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THE FULL C^*-ALGEBRA OF THE FREE GROUP ON TWO GENERATORS

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$C^*(F_2)$ is a primitive C^*-algebra with no nontrivial projection. $C^*(F_2)$ has a separating family of finite-dimensional representations.

1. **Introduction.** We present a "generic" C^*-algebra in illustration of several peculiar phenomena that may occur in the theory of representations.

Let $\boldsymbol{F}_2$ denote the free group on two generators. If π is the universal unitary representation of $\boldsymbol{F}_2$ on a Hilbert space $\mathscr{H}$, then the *full group C^*-algebra* $C^*(\boldsymbol{F}_2)$ is the C^*-subalgebra of $\mathscr{B}(\mathscr{H})$ generated by the set $\{\pi(g): g \in \boldsymbol{F}_2\}$ (see [4, § 13.9]). Alternatively, we can re-define $C^*(\boldsymbol{F}_2)$, in an operator-theoretical setting, as follows:

DEFINITION. Let U, V be two unitary operators on a Hilbert space $\mathscr{H}$. We say that (U, V) is a *universal pair of unitaries* iff for each pair of unitary operators (U_1, V_1) on a Hilbert space $\mathscr{H}_1$, the assignment

$$\begin{cases} U \longmapsto U_1 \\ V \longmapsto V_1 \end{cases}$$

extends to a *-homomorphism from $C^*(U, V)$ onto $C^*(U_1, V_1)$.

DEFINITION. We let $C^*(\boldsymbol{F}_2)$ denote the abstract C^*-algebra which is *-isomorphic with the C^*-algebra generated by a universal pair of unitaries.

Obviously, the universal pairs of unitaries are *unique* up to algebraic *-isomorphic equivalence. To see the existence of a universal pair of unitaries, we may simply let

$$(*) \qquad U = \bigoplus U_\nu, \quad V = \bigoplus V_\nu$$

where (U_ν, V_ν) runs through all possible pairs of unitary operators on a fixed separable Hilbert space. By some judicious selection, it suffices to let ν run through only a countable index set. [In fact, for a general separable C^*-algebra $\mathfrak{A} \subseteq \mathscr{B}(\mathscr{H})$. There is always a projection P of countable dimension, such that $A \mapsto \mathrm{PAP}$ is a *-isomorphism from $\mathfrak{A}$ onto $P\mathfrak{A}P \subseteq \mathscr{B}(P\mathscr{H})$.]

The main result of this paper is concerned with various expressions for the universal pairs of unitaries. On one hand, we can just take the expression $(*)$ above, such that each (U_ν, V_ν) is a pair of finite-dimensional unitary matrices (Theorem 7). On the other hand, there exist a universal pair of unitary operators that do not have a common nontrivial reducing subspace (Theorem 6). These two apparently opposite constructions induce two important representations of $C^*(\boldsymbol{F}_2)$.

Now, we examine $C^*(\boldsymbol{F}_2)$ in regard to its operator-algebraic structure. First and foremost, $C^*(\boldsymbol{F}_2)$ is a *primitive* C^*-algebra; i.e., $C^*(\boldsymbol{F}_2)$ has a faithful irreducible representation (Theorem 6). This yields the key information that is indispensable to the study of Prim $(\boldsymbol{F}_2)$ (cf. [7, Proposition 6.1]). Furthermore, by the universal property of $C^*(\boldsymbol{F}_2)$, all C^*-algebras generated by two unitaries (including all C^*-algebras generated by single operators) are *-homomorphic images of $C^*(\boldsymbol{F}_2)$. It may be surprising to see that such a "tremendous" C^*-algebra has no nontrivial projection (Theorem 1), and no nonnormal hyponormal element (Corollary 8); indeed, $C^*(\boldsymbol{F}_2)$ even has a faithful tracial state (Corollary 9).

In short, $C^*(\boldsymbol{F}_2)$, being faithfully irreducibly represented, serves as an example for each of the following unusual conditions:

(i) an irreducible C^*-algebra with (the most) abundant ideals.

(ii) an irreducible C^*-algebra with no nontrivial projection (cf. another example given by Philip Green [5]).

(iii) an irreducible C^*-algebra that admits a separating family of finite-dimensional representations.

Finally, we remark that all results of this paper can be extended to $C^*(\boldsymbol{F}_n)$ (for $n > 1$) and $C^*(\boldsymbol{F}_\infty)$. Readers are also referred to [2, Lemma 4.4-Theorem 4.5, pp. 1108-1109; 9, Proposition 2.7, p. 250; 10, § 3; 11, Theorem 12] for some other unusual aspects of $C^*(\boldsymbol{F}_2)$.

2. The author is indebted to Robert Powers for helpful communication leading to the following theorem. (The idea of the proof is actually originated by Joel Cohen [3].)

THEOREM 1. *$C^*(\boldsymbol{F}_2)$ has no nontrivial projection.*

Proof. By a faithful representation, we may write $C^*(\boldsymbol{F}_2) = C^*(U, V)$, where U, V are a universal pair of unitary operators on a Hilbert space $\mathscr{H}$. Let

$$\mathfrak{A} = \left\{\begin{array}{l}\text{all norm-continuous functions } \Phi\colon [0, 1] \longrightarrow \mathscr{B}(\mathscr{H}) \\ \text{such that } \Phi(0) \text{ are scalar operators}\end{array}\right\}.$$

Then $\mathfrak{A}$ is a C^*-algebra with no nontrivial projections. In fact, if $\Phi \in \mathfrak{A}$ is a projection, then $\Phi(0)$ is 0 or I and by continuity, the projections $(\Phi(t))_{t\in[0,1]}$ must be all 0 or all I. Now we claim that $C^*(\boldsymbol{F}_2)$ can be imbedded into $\mathfrak{A}$ as a C^*-subalgebra and consequently, $C^*(\boldsymbol{F}_2)$ has no nontrivial projection either.

To see the claim, we first choose, by the spectral theorem, two hermitian operators $A, B \in \mathscr{B}(\mathscr{H})$ such that $U = e^{iA}$, $V = e^{iB}$. Next, define two unitary elements $\Phi_U, \Phi_V \in \mathfrak{A}$ by

$$\Phi_U(t) = e^{itA}, \quad \Phi_V(t) = e^{itB} .$$

Then obviously, the evaluation map $\Phi \mapsto \Phi(1)$ is a *-homomorphism from $C^*(\Phi_U, \Phi_V)$ onto $C^*(\boldsymbol{F}_2)$. On the other hand, by the universal property of $C^*(\boldsymbol{F}_2)$, the assignment $U \mapsto \Phi_U$, $V \mapsto \Phi_V$ determines a *-homomorphism from $C^*(\boldsymbol{F}_2)$ onto $C^*(\Phi_U, \Phi_V)$. Hence, the two *-homomorphisms above, being inverse to each other, must be *-isomorphisms. Therefore, $C^*(\boldsymbol{F}_2)$ can be imbedded into $\mathfrak{A}$ as claimed.

COROLLARY 2. *If π is a faithful representation of $C^*(\boldsymbol{F}_2)$ on a Hilbert space $\mathscr{H}$, then $\pi(C^*(\boldsymbol{F}_2))$ contains no nonzero compact operator.*

Proof. Any C^*-algebra, containing a nonzero compact operator K, must also contain K^*K and, thus, the finite-rank spectral projections of K^*K. Since $\pi(C^*(\boldsymbol{F}_2)) \simeq C^*(\boldsymbol{F}_2)$ contains no nontrivial projection, we have that $\pi(C^*(\boldsymbol{F}_2))$ contains no nonzero compact operator, either.

We proceed to construct a universal pair of unitary operators that do not have a common nontrivial reducing subspace. The main technique below is a variant of Radjavi-Rosenthal's treatment on nonexistence of common invariant subspace [8, Theorem 7.10, p. 121; Theorem 8.30, p. 162].

LEMMA 3. *Let A, B be two infinite matrices standing for operators on a separable Hilbert space $\mathscr{H}$ endowed with a fixed orthonormal basis $\{e_n\}_{n=1}^{\infty}$. If A is a diagonal operator with all distinct diagonal entries, and if all first-column entries of B are nonzero, then A, B do not have a common nontrivial reducing subspace.*

Proof. By simple evaluation on infinite matrices, we deduce that the commutant of A consists of diagonal operators only, and, diagonal operators commuting with B must be scalar operators. Hence, the projections commuting with both A and B are trivial projections. Therefore, A, B do not have a common nontrivial reducing subspace.

In the following two lemmas, we deal with the compact perturbations of unitary operators.

LEMMA 4. *Let U be a unitary operator on a separable Hilbert space $\mathscr{H}$. Then there exists a compact operator K, and a unitary diagonal operator D with respect to an orthonormal basis $\{e_n\}_{n=1}^{\infty}$, such that $U = D + K$ and all diagonal entries of D are distinct.*

Proof. From [6], every normal operator U can be written as $D_0 + K_0$, where K_0 is a compact operator and D_0 is a diagonal operator with respect to an orthonormal basis $\{e_n\}_{n=1}^{\infty}$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be the diagonal entries of D_0. Since $U = D_0 + K_0$ is unitary, we derive that $\lim_{n\to\infty} |\alpha_n| = 1$. It is easy to choose a complex sequence $\{\beta_n\}_{n=1}^{\infty}$ such that

$$\begin{cases} |\beta_n| = 1 \text{ for all } n \\ \beta_i \neq \beta_j \text{ whenever } i \neq j \\ \lim_{n\to\infty} (\alpha_n - \beta_n) = 0 . \end{cases}$$

Denoting by D for the diagonal operator with the diagonal entries $\{\beta_n\}_{n=1}^{\infty}$, we have that $D_0 - D$ is a diagonal compact operator; thus $K = K_0 + D_0 - D$ is a compact operator and $U = D + K$ as desired.

LEMMA 5. *Let U be a unitary operator on a separable Hilbert space $\mathscr{H}$ endowed with a fixed orthonormal basis $\{e_n\}_{n=1}^{\infty}$. Then there exists a compact operator $K \in \mathscr{B}(\mathscr{H})$, such that $U - K$ has the infinite matrix expression with all first-column entries being nonzero and with $U - K$ unitary.*

Proof. Let v be a unit vector with all nonzero co-ordinates with respect to the orthonormal basis $\{e_n\}_{n=1}^{\infty}$, and let $\mathscr{S}$ denote the linear span of v and Ue_1(= the first-column vector of U). Choose any unitary operator $V \in \mathscr{B}(\mathscr{H})$ such that

$$\begin{cases} V(\mathscr{S}) = \mathscr{S} \text{ with } V(Ue_1) = v , \\ V|_{\mathscr{S}^\perp} = I|_{\mathscr{S}^\perp} . \end{cases}$$

Then $V - I$ is an operator of rank $\leqq 2$ and the first column vector of VU is v; thus

$$VU = U + (V - I)U$$

is a compact perturbation of U as desired.

THEOREM 6. *$C^*(\boldsymbol{F}_2)$ is a primitive C^*-algebra;* i.e., *$C^*(\boldsymbol{F}_2)$ has a faithful irreducible representation.*

Proof. Since $C^*(\boldsymbol{F}_2)$ is separable, we may write $C^*(\boldsymbol{F}_2) = C^*(U, V)$, where U, V are a universal pair of unitary operators on a separable Hilbert space $\mathscr{H}$. Applying Lemmas 4–5 to U, V, we have that

$$U = U_0 + \text{compact}, \quad V = V_0 + \text{compact},$$

and with respect to a suitable orthonormal basis, U_0 is a unitary diagonal operator with distinct diagonal entries, and V_0 is a unitary operator with all first-column entries nonzero. From the universal property, the assignment

$$U \longmapsto U_0, \quad V \longmapsto V_0$$

defines a representation

$$\pi\colon C^*(\boldsymbol{F}_2) \longrightarrow C^*(U_0, V_0) \subseteqq \mathscr{B}(\mathscr{H}) .$$

By Lemma 3, U_0, V_0 do not have a common nontrivial reducing subspace; thus $C^*(U_0, V_0)$ is an irreducible C^*-algebra, and π is an irreducible representation.

It remains to show that π is faithful. Letting $\mathscr{K}(\mathscr{H})$ be the ideal of compact operators and $\eta\colon \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})/\mathscr{K}(\mathscr{H})$ be the natural quotient map, we have then

$$\eta(C^*(U_0, V_0)) = C^*(\eta(U_0), \eta(V_0)) = C^*(\eta(U), \eta(V)) .$$

But from Corollary 2, η restricted to $C^*(U, V)$ is an *-isomorphism. The composition of the canonical *-homomorphisms

$$\begin{aligned} C^*(\boldsymbol{F}_2) \xrightarrow{\pi} C^*(U_0, V_0) \xrightarrow{\eta} C^*(\eta(U_0), \eta(V_0)) &= C^*(\eta(U), \eta(V)) \\ \simeq C^*(U, V) &= C^*(\boldsymbol{F}_2) \end{aligned}$$

leads to the identity map on $C^*(\boldsymbol{F}_2)$; therefore π is a *-isomorphism as desired.

For a general C^*-algebra $\mathfrak{A} \subseteqq \mathscr{B}(\mathscr{H})$ with separable $\mathscr{H}$, we can construct a "completely order injection" φ from $\mathfrak{A}$ into $\bigoplus_{n=1}^{\infty} M_n$, the direct sum of full matrix algebras, by letting

$$\varphi(A) = \bigoplus\nolimits_{n=1}^{\infty} P_n A P_n$$

where $\{P_n\}$ is a sequence of finite-rank projections approaching strongly to I. In case $\mathfrak{A} = C^*(\boldsymbol{F}_2)$, we will modify φ to get actually an "algebraic *-isomorphism".

THEOREM 7. *$C^*(\boldsymbol{F}_2)$ has a separating family of finite-dimensional representations.*

Proof. Since $C^*(\boldsymbol{F}_2)$ is separable, we may assume that $C^*(\boldsymbol{F}_2) = C^*(U, V)$ where U, V are a universal pair of unitary operators on a separable Hilbert space $\mathscr{H}$. Let $\{P_n\}_n$ be a sequence of increasing projections in $\mathscr{B}(\mathscr{H})$ approaching strongly to the identity operator I with rank $P_n = n$. Write

$$A_n = P_n U P_n, \qquad B_n = P_n V P_n,$$

$$U_n = \begin{bmatrix} A_n & (P_n - A_n A_n^*)^{1/2} \\ (P_n - A_n^* A_n)^{1/2} & -A_n^* \end{bmatrix},$$

$$V_n = \begin{bmatrix} B_n & (P_n - B_n B_n^*)^{1/2} \\ (P_n - B_n^* B_n)^{1/2} & -B_n^* \end{bmatrix}.$$

By identifying $P_n \mathscr{H} P_n$ with M_n, we may regard P_n as the identity $n \times n$ matrix, and U_n, V_n as $2n \times 2n$ unitary matrices. From the universal property of $C^*(\boldsymbol{F}_2)$, the assignment

$$U \longmapsto U_n, \quad V \longmapsto V_n$$

defines a representation $\pi_n: C^*(\boldsymbol{F}_2) \to M_{2n}$. Now, we *claim* that $\{\pi_n\}_{n=1}^\infty$ is a separating family of representations; in other words, the *-homomorphism

$$\pi: C^*(\boldsymbol{F}_2) \longrightarrow \textstyle\bigoplus_{n=1}^\infty M_{2n},$$

defined by

$$\pi(A) = \textstyle\bigoplus_{n=1}^\infty \pi_n(A),$$

is actually a *-isomorphism.

Note that in the strong topology, U_n, U_n^*, V_n, and V_n^* converge to

$$\begin{bmatrix} U & 0 \\ 0 & -U^* \end{bmatrix}, \quad \begin{bmatrix} U^* & 0 \\ 0 & -U \end{bmatrix}, \quad \begin{bmatrix} V & 0 \\ 0 & -V^* \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} V^* & 0 \\ 0 & -V \end{bmatrix}$$

respectively. Hence, if $F(\ ,\ )$ is a finite linear combination of words in two free variables, then $F(U_n, V_n)$ also converges to

$$\begin{bmatrix} F(U, V) & 0 \\ 0 & F(-U^*, -V^*) \end{bmatrix}$$

in the strong topology. Therefore, for any $\varepsilon > 0$, and given $\|F(U, V)\| = 1$, we have that

$$\|F(U_n, V_n)\| \geqq 1 - \varepsilon$$

for all sufficiently large n; thus

$$\|\pi(F(U, V))\| \geqq \|\pi_n(F(U, V))\| = \|F(U_n, V_n)\| \geqq 1 - \varepsilon .$$

Since ε is arbitrary, we conclude that π, restricted to the pre-C^*-algebra generated by U, V, is an isometry. By continuity, π is an isometry and, thus, a $*$-isomorphism as desired.

We say that an operator A is *hyponormal* iff $A^*A \geqq AA^*$.

COROLLARY 8. *Every hyponormal operator in $C^*(\boldsymbol{F}_2)$ is normal.*

Proof. From the theorem above, we may imbed $C^*(\boldsymbol{F}_2)$ into $\bigoplus_{n=1}^{\infty} M_{2n}$ as a C^*-subalgebra. Since every hyponormal matrix is normal, we have then for each $A = \bigoplus A_n \in \bigoplus M_{2n}$,

$$\begin{aligned} A \text{ is hyponormal} &\Longrightarrow A_n \text{ is hyponormal for each } n \\ &\Longrightarrow A_n \text{ is normal for each } n \\ &\Longrightarrow A \text{ is normal} \end{aligned}$$

as desired.

COROLLARY 9. *$C^*(\boldsymbol{F}_2)$ has a faithful tracial state.*

Proof. By Theorem 7, we can imbed $C^*(\boldsymbol{F}_2)$ into $\bigoplus_{n=1}^{\infty} M_{2n}$ as a C^*-subalgebra. Let τ_n be the faithful tracial state of M_{2n}. Then $\tau\colon \bigoplus M_{2n} \to \boldsymbol{C}$, defined by

$$\tau(\bigoplus A_n) = \sum (\tau_n(A_n)/2^n) ,$$

is a faithful tracial state as desired.

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