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AXIOMS FOR CLOSED LEFT IDEALS IN A C^*-ALGEBRA

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A set of axioms is formulated to describe the conditions under which a Banach algebra may be embedded as a closed left ideal in a C^*-algebra.

In this paper we attempt to characterize the class of all closed left ideals in a C^*-algebra as a class of Banach algebras equipped with a certain (nonassociative) multiplication structure. To describe such a multiplication, we formulate a set of axioms which extracts the essential properties of the binary operation

$$(x, y) \longrightarrow y^* x$$

taking place in a closed left ideal of a C^*-algebra. Following the notion of centralizers of C^*-algebras introduced by B. E. Johnson [3] and R. C. Busby [1] we are able to show that the axioms are indeed suitable for our purpose: in order that a Banach algebra L fulfills the conditions of the axiom, it is necessary and sufficient that L can be identified with a closed left ideal of some C^*-algebra. This paper is taken from parts of author's Ph. D. thesis under the supervision of C. Akemann.

AXIOM 1. Let $(L, \|\ \|)$ be a complex Banach algebra which contains a closed subalgebra B that has a C^*-algebra structure, i.e., besides the algebraic and the norm structures inherited from L, B has an involution $*$ so that B is a Banach $*$-algebra satisfying $\|x^*x\| = \|x\|^2$ for $x \in B$. Suppose that

$$[\cdot, \cdot]: L \times L \longrightarrow B$$

is a function such that for elements x, y, z in L and for each complex scalar λ the following rules hold:

(i) $[x, y] = [y, x]^*$
(ii) $[x + y, z] = [x, z] + [y, z]$
(iii) $[\lambda x, y] = \lambda[x, y]$
(iv) $[x, x]$ is a positive element of the C^*-algebra B
(v) $\|[x, x]\| = \|x\|^2$
(vi) $\|[x, y]\| \leqq \|x\| \|y\|$
(vii) $[xy, z] = [y, [z, x]]$
(viii) $[x, y] = y^*x$ for x, y in B.

We now exhibit a situation in which the conditions stated in the above axiom hold naturally.

PROPOSITION 2. *Suppose that L is a closed left ideal of a C^*-algebra A. Set $B = L \cap L^*$. Let $[\cdot, \cdot]: L \times L \to B$ be defined by $[x, y] = y^*x$. Then $L, B, [\cdot, \cdot]$ satisfy the conditions of Axiom* 1.

Proof. Clearly B is a C^*-algebra and condition (viii) is satisfied. Conditions (i) $\sim$ (v) reflect the basic properties of A. For x, y in A, $\|y^*x\| \leqq \|y\|\,\|x\|$, thus (vi). Condition (vii) is the consequence of the associative law of multiplication: for x, y, z in L, we have

$$z^*(xy) = (z^*x)y = (x^*z)^*y .$$

We remark that the binary operation $[\cdot, \cdot]$ is not required to be associative. As in the situation of Proposition 2, for arbitrary elements x, y, z in A, $[[x, y], z] = z^*(y^*x)$ is usually not the same as $(y^*z)^*x = [x, [y, z]]$.

Conditions (i) $\sim$ (v) resemble rules of the scalar product defined on vector spaces. Indeed we are able to derive consequences similar to those of the inner product.

PROPOSITION 3. *Under Axiom* 1 *the following hold*:

(1) $[x, y + z] = [x, y] + [x, z]$ *for* x, y, z *in* L.

(2) $[x, \lambda y] = \bar{\lambda}[x, y]$ *for complex scalars* λ *and elements* x, y *in* L.

(3) $\|x\| = \sup\{\|[x, y]\|: y \in L, \|y\| \leqq 1\}$ *for* $x \in L$.

(4) *The condition "$x \in L$ and $[x, y] = 0$ for all $y \in L$" implies* $x = 0$.

(5) *For a, b, x, y in L, we have*

$$[ax, by] = [[a, b]x, y] .$$

(6) *For $x \in L$, we have*

$$\|x\| = \sup\{\|[xy, z]\|: y, z \in L, \|y\| \leqq 1, \|z\| \leqq 1\} .$$

Proof. For x, y, z in L and for each complex scalar λ, we have

$$\begin{aligned} [x, y + z] &= [y + z, x]^* = [y, x]^* + [z, x]^* \\ &= [x, y] + [x, z] , \\ [x, \lambda y] &= [\lambda y, x]^* = \bar{\lambda}[y, x]^* \\ &= \bar{\lambda}[x, y] . \end{aligned}$$

Thus (1) and (2).
Now for x, y in L with $\|y\| \leqq 1$, we have

$$\|[x, y]\| \leqq \|x\| \|y\| \leqq \|x\| .$$

Thus if $x = 0$ then clearly $\|[x, y]\| = 0$. If $x \neq 0$, then

$$\left\| \left[x, \frac{x}{\|x\|} \right] \right\| = \frac{\|x\|^2}{\|x\|} = \|x\| .$$

Therefore (3).

Condition (4) follows from (3). To see condition (5), we repeat rule (vii) to obtain:

$$\begin{aligned}[ax, by] &= [x[by, a]] = [x, [y, [a, b]]] \\ &= [[a, b]x, y] .\end{aligned}$$

To see (6), notice that if $x \neq 0$, then

$$\begin{aligned}\left\| \left[x\frac{[x, x]}{\|x\|^2}, \frac{x}{\|x\|} \right] \right\| &= \left\| \frac{[x, x]}{\|x\|^2}, \frac{[x, x]}{\|x\|} \right] \right\| \\ &= \|x\|^4/\|x\|^3 = \|x\| .\end{aligned}$$

The following definition is a slight modification of a concept first introduced by B. E. Johnson and was later investigated by R. C. Busby in the case of C^*-algebras (see [3], [1]).

L is assumed to satisfy Axiom 1 from now on.

DEFINITION 4. A bracket centralizer on L is a pair (T', T'') of functions from L to L such that $[T'x, y] = [x, T''y]$ for x, y in L. We denote the set of all bracket centralizers of L by $M(L)$.

PROPOSITION 5. *Let $(T', T'') \in M(L)$. Then*

(1) $(T'', T') \in M(L)$.

(2) *T' and T'' are continuous linear maps from L to L.*

(3) *$T'(xy) = T'(x)y$, $T''(xy) = T''(x)y$ for all x, y in L.*

Proof. (1) For all x, y in L, we have

$$[T''x, y] = [y, T''x]^*$$

and

$$[T'y, x]^* = [x, T'y] .$$

Hence $(T', T'') \in M(L)$ iff $(T'', T') \in M(L)$.

(2) Fix $z \in L$ and for each x, y in L and complex scalars α, β, we have

$$\begin{aligned}[T'(\alpha x + \beta y), z] &= [\alpha x + \beta y, T''z] = \alpha[x, T''z] + \beta[y, T''z] \\ &= \alpha[T'x, z] + \beta[T'y, z] \\ &= [\alpha T'x + \beta T'y, z] .\end{aligned}$$

Hence $\alpha T'x + \beta T'y = T'(\alpha x + \beta y)$, by Proposition 3 (4). Consequently T' is a linear map. Since T'' plays the same role as T' by (1), we conclude that T'' is also a linear map.

Suppose that $\{x_n\}$ is a sequence in L and y is an element of L such that

$$\lim_n ||x_n - x|| = 0 = \lim_n ||T'x_n - y|| \, .$$

Then for each fixed z in L, we have

$$\begin{aligned}||[T'x - y, z]|| &= ||[T'x, z] - [y, z]|| \\ &\leqq ||[T'x, z] - [T'x_n, z]|| + ||[T'x_n, z] - [y, z]|| \\ &= ||[x, T''z] - [x_n, T''z]|| + ||[T'x_n - y, z]|| \\ &\leqq ||x - x_n|| \, ||T''z|| + ||T'x_n - y|| \, ||z|| \\ &\longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty \, .\end{aligned}$$

As a result of Proposition 3 (4), we have $T'x = y$. By the closed graph theorem, T' is continuous. By symmetry, T'' is continuous.

(3) Let $x, y, z \in L$. Then

$$\begin{aligned}[T'(xy), z] &= [xy, T''z] = [y, [T''z, x]] = [y, [z, T'x]] \\ &= [T'(x)y, z] \, ,\end{aligned}$$

by condition (vii) of Axiom 1. Therefore, $T'(xy) = T'(x)y$.

The above proposition has the following interesting byproduct:

COROLLARY 6 (*see* [4; p. 296]). *Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. A function $T: H \to H$ is a bounded linear operator on H iff there exists some function $T^*: H \to H$ so that $\langle Tx, y \rangle = \langle x, T^*y \rangle$ holds for all x, y in H.*

Proof. The "only if" part follows from the fact that corresponding to each bounded linear operator T there exists an adjoint operator T^* such that $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for all x, y in H.

We now prove the "if" part. Fix an orthonormal basis $\{\xi_\alpha\}_{\alpha \in \Gamma}$ for H. Imagine H as being embedded in some fixed column of matrices of the size $\Gamma \times \Gamma$, i.e., we fix an index γ_0 in Γ and identify $\sum_{\alpha \in \Gamma} c_\alpha \xi_\alpha \in H$ with the complex matrix $(b_{\beta\gamma})$, where $b_{\beta\gamma} = 0$ for $\gamma \neq \gamma_0$ and $b_{\beta\gamma_0} = c_\beta$ for $\beta \in \Gamma$. Notice that the norm is preserved under this identification. H is stable under the matrix multiplication so induced and thus becomes a Banach algebra which contains a one-dimensional C^*-subalgebra B, where

$$\begin{aligned}B &= \{(b_{\beta\gamma}) : b_{\beta\gamma} = 0 \text{ if } \beta \neq \gamma_0 \text{ or } \gamma \neq \gamma_0\} \\ &= \text{the scalar multiples of the matrix } e = (e_{\beta\gamma}) \, , \\ &\quad\ \text{where } e_{\beta\gamma} = 0 \text{ if } \beta \neq \gamma_0 \text{ or } \gamma \neq \gamma_0, \ e_{\gamma_0\gamma_0} = 1 \, .\end{aligned}$$

Then the binary operation

$$[\cdot, \cdot]: H \times H \longrightarrow B$$

defined by

$$[x, y] = y^*x = \langle x, y\rangle e$$

(y^* is the conjugate transpose of y) satisfies all the conditions listed in Axiom 1. (Notice that condition (vii) is a result of the associative law of matrix multiplication.) Thus if $T: H \to H$ is a function with the property that there is some function $T^*: H \to H$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all x, y in H, then

$$\begin{aligned}[Tx, y] &= \langle Tx, y\rangle e = \langle x, T^*y\rangle e \\ &= [x, T^*y] .\end{aligned}$$

It follows from Proposition 5 (2) that $T: H \to H$ is a bounded linear operator.

The next corollary is a slight generalization of the previous one.

COROLLARY 7. *Let H be a Hilbert space with inner product $\langle \cdot, \cdot\rangle$. Suppose that $T, T^*: H \to H$ is a pair of functions such that*

$$\{\langle Tx, y\rangle - \langle x, T^*y\rangle : x, y \in H\}$$

is a bounded subset of complex numbers. Then T and T^* are bounded linear operators on H. Furthermore, T^* is indeed the adjoint of T.

Proof. Assume that M is a positive real number such that

$$|\langle Tx, y\rangle - \langle x, T^*y\rangle| \leqq M$$

for all x, y in H. Replacing x by λx ($\lambda \in \boldsymbol{C}$) we have

$$|\langle T(\lambda x), y\rangle - \langle \lambda x, T^*y\rangle| \leqq M .$$

Thus

$$\left|\frac{1}{\lambda}\langle T(\lambda x), y\rangle - \langle x, T^*y\rangle\right| \leqq \frac{M}{\lambda}$$

for $\lambda > 0$ and x, y in H. Let $\varepsilon > 0$ be given. Fix $\beta > 0$. Choose $\lambda > 0$ so that

$$\beta M/\lambda < \varepsilon/2 \quad \text{and} \quad M/\lambda < \varepsilon/2 .$$

Hence

$$\left|\frac{1}{\lambda}\langle T(\lambda x), \beta y\rangle - \langle x, T^*(\beta y)\rangle\right| \leqq \frac{M}{\lambda} .$$

In view of the equality

$$\frac{1}{\lambda}\langle T(\lambda x),\ \beta y\rangle = \frac{\beta}{\lambda}\langle T(\lambda x),\ y\rangle$$

and the inequality

$$\left|\frac{\beta}{\lambda}\langle T(\lambda x),\ y\rangle - \langle x,\ \beta T^* y\rangle\right| \leqq \frac{\beta M}{\lambda}\ ,$$

we have

$$|\langle x,\ T^*(\beta y) - \beta T^* y\rangle| \leqq \frac{M}{\lambda} + \frac{\beta M}{\lambda} < \varepsilon/2 + \varepsilon/2 = \varepsilon\ .$$

This shows $T^*(\beta y) = \beta T^* y$ for all y in H and for all $\beta > 0$. Now for all x, y in H, we have

$$|\langle Tx,\ \beta y\rangle - \langle x,\ T^*(\beta y)\rangle| \leqq M$$

and so

$$|\langle Tx,\ y\rangle - \langle x,\ T^* y\rangle| \leqq M/\beta$$

for all x, y in H and all $\beta > 0$. Hence

$$\langle Tx,\ y\rangle = \langle x,\ T^* y\rangle$$

for all x, y in H. The desired conclusion follows frow Corollary 6.

PROPOSITION 8. *Let* (T', T'') *be in* $M(L)$. *If we regard* T' *and* T'' *as bounded linear operators on the Banach space* L, *then*

$$\|T'\| = \|T''\|\ .$$

Proof. Let $x \in L$, $\|x\| < 1$. Considering Proposition 3 (3), we have

$$\begin{aligned}\|T'x\| &= \sup_{\substack{y \in L\\ \|y\| \leqq 1}} \|[T'x,\ y]\| = \sup_{\substack{y \in L\\ \|y\| \leqq 1}} \|[x,\ T''y]\| \\ &\leqq \sup_{\substack{y \in L\\ \|y\| \leqq 1}} \|T''y\| = \|T''\|\ .\end{aligned}$$

Hence $\|T'\| \leqq \|T''\|$. By symmetry, we also have $\|T''\| \leqq \|T'\|$. Thus $\|T'\| = \|T''\|$.

PROPOSITION 9. *If* $(T', T''), (S', S'') \in M(L)$, *then* $(T'S', S''T'') \in M(L)$.

Proof. For $x, y \in L$,

$$[T'S'x,\ y] = [S'x,\ T''y] = [x,\ S''T''y]\ .$$

THEOREM 10. *$M(L)$ equipped with the norm and algebraic operations defined as follows becomes a C^*-algebra with identity. For (T', T''), $(S', S'') \in M(L)$ and complex scalar α, set*

(1) $(T', T'') + (S', S'') = (T' + S', T'' + S'')$

(2) $\alpha(T', T'') = (\alpha T', \bar{\alpha} T'')$

(3) $(T', T'')(S', S'') = (T'S', S''T'')$

(4) $(T', T'')^* = (T'', T')$

(5) $\|(T', T'')\| =$ *the operator norm of T' on L(= the operator norm of T'' on L, by Proposition* 8).

Proof. It is clear that $M(L)$ is an involutive normed algebra with respect to the above operations. We now show $M(L)$ is complete under the norm given by (5). Let $\{(T'_n, T''_n)\}_{n\geqq 1}$ be a Cauchy sequence in $M(L)$. Then $\{T'_n\}_{n\geqq 1}$ and $\{T''_n\}_{n\geqq 1}$ are Cauchy sequences in the Banach space $B(L)$ of all bounded linear transformations on L. Thus there are elements T'_∞ and T''_∞ in $B(L)$ such that T'_∞ and T''_∞ are the uniform limits of $\{T'_n\}_{n\geqq 1}$ and $\{T''_n\}_{n\geqq 1}$ respectively. If $x, y \in L$, then

$$\begin{aligned}[T'_\infty y, x] &= \lim_n [T'_n y, x] = \lim_n [y, T''_n x] \\ &= [y, T''_\infty x] .\end{aligned}$$

Hence $(T'_\infty, T''_\infty) \in M(L)$ and (T'_n, T''_n) is convergent to (T'_∞, T''_∞).

It remains to check the C^*-norm condition:

$$\begin{aligned}\|(T', T'')^*(T', T'')\| &= \|(T''T', T'T'')\| = \|T''T'\| \\ &= \sup\{\|[T''T'x, y]\|: \|x\| \leqq 1, \|y\| \leqq 1, x, y \in L\} \\ &= \sup\{\|[T'x, T'y]\|: \|x\| \leqq 1, \|y\| \leqq 1, x, y \in L\} \\ &\geqq \sup\{\|[T'x, T'x]\|: \|x\| \leqq 1, x \in L\} \\ &= \|T'\|^2 = \|T'\| \, \|T''\| \geqq \|T''T'\| \\ &= \|(T''T', T'T'')\| = \|(T', T'')^*(T', T'')\| .\end{aligned}$$

Therefore

$$\|(T', T'')^*(T', T'')\| = \|(T', T'')\|^2 .$$

We are now ready to define an embedding of L satisfying Axiom 1 onto a closed left ideal of the C^*-algebra $M(L)$. For each a in L, let $\pi'(a)$ (respectively $\pi''(a)$) be the function from L to L defined by $\pi'(a)x = ax$ (respectively $\pi''(a)x = [x, a]$) for x in L. Condition (vii) of Axiom 1 guarantees that the pair $(\pi'(a), \pi''(a))$ belongs to $M(L)$ for each a in L.

THEOREM 11. *There is a closed left ideal J of the C^*-algebra*

$M(L)$ and an isometric linear map π from L onto J with the following properties:

(1) $\pi(B) = J \cap J^*$.

(2) $\pi|_B$ *is a *-isomorphism of C^*-algebras.*

(3) $\pi(xy) = \pi(x)\pi(y)$ *for x, y in L.*

(4) $\pi([x, y]) = \pi(y)^*\pi(x)$ *for x, y in L.*

Proof. As noticed above, the pair $\pi(a) = (\pi'(a), \pi''(a)) \in M(L)$ for $a \in L$. We shall show that

$$J = \{\pi(a) : a \in L\}$$

is a closed left ideal of $M(L)$ and $\pi: L \to J$ indeed fulfills conditions (1) $\sim$ (4).

First we observe that, when regarded as a map from L into $M(L)$, π is linear. Thus J is a linear subspace of M. As a result of Proposition 3(6), we have

$$||\pi(a)|| = \sup_{\substack{||x|| \leq 1, x \in L \\ ||y|| \leq 1, y \in L}} ||[ax, y]|| = ||a|| .$$

Therefore $\pi(L) = J$ is a complete linear subspace of $M(L)$ and so is uniformly closed. Suppose that $(T', T'') \in M(L)$. Then for a in L, $b = T'a$ is an element of L. Thus for x in L we have

$$\begin{aligned} [T' \circ \pi'(a)](x) &= T'(ax) = T'(a)x = bx = \pi'(b)(x) ; \\ [\pi''(a) \circ T''](x) &= \pi''(a)(T''x) = [T''x, a] = [x, T'a] \\ &= [x, b] = \pi''(b)(x) . \end{aligned}$$

Consequently, $(T', T'')\pi(a) = \pi(b)$. This shows that $\pi(L) = J$ is a surjective linear isometry.

For x, y, v, w in L, by Proposition 3 (5), we have

$$[[x, y]v, w] = [xv, yw] = [[xv, y], w] .$$

Therefore $\pi([x, y]) = \pi(y)^*\pi(x)$, so (4) is proved. In particular, $\pi([x, x])$ is a positive element of J. Since every positive element of B is of the form $[x, x]$ for some x in L (by condition (vii)) and since every element of B is a linear combination of positive ones, we conclude that $\pi(B) \subset J \cap J^*$. On the other hand, every positive element of $J \cap J^*$ is of the form $\pi(x)^*\pi(x) = \pi(x^*x)$ for some x in L, we see that $\pi(B) = J \cap J^*$. Thus (1) holds.

Condition (3) is clear. Condition (2) follows from conditions (1) and (3) and the fact that π is isometric. This completes the proof.

There is an alternative method of embedding a Banach algebra satisfying Axioms 1 into a C^*-subalgebra of $B(H)$, the C^*-algebra of all bounded linear operators on some Hilbert space H [2; p. 41]. Based on the characterization of Jordan and von Neumann, it is shown in [2; p. 45] that parts of Axiom 1 may be formulated differently.

We conclude with the following summary of the main result:

THEOREM 12. *A Banach algebra can be isometrically embedded as a closed left ideal of a C^*-algebra if and only if conditions of Axiom* 1 *hold.*

REFERENCES

1. R. C. Busby, *Double centralizers and extensions of C^*-algebras*, Trans. Amer. Math. Soc., **132** (1968), 79–99.

2. J. C. Chuan, *One-sided ideals in a C^*-algebra*, dissertation, University of California, Santa Barbara, 1977.

3. B. E. Johnson, *An introduction to the theory of centralisers*, Proc. London Math. Soc., (3), **14** (1964), 299–320.

4. F. Riesz and B. Sz.-Nagy, *Functional Analysis*, Frederick Ungar, New York, 1955.

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