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RESOLUTIONS ON THE LINE

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It has been shown previously that under suitable conditions a bounded open set on the line may be resolved into a countable union of disjoint open intervals. Here, such a resolution is obtained for an unbounded open set; it requires the introduction of a suitable system of extended real numbers. The methods used are those of modern constructive analysis.

1. Introduction. A recurring procedure in analysis is that of measuring the distance from a point to a set. It is *not* always possible to do this constructively. Often enough, however, it is possible. Subsets G of a metric space X, for which the distance

$$\rho(x, G) \equiv \inf \{\rho(x, y): y \in G\}$$

exists for every x in X, are called *located*. Metric spaces commonly used have sufficiently many located sets to allow the constructivization of analysis to be carried out in Bishop's *Foundations of Constructive Analysis* [1] and in the work of many others. The *metric complement* of a located set G is the set

$$-G \equiv \{x \in X: \rho(x, G) > 0\} .$$

These metric complements are called *colocated*; they are the open sets with which we shall work, and may be considered to be those open sets having constructive significance.

Our main result (Theorem 6) is that every colocated set on the line is a countable union of disjoint open intervals. Classically (i.e., nonconstructively) this is true for every open set, but *not* constructively; a counterexample is given in [3, §7]. The resolution of a colocated set given here will be developed further in [4] to yield a complete characterization of located sets on the line and an explicit procedure for their construction.

The italicized word "*not*" used in the above paragraphs has a special meaning in constructive mathematics. It means that a counterexample exists in the sense of Brouwer. Such a counterexample consists of a proof showing that the statement in question implies one of several principles which are constructively unbelievable. That is, no proofs of the principles are known, and it seems very unlikely that constructive proofs will ever be found. For example, the statement "every open set on the line is a countable union of disjoint open intervals" implies the *limited principle of* *omniscience* (LPO). LPO states that there is a general procedure which applies to any given sequence of integers and determines, by a finite process, either that each term of the sequence is zero or that some (specifically presented) term is nonzero. A moment's reflection is usually sufficient to produce complete conviction that such a procedure will never be discovered. Therefore, because of the implication given in the counterexample, we do not believe that a proof of the statement in quotes will ever be found. Further discussion of Brouwerian counterexamples is found in Chapters 1 and 2 of [1].

The resolution of a *bounded* colocated set on the line into a countable union of disjoint open intervals is given in Theorem 1 of [3]. It depends on the fundamental lemma (L) of [2]. L shows that the components of a bounded colocated set are totally bounded (and thus their endpoints may be calculated). The extension of the resolution to *unbounded* (i.e., not known to be bounded) colocated sets requires two steps. First, an extension of L to unbounded sets is required. Second, a generalization of the concept of extended real number is needed for the calculation of the endpoints of the components.

The first step is carried out in §2. Theorem 2, which shows that components are located, is the extension of L required. The proof of Theorem 2 proceeds by a reduction to the bounded case; it uses L and Theorem 1. Theorem 1 characterizes located sets in terms of totally bounded sets; its proof is also based on L.

The second step is required because of an example in [3, §11]. The example shows that an unbounded colocated set need *not* have a resolution into disjoint open intervals, when open intervals have endpoints that are extended real numbers in the sense of real numbers or $\pm\infty$. The colocated set constructed in the example is based on an increasing sequence $\{a_n\}$ of zeros and ones. One of the components would be $J \equiv \cup\{(0, n+1): a_n = 0\}$. This component can *not* be expressed as an open interval (a, b), where b is the usual sort of extended real number. That is, J does *not* have a supremum in the existing system of extended real numbers. However, the corresponding set $\bar{J}$ in $(-1, 1)$ does have a supremum in $[-1, 1]$.

Thus it seems natural to redefine the extended real numbers so that R^∞ will be similar to $[-1, 1]$. Then components will have suprema in R^∞. There is no more reason for being able to decide whether an extended real number is finite or infinite than for deciding whether a point of $[-1, 1]$ lies in $(-1, 1)$ or is ± 1.

The construction of R^∞ is carried out in §4. Extended real numbers are defined in terms of $[-1, 1]$; they are characterized in terms of R by Theorem 3. Theorem 4 constructs the required suprema of convex located sets. However, *not* all located sets will have suprema in R^{∞}; those which do are characterized in [5].

Colocated sets are resolved into countable unions of disjoint open intervals in §5.

If a set S has the property that, for any x, the condition $x \in S$ is contradictory, then, as usual, S is said to be *void*. If, on the other hand, it is known that there exists some element x in S (i.e., x is explicitly constructed and $x \in S$ is proved), then we shall say that S is *fixed*. (The negative term "nonvoid" could be used for the negativistic concept "'S is void' is contradictory"; however, this concept is of little constructive interest.) In general, it can *not* be determined whether a given set is void or fixed. We require sets (in particular, open intervals) which are known to be either void or fixed; such sets, and families of such sets, will be called *fixative*.

2. Located sets.

LEMMA 1. *The closure of a located subset of a metric space X is also located. A dense subset of a located subset is also located.*

Proof. When S is either the closure of a located subset G, or is dense in G, it is easy to show that $\rho(x, S) = \rho(x, G)$ for any $x \in X$.

LEMMA 2. *Any finite union of located subsets of a metric space X is also located.*

Proof. Let G_i $(1 \leqq i \leqq n)$ be located, and put $G \equiv \bigcup_i G_i$. Let $x \in X$ and put $\rho \equiv \bigwedge_i \rho(x, G_i)$. Clearly $\rho \leqq \rho(x, y)$ for all $y \in G$. Let $\varepsilon > 0$, choose i so that $\rho(x, G_i) < \rho + \varepsilon/2$, and construct $y \in G_i$ such that $\rho(x, y) < \rho(x, G_i) + \varepsilon/2$; then $\rho(x, y) < \rho + \varepsilon$. Hence $\rho(x, G) = \rho$.

LEMMA 3. *If G is a subset of a metric space X, and $\rho(x, G)$ exists for all x in some dense subset S of X, then G is located.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in S. Then $\rho(x_n, G) \leqq \rho(x_n, x_m) + \rho(x_m, G)$; hence $|\rho(x_n, G) - \rho(x_m, G)| \leqq \rho(x_n, x_m)$ for all n and m, and thus $\{\rho(x_n, G)\}$ is a Cauchy sequence. Let $x \in X$, construct $\{x_n\}$ in S with $x_n \to x$, and put $\rho \equiv \lim \rho(x_n, G)$. For $y \in G$, since $\rho(x_n, G) \leqq \rho(x_n, y)$ for all n, it follows that $\rho \leqq \rho(x, y)$. Now let $\varepsilon > 0$, choose n so $\rho(x, x_n) < \varepsilon/3$, and construct $y \in G$ with $\rho(x_n, y) < \rho(x_n, G) + \varepsilon/3$. Then $\rho(x, y) \leqq \rho(x, x_n) + \rho(x_n, y) < \rho(x_n, G) + 2\varepsilon/3 < \rho + \varepsilon$. It follows that $\rho(x, G) = \rho$.

LEMMA 4. *If G is a located set on the line and $a \in G$, then*

$G \cap (-\infty, a]$ is also located.

Proof. Put $H \equiv G \cap (-\infty, a]$, let $x \in -\{a\}$, and put $\rho \equiv (x - a) \vee \rho(x, G)$. (1) If $x > a$, then $\rho = x - a$ and clearly $\rho(x, H) = \rho$. (2) If $x < a$, then $\rho = \rho(x, G)$, so $\rho \leqq \rho(x, y)$ for all $y \in H$. For any $\varepsilon > 0$, construct $y \in G$ such that $\rho(x, y) < \rho(x, G) + \varepsilon/2$. (2.1) If $y < a$, then $y \in H$ with $\rho(x, y) < \rho + \varepsilon$. (2.2) If $y > a - \varepsilon/2$, then $\rho(x, a) < \rho(x, y) + \varepsilon/2 < \rho + \varepsilon$. Hence $\rho(x, H) = \rho$. It follows from Lemma 3 that H is located.

COROLLARY 1. *In the situation of the lemma, $\rho(x, G \cap (-\infty, a]) = (x - a) \vee \rho(x, G)$, for all x in R.*

LEMMA 5. *If G is a located set on the line, $a \in -G$, and G meets $(-\infty, a]$, then $G \cap (-\infty, a]$ is also located.*

Proof. Put $H \equiv G \cap (-\infty, a]$, construct $b \in H$, and put $G_1 \equiv (-\infty, b - 1] \cup G \cup [a + 1, +\infty)$. Then $U \equiv -G_1$ is a bounded colocated set with $a \in U$. Applying [2, Theorem 4], construct the component (c, d) of a in U; Then c lies in the closure of G_1, hence in the closure K of G. By Lemma 4, $K \cap (-\infty, c]$ is located. Thus, since H is dense in $K \cap (-\infty, c]$, it follows from Lemma 1 that H is also located.

COROLLARY 2. *In the situation of the lemma, $\rho(x, G \cap (-\infty, a]) = (x - c) \vee \rho(x, G)$, for all $x \in R$, where c is the left-hand endpoint of the component of a in $-G$.*

Now consider the intersection of a located set G with open infinite intervals. In the case $a \in -G$, since $G \cap (-\infty, a) = G \cap (-\infty, a]$, Lemma 5 applies. For the case $a \in G$, however, we construct a located set G and a point a in G such that G meets $(-\infty, a)$, but $G \cap (-\infty, a)$ is *not* located, in the sense of a Brouwerian counterexample. Let $\alpha \geqq 0$ and put $G \equiv \{0, 1 - \alpha, 1\}$; then $G \cap (-\infty, 1)$ is *not* located, for if it were we would have a procedure determining either $\alpha > 0$ or $\alpha = 0$, yielding an equivalent form of LPO.

The next lemma is contained in [1, p. 89].

LEMMA 6. *A set on the line is totally bounded if and only if it is located and bounded.*

LEMMA 7. *If G is a located set on the line, and a and b are points of $G \cup -G$ such that G meets $[a, b]$, then $G \cap [a, b]$ is totally bounded.*

Proof. Apply Lemmas 4 and 5 in succession.

THEOREM 1. *For any fixed set G on the line, the following are equivalent*:

(1) *G is located.*

(2) *For every x and y in R such that G meets $[x, y]$, and for every $\varepsilon > 0$, there exist a within ε of x, and b within ε of y, such that $G \cap [a, b]$ is totally bounded.*

(3) *For every n in Z^+ there exist $a < -n$ and $b > n$ such that $G \cap [a, b]$ is totally bounded.*

Proof. (1) implies (2). If $\rho(x, G) > 0$, put $a \equiv x$, while if $\rho(x, G) < \varepsilon$, construct a in G such that $\rho(x, a) < \varepsilon$. Construct b similarly. It is easily seen that G meets $[a, b]$; apply Lemma 7.

(2) implies (3) is easily seen.

(3) implies (1). Fix y in G and let $x \in R$. Choose $n > 3(|y| \vee |x|)$ and construct corresponding numbers a and b. Put $\rho \equiv \rho(x, G \cap [a, b])$, and let $z \in G$. (1) If $a < z < b$, then $\rho \leqq \rho(x, z)$. (2) If $|z| > n$, then $\rho \leqq \rho(x, y) < 2n/3 < \rho(x, z)$. Hence $\rho(x, G) = \rho$.

The next theorem extends the fundamental lemma of [2], and completes the first step in the construction of unbounded components. The second step is Theorem 5 in §4, which calculates $\sup H$.

THEOREM 2. *Let U be a colocated set on the line with $a \in U$. Then*

$$H \equiv \{x \in R : x \geqq a \text{ and } [a, x] \subset U\}$$

is located.

Proof. Let $n \in Z^+$, and choose $b > n \vee |a|$. Then $a < b$ and $H \cap [-b, b] = \{x \in [a, b] : [a, x] \subset U\}$, which by [2, Lemma] is totally bounded. It follows from Theorem 1 that H is located.

3. Line and interval. In the next section R^∞ is constructed so that it is similar to $[-1, 1]$. In order to do this, R must first be related with $(-1, 1)$. The mapping $x \to x/(|x| + 1)$ will be used; a tedious computation results.

DEFINITION 1. For any real number x, put $\bar{x} \equiv x/(|x| + 1)$.

LEMMA 8. *For any x, y in R, $|\bar{x} - \bar{y}| \leqq |x - y|$.*

Proof. First note that for any $x \in R$, we have $|\bar{x}| \leqq |x|$. Now let $x, y \in R$ and $\varepsilon > 0$. (1) $|x| < \varepsilon/2$. Then $|\bar{x} - \bar{y}| < |\bar{y}| + \varepsilon/2$. Also $|x - y| \geqq |y| - |x| > |\bar{y}| - \varepsilon/2$, and hence $|\bar{x} - \bar{y}| < |x - y| + \varepsilon$. (2) $|y| < \varepsilon/2$. Similar. (3) $x \neq 0$ and $y \neq 0$. (3.1) $x > 0$ and $y > 0$. Then $|\bar{x} - \bar{y}| = |x - y|/(x + 1)(y + 1) \leqq |x - y|$. (3.2) $y < 0 < x$. Then $\bar{y} < 0 < \bar{x}$, so that $|\bar{x} - \bar{y}| = \bar{x} - \bar{y} \leqq x - y = |x - y|$. The remaining cases reduce to these.

COROLLARY 3. *If $x_n \to x$, then $\bar{x}_n \to \bar{x}$.*

LEMMA 9. *If $x, y \in R$ and $x < y$, then $x|y| \leqq |x|y$.*

Proof. (1) If $0 > x$, then $x|y| \leqq x(-y) = (-x)y = |x|y$. (2) If $0 < y$, then $x|y| = xy \leqq |x|y$.

LEMMA 10. *If $x, y \in R$, then $x < y$ if and only if $\bar{x} < \bar{y}$.*

Proof. Let $x < y$. Adding this to the inequality in Lemma 9, we obtain $x|y| + x < |x|y + y$; thus $\bar{x} < \bar{y}$. Now let $\bar{x} < \bar{y}$. First suppose $y < x$, then $\bar{y} < \bar{x}$, a contradiction, hence $x \leqq y$. Now put $\varepsilon \equiv \bar{y} - \bar{x} > 0$ and suppose $y < x + \varepsilon$. Then $\varepsilon = |\bar{x} - \bar{y}| \leqq |x - y| < \varepsilon$, a contradiction; hence $y \geqq x + \varepsilon > x$.

COROLLARY 4. *If $x, y \in R$, then $x \leqq y$ if and only if $\bar{x} \leqq \bar{y}$.*

LEMMA 11. *For any $M > 0$, if $|x| \leqq M$ and $|y| \leqq M$, then $|x - y| \leqq (M + 1)^2|\bar{x} - \bar{y}|$.*

Proof. First consider the case $x \neq 0$ and $y \neq 0$. (1) $x > 0$ and $y > 0$. Then $|\bar{x} - \bar{y}| = |x - y|/(x + 1)(y + 1) \geqq |x - y|/(M + 1)^2$. (2) $x < 0 < y$. Then $y - x > 0$ and $-2xy > 0$; hence $|\bar{x} - \bar{y}| = |x/(-x + 1) - y/(y + 1)| = |2xy - y + x|/(-x + 1)(y + 1) \geqq |y - x - 2xy|/(M + 1)^2 > |y - x|/(M + 1)^2$. Now for arbitrary x and y, choose $x_n \to x$ and $y_n \to y$ with $0 < |x_n| \leqq M$ and $0 < |y_n| \leqq M$, and apply Corollary 3.

COROLLARY 5. *Let $0 < b < 1$. If $|\bar{x}| \leqq b$ and $|\bar{y}| \leqq b$, then $|x - y| \leqq (1/(1 - b)^2)|\bar{x} - \bar{y}|$.*

LEMMA 12. *For any y in $(-1, 1)$, there exists x in R such that $\bar{x} = y$.*

Proof. Put $x \equiv y/(1 - |y|)$.

PROPOSITION 1. *The mapping $x \to \bar{x}$ is an order preserving bicontinuous correspondence of R with $(-1, 1)$.*

4. Extended real numbers.

DEFINITION 2. A sequence $\{x_n\}$ of real numbers is an *extended Cauchy sequence* if $\{\bar{x}_n\}$ is a Cauchy sequence in $(-1, 1)$. An *extended real number* is an extended Cauchy sequence of real numbers. The set of extended real numbers is denoted R^∞. For any $x \equiv \{x_n\}$ in R^∞, we put $\bar{x} \equiv \lim \bar{x}_n$. For any x and y in R^∞, define $x = y$, $x \leqq y$, $x < y$, or $x \neq y$ if the corresponding relation holds between $\bar{x}$ and $\bar{y}$ in $[-1, 1]$. For any real number x, put $x^* \equiv \{x, x, x, \cdots\}$.

The map $x \to x^*$ is an order-preserving injection of R into R^∞; thus we identify R with a subset of R^∞. The map $x \to \bar{x}$ is an order-preserving correspondence of R^∞ with $[-1, 1]$. If $x \in R^\infty$ and $\bar{x} = 1$ (resp. $\bar{x} = -1$), we write $x = +\infty$ (resp. $x = -\infty$).

THEOREM 3. *A sequence $\{x_n\}$ of real numbers is an extended Cauchy sequence if and only if there exists an increasing sequence $\{\sigma_k\}$ of* 0's *and* 1's, *and a sequence $\{M_k\}$ of positive integers, such that*

(a) *if $\sigma_k = 0$, then either $x_n \geqq k$ $(n \geqq M_k)$, or $x_n \leqq -k$ $(n \geqq M_k)$, and*
(b) *if $\sigma_k = 1$, then $|x_m - x_n| \leqq 1/k$ $(m, n \geqq M_k)$.*

Proof. We first prove the sufficiency. Let $m, n \geqq M_k$. (1) If $\sigma_k = 0$, then, in the first case, $\bar{x}_n \geqq \bar{k} = k/(k+1) > 1 - 1/k$, and similarly for $\bar{x}_m$; thus $|\bar{x}_m - \bar{x}_n| \leqq 1/k$. The other case is similar. (2) If $\sigma_k = 1$, then by Lemma 8, $|\bar{x}_m - \bar{x}_n| \leqq |x_m - x_n| \leqq 1/k$.

Conversely, let $\{\bar{x}_n\}$ be a Cauchy sequence in $(-1, 1)$ and put $y \equiv \lim \bar{x}_n$. Construct an increasing sequence $\{\sigma_n\}$ of 0's and 1's such that (a) $|y| > k/(k+1)$ when $\sigma_k = 0$, and (b) $|y| < 1$ when $\sigma_k = 1$. (1) If $\sigma_k = 0$, consider the case $y > k/(k+1)$. Choose M_k so that $\bar{x}_n \geqq k/(k+1)$ for $n \geqq M_k$, thus $x_n \geqq k$ $(n \geqq M_k)$. The other case is similar. (2) If $\sigma_k = 1$, choose b so $|y| < b < 1$. Then $|\bar{x}_n| \leqq b$ eventually; hence by Corollary 5 we have $|x_m - x_n| \leqq (1/(1-b)^2)|\bar{x}_m - \bar{x}_n|$ eventually. Since $\{\bar{x}_n\}$ is a Cauchy sequence, there exists M_k such that $|x_m - x_n| \leqq 1/k$ $(m, n \geqq M_k)$.

DEFINITION 3. A set S on the line is *convex* if $x \in S$ whenever $a, b \in S$ and $a \leqq x \leqq b$.

THEOREM 4. *Any convex located set on the line has a supremum and an infimum in R^∞.*

Proof. By Theorem 1, construct a decreasing sequence $\{a_n\}$ and an increasing sequence $\{b_n\}$ such that $a_n < -n$, and $b_n > n$, and $G_n \equiv G \cap [a_n, b_n]$ is totally bounded. Put $s_n \equiv \sup G_n$; clearly $\{s_n\}$ is increasing. If for some n, we have $s_n < b_n$, then $s_m = s_n$ $(m \geqq n)$. For, let $x \in G$ and suppose $x > s_n$. Choose y so $s_n < y < b_n \wedge x$; then $y \in G_n$, so that $y \leqq s_n$, a contradiction. Hence $x \leqq s_n$ for all $x \in G$, and it follows that $s_m = s_n$ $(m \geqq n)$. Thus we may construct an increasing sequence $\{\sigma_n\}$ of 0's and 1's such that (a) $s_n > n$ when $\sigma_n = 0$ and (b) $s_n < b_n$ when $\sigma_n = 1$. (1) If $\sigma_n = 0$, then $s_m \geqq n$ $(m \geqq n)$, because $\{s_n\}$ is increasing. (2) If $\sigma_n = 1$, then $s_m = s_n$ $(m \geqq n)$, as shown above. By Theorem 3, $\{s_n\}$ is an extended Cauchy sequence and $s \equiv \{s_n\}$ is in R^∞; clearly $s = \sup G$.

The convexity in the above theorem is essential. A located set on the line need *not* have a supremum in R^∞. For example, let $\{a_n\}$ be an increasing sequence of zeros and ones, and let the set G consist of the point 0 and also, in the event that some $a_n = 1$, of the least such integer n. Then G is located but does *not* have a supremum in R^∞; see [3, §11]. This problem is considered further in [5].

COROLLARY 6. *Every open (resp. closed) located convex set on the line is an open (resp. closed) interval.*

Proof. Let G be a located convex set; put $r \equiv \inf G$ and $s \equiv \sup G$. If G is open then clearly $G = (r, s)$. Now let G be closed; clearly $G \subset [r, s]$. (Note that $[r, s]$ consists of all *finite* numbers x with $r \leqq x \leqq s$.) Let $x \in [r, s]$; first construct a point $y \in G$ such that $x \leqq y$, as follows. (1) If $s > x$, construct $y \in G$ such that $y > x$. (2) If $s < x + 1$, then s is finite, so $s \in G$; put $y \equiv s$. Similarly, construct $z \in G$ such that $z \leqq x$. Hence $x \in G$.

Not every located convex set on the line is an interval. A Brouwerian counterexample is easily given; it shows that the assertion "every located convex set on the line is an interval" implies the (unbelievable) *limited principle of existence* (LPE). LPE states that if $\{a_n\}$ is any sequence of integers such that it is contradictory that all $a_n = 0$, then some $a_n \neq 0$; or, equivalently, if $t \geqq 0$ and $t = 0$ is contradictory, then $t > 0$. The set used in the counterexample is $\{x \in [0, 1): x = 0 \text{ is contradictory}\}$.

We now calculate the endpoints of the components of a colocated set.

THEOREM 5. *Let U be a colocated set on the line with $a \in U$. Then*

$$H \equiv \{x \in R: x \geqq a \text{ and } [a, x] \subset U\}$$

has a supremum in R^∞.

Proof. Theorem 2 shows that H is located; thus Theorem 4 applies.

5. Resolution of colocated sets.

THEOREM 6. *Any colocated set* U *on the line is a countable union of fixative disjoint open intervals. Thus,* $U = \bigcup_{k=1}^{\infty} I_k$, *where*
(1) *each open interval* I_k *is either void or fixed, and*
(2) I_j *and* I_k *are disjoint whenever* $j \neq k$.

Proof. In the bounded case, this is Corollary 1 of [3]. The boundedness restriction is used there at only two points: (a) The lemma in [2], where we now use Theorem 5 above, and (b) Lemma 1 in [3], where we now examine the images of the intervals in $[-1, 1]$.

The proofs of the following applications proceed as for the bounded case given in [3].

COROLLARY 7. *If two fixed open subsets of the line are disjoint, with one colocated, then there exists a point on the line that belongs to neither subset.*

COROLLARY 8. *A fixed closed colocated subset of the line is the entire line.*

For Corollary 8 to have any content, the concept of located set must be extended so as to include the void set $\varnothing$, from which the distance $\rho(x, \varnothing)$ of any point x is $+\infty$. More to the point, we must consider as located those sets from which distances exist in R^∞, but which are not known to be fixative. For example, if $\{a_n\}$ is an increasing sequence of zeros and ones, then the set $\{n: a_n = 1\}$ is, in general, *not* fixative, and yet is located in the sense of distance in R^∞. This more general concept of located set will be considered in [4] and [5].

COROLLARY 9. *Every fixed colocated convex set on the line is an open interval.*

It follows that a fixed colocated convex set on the line is located. However, *not* every fixed colocated set is located; consider $U \equiv$

$(0, 1) \cup (2, 2 + 1/k)$, where $\{a_n\}$ is an increasing sequence of zeros and ones and k is the least of the integers n such that $a_n = 1$, if any exist.

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