) = l(\mathrm{im}\, f) + l(\mathrm{im}\, g) \leqq l(T^{p_2}) + l(T^{p_0})$, where $l(X)$ denotes the length of X. Therefore $p_1 \leqq p_2 + p_0$. Now from (*) we have

$$P_R^N(t) = \left(\sum_{i=1}^{\infty} c_i t^i\right)\left(\sum_{j=0}^{\infty} p_j t^j\right) = \sum_{k=0}^{\infty} \beta_k t^k .$$

Thus $\beta_1 = c_0 p_1 + c_1 p_0 = p_1$, and $\beta_2 = c_0 p_2 + c_1 p_1 + c_2 p_0 = p_2 + b_2 p_0$. Since $b_2 \geqq d \geqq 2$, $\beta_2 \geqq p_2 + p_0 + (d-1)p_0 \geqq p_1 + (d-1)p_0 = \beta_1 +$

$(d-1)p_0$. So $\beta_2 - \beta_1 \geqq (d-1)p_0 \geqq d-1 \geqq 1$.

2. We now remove the restriction that n be "sufficiently large". Our starting point is [3, Theorem 3.4]: Let $(S, \mathfrak{n})$ be a noetherian local domain and let J be any nonzero ideal. Let $R = S/\mathfrak{n}J$. Then for any finitely generated R-module M, the sequence $\{\beta_i(M)\}_{i\geqq 1}$ is nondecreasing.

The proof of this result was a minor modification of the proof of [3, Theorem 3.2]. A further modification yields:

PROPOSITION 2.1. *Let $(S, \mathfrak{n})$ be a noetherian local ring and let J be a nonnilpotent ideal. Let $R = S/\mathfrak{n}J$. Then for any finitely generated R-module M, the sequence $\{\beta_i(M)\}_{i\geqq 1}$ is nondecreasing.*

Proof. Following the proof of [3, Theorem 3.4] we obtain an S-module A such that $JS^p \subset A \subset S^p$, where $p = \beta_1(M)$, and $\beta_2(M) =$ the minimal number of generators of A. Thus we must show that A can not be generated by $p-1$ elements.

Let $x \in J$ be a nonnilpotent element, and let T be the localization of S at the multiplicative set $\{x^i \mid i \geqq 0\}$. Then

$$JS^p \otimes_S T \subset A \otimes_S T \subset S^p \otimes_S T = T^p .$$

Since J meets the multiplicative set, $JS^p \otimes_S T = T^p$. Hence $A \otimes_S T = T^p$. Now the minimal number of generators of A as an S-module is at least the minimal number of generators of $A \otimes_S T$ as a T-module, and since a free module of rank p can not be generated by $p-1$ elements, we are done.

As an easy consequence we have:

COROLLARY 2.2. *Let $(S, \mathfrak{n})$ be a noetherian local ring of Krull dimension $\geqq 1$, and let $R = S/\mathfrak{n}^n$. Then for any finitely generated R-module M, the sequence $\{\beta_i(M)\}_{i\geqq 1}$ is nondecreasing.*

Proof. When $n = 1$, R is a field and all the Betti numbers in the sequence are 0. For $n \geqq 2$, let $J = \mathfrak{n}^{n-1}$. Since Krull dim $S \geqq 1$, J is not nilpotent, and the preceding proposition applies.

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