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ON THE BEHAVIOR OF A CAPILLARY SURFACE AT A RE-ENTRANT CORNER

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Changes in a domain's geometry can force striking changes in the capillary surface lying above it. Concus and Finn [1] first studied capillary surfaces above domains with corners, in the presence of gravity. Above a corner with interior angle θ satisfying $\theta < \pi - 2\gamma$, they showed that a capillary surface making contact angle γ with the bounding wall must approach infinity as the vertex is approached. In contrast, they showed that for $\theta \geqq \pi - 2\gamma$ the solution $u(x, y)$ is bounded, uniformly in θ as the corner is closed. Since their paper appeared, the continuity of u at the vertex has been an open problem in the bounded case. In this note we show by example that for any $\theta > \pi$ and any $\gamma \neq \pi/2$ there are domains for which u does not extend continuously to the vertex. This is in contrast to the case $\pi > \theta > \pi - 2\gamma$; here independent results of Simon [5] show that u actually must extend to be C^1 at the vertex.

We consider bounded domains Ω in $\boldsymbol{R}^2$ with piecewise smooth boundaries $\partial\Omega$, and functions $u(x, y)$ satisfying

(i) $\operatorname{div} Tu = 2H(u) = \kappa u$ in Ω; $Tu = Du/\sqrt{1 + Du^2}$, $H(u) =$ mean curvature of the surface $z = u(x, y)$, $\kappa > 0$.

(ii) $Tu \cdot n = \cos\gamma$ on the smooth part of $\partial\Omega$; $0 \leqq \gamma \leqq \pi$, $n =$ exterior normal to $\partial\Omega$.

Physically u describes the capillary surface formed when a vertical cylinder with horizontal cross section Ω is placed in an infinite reservoir of liquid having rest height $z = 0$. Then

$$\kappa = \frac{\rho g}{\sigma},$$

where

$\rho =$ density of liquid
$g =$ (downward) acceleration of gravity
$\sigma =$ surface tension between liquid and air.

$$\cos\gamma = \frac{\sigma_1}{\sigma},$$

where

$\sigma_1 =$ surface attraction between liquid and cylinder.

Geometrically γ is the contact angle between the capillary surface and the bounding cylinder; it is the angle between the downward normal of the surface $z = u(x, y)$, and the exterior normal of the cylinder $\partial\Omega \times \boldsymbol{R}$.

If $\gamma = \pi/2$, the only solution to (i) and (ii) is $u \equiv 0$. If $\gamma \neq \pi/2$, by considering either u or $-u$, we make the usual assumption that $0 \leqq \gamma < \pi/2$. This is the case in which the surface rises to meet the cylinder, or "wets" it.

Let θ and γ satisfy

$$\pi < \theta \leqq 2\pi\ , \qquad 0 < \gamma < \pi/2\ .$$

We will construct a domain for which a bounded solution u to (i) and (ii) exists, but having a corner of interior angle θ at which there is a jump discontinuity in u. (The arguments can be modified to include the case $\gamma = 0$.)

Determine the domain scale by fixing $R > 0$ (Fig. 1). Since $\theta > \pi$, we can pick θ_1 and θ_2, satisfying

$$\theta_1 > \pi - \gamma\ , \qquad \pi > \theta_2 > \gamma\ , \qquad \theta_1 + \theta_2 = \theta\ .$$

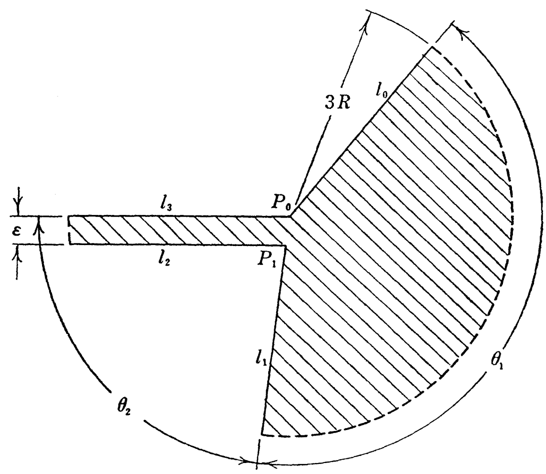

FIGURE 1. The intersection of Ω_ε with the disc of radius $3R$

$\theta_1 > \pi - \gamma$	$P_0 = (0, 0)$	$l_0 = \{y \cos\theta = x \sin\theta\}$
$\pi > \theta_2 > \gamma$	$P_1 = (-\varepsilon \cot\theta_2, -\varepsilon)$	$l_1 = \{y \cos\theta_2 = x \sin\theta_2\}$
$\theta_1 + \theta_2 = \theta > \pi$		$l_2 = \{y = -\varepsilon\}$
		$l_3 = x$-axis

For positive ε less than $R \sin\theta_2$, let Ω_ε be a bounded domain, of which the intersection with $B_{3R}(0)$ is shown in Fig. 1, and which has C^4 boundary except at P_0 and P_1. ($B_{3R}(0)$ is the disc of radius $3R$ centered at the origin.)

LEMMA 1. *There exists a unique solution to* (i) *and* (ii) *in any* Ω_ε. *It is bounded above and nonnegative.*

Proof. Because Ω_ε is C^2, except for a finite number of re-entrant corners, it satisfies a uniform internal sphere condition with contact angle γ, for any γ. Therefore it is admissible in the sense of Finn and Gerhardt [4]. Thus there is a bounded, nonnegative, real analytic function $u_\varepsilon(x, y)$ in Ω_ε, satisfying (i). Because u is energy minimizing in the sense of Emmer [3], the regularity theory of Simon and Spruck [6] implies that everywhere the boundary is C^4, u_ε extends to be at least C^2, and satisfies (ii). Uniqueness follows from a maximum principle of Concus and Finn [2].

We are interested in the behavior of u_ε near P_0, as ε approaches 0. Lemma 2 will show that u_ε stays uniformly bounded in one sector near P_0, and Lemma 3 show that in another sector it gets uniformly large. It follows that u_ε eventually has a jump discontinuity at P_0.

Let I_ε be the subdomain of Ω_ε shown in Fig. 2. Then we have

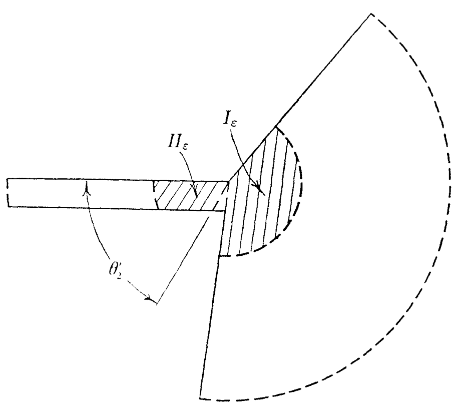

FIGURE 2. The subdomains I_ε and II_ε

$\theta_2 > \theta_2' > \gamma \qquad B_R(0) = \{x^2 + y^2 < R^2\}$

$$I_\varepsilon = B_R(0) \cap \{y \cos\theta > x \sin\theta\} \cap \{y \cos\theta_2 < x \sin\theta_2\}$$
$$II_\varepsilon = B_R(0) \cap \{y < 0\} \cap \{y > -\varepsilon\} \cap \{y \cos\theta_2' > x \sin\theta_2'\}$$

LEMMA 2. *u_ε is uniformly bounded in I_ε, independently of ε.*

Proof. In this and the following lemma the basic tool is a comparison method of Concus and Finn [2] for surfaces of known mean curvature and contact angle.

Consider circles of radius R which either lie entirely in Ω_ε or contact $\partial\Omega_\varepsilon$ only at a point of tangency. (In particular, do not allow them to have contact at P_0 or P_1.) If $\theta_1 < \pi$, also allow circles which intersect $\partial\Omega_\varepsilon$ at two points on $l_0 - P_0$, making an angle of no more than $\pi - \theta_1$ with l_0 at these intersections. Every point in I_ε lies interior to at least one of these circles (see Fig. 3).

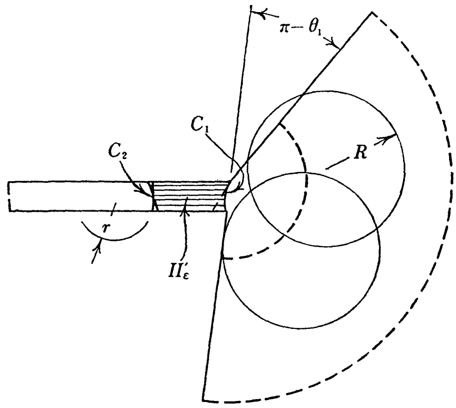

FIGURE 3. Equatorial circles near I_ε
The region II'_ε above which v is defined.

In $\boldsymbol{R}^3$ consider a closed lower hemisphere L with equatorial circle E, so that the projection $\pi(E)$ of E onto $\boldsymbol{R}^2$ is one of the above circles (see Fig. 4). If L contacts $l_0 \times \boldsymbol{R}$, then along the arc of intersection A the contact angle γ_L equals the angle between $\pi(E)$ and l_0. Thus $\gamma_L \leqq \pi - \theta_1 < \gamma$. Because P_0 and P_1 are the only two boundary points at which u_ε may not be C^2, u_ε is C^2 on $\overline{\pi(L) \cap \Omega_\varepsilon}$.

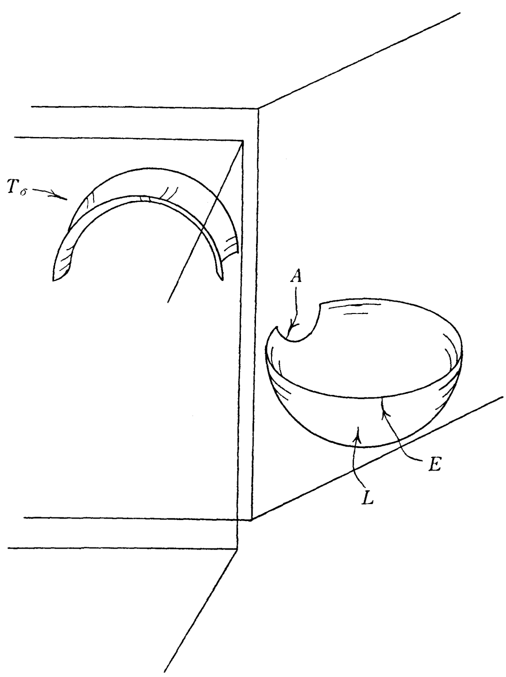

FIGURE 4. A lower hemisphere L contacting $\partial\Omega_\varepsilon \times \boldsymbol{R}$ along A, with contact angle less than γ. The "undeside" T_δ of a torus, contacting $\partial\Omega_\varepsilon \times \boldsymbol{R}$ with contact angle greater than γ.

Raise L until it lies above the bounded surface $\{z = u_\varepsilon(x, y)\}$. Lower L until the two surfaces first contact each other. Let $Q_0 = (x_0, y_0, u_\varepsilon(x_0, y_0))$ be a point of first contact.

Q_0 is not on E. This is because L is vertical along E whereas u_ε is C^2.

Q_0 is not on A: The end points of A are on E and are already excluded. If Q_0 was not an end point, the traces of the two surfaces on $l_0 \times \boldsymbol{R}$ would be tangent there. Since L contacts $l_0 \times \boldsymbol{R}$ at a steeper angle than the capillary surface, it would follow that L was actually below the surface in the interior normal direction from Q_0. Thus Q_0 would not be a point of first contact.

Thus (x_0, y_0) lies in the interior of $\pi(L) \cap \Omega_\varepsilon$. Since Q_0 is an interior point of first contact, the two surfaces are tangent there, and since L is nowhere below $\{z = u_\varepsilon(x, y)\}$, it follows that

$$H(u_\varepsilon)(x_0, y_0) \leqq \frac{1}{R} \quad \left(\text{since } \frac{1}{R} \text{ is the mean curvature of } L\right).$$

Using (i) gives:

$$u_\varepsilon(x_0, y_0) \leqq \frac{2}{\kappa R}.$$

Since L varies in height by R,

$$u_\varepsilon(x, y) \leqq \frac{2}{\kappa R} + R \quad \text{for all} \quad (x, y) \in \pi(L) \cap \Omega_\varepsilon .$$

By our previous comments this estimate holds in all of I_ε.

Fix θ_2' with $\gamma < \theta_2' < \theta_2$ and let II_ε be the subregion of Ω_ε as described in Fig. 2. Then we have

LEMMA 3. *$u_\varepsilon(x, y)$ approaches ∞ uniformly in II_ε, as ε approaches 0.*

Proof. Consider the unique circle C_1, containing P_0, making an angle θ_2' with l_3 and going through P_1 if $\theta_2 \leqq \pi/2$, or through $(0, -\varepsilon)$ if $\theta_2 > \pi/2$. Let C_2 be a circle of the same radius translated $2R$ units to the left.

There is a unique torus in $\boldsymbol{R}^3$ containing C_1 and C_2. It is generated by rotating C_1 about an axis parallel to the y-axis and going through Q_1, the point midway between C_1 and C_2. Let II_ε' be the part of $\bar{\Omega}_\varepsilon$ on or to the left of C_1, and on or to the right of C_2 (see Fig. 3). Then in II_ε', the "underside" T of the torus is given by

$$v(x, y) = [(R - \sqrt{r^2 - (y - y_1)^2})^2 - (x - x_1)^2]^{1/2} ,$$

where $(x_1, y_1) = Q_1$ (see Fig. 4). T contacts $l_3 \times \boldsymbol{R}$ with contact angle $\theta_2' > \gamma$, and contacts $l_2 \times \boldsymbol{R}$ with contact angle of at least θ_2'. It is vertical at C_1 and C_2.

Let any $\delta > 0$ be given. In order to avoid P_0 and P_1 translate T δ units to the left and call it T_δ, as in Fig. 4. Lower T_δ beneath $\{z = u_\varepsilon(x, y)\}$, and raise it until the first contact is made. By reasoning as in Lemma 2 it follows that if $(x_0, y_0, u_\varepsilon(x_0, y_0))$ is a point of first contact, then it does not occur on the boundary of T_δ. Thus it is a point of tangency and since T_δ is nowhere above $\{z = u_\varepsilon(x, y)\}$, the mean curvature of T_δ is no bigger than that of u_ε at $(x_0, y_0, u_\varepsilon(x_0, y_0))$. But by looking at the normal curvatures for a torus, one can calculate the following inequality:

$$H(v)(x, y) \geqq \frac{1}{2}\left(\frac{1}{r} - \frac{1}{R - r}\right) \quad (x, y) \in II_\varepsilon'$$

so that

$$\operatorname{div} Tu_\varepsilon(x_0, y_0) \geqq \left(\frac{1}{r} - \frac{1}{R - r}\right)$$

or

$$u_\varepsilon(x_0, y_0) \geqq \frac{1}{\kappa}\left(\frac{1}{r} - \frac{1}{R - r}\right).$$

Since T_δ varies in height by at most R, and since δ can be chosen arbitrarily small,

$$u_\varepsilon(x, y) \geqq \frac{1}{\kappa}\left(\frac{1}{r} - \frac{1}{R - r}\right) - R \quad \text{for } (x, y) \text{ in } II_\varepsilon'.$$

Since $II_\varepsilon \subset II_\varepsilon'$ for ε small enough, the last inequality eventually holds in II_ε. Noticing that r is proportional to ε and R is fixed, the result follows.

Combining the three lemmas yields the desired result:

THEOREM. *For ε sufficiently small, the solution $u_\varepsilon(x, y)$ to the capillary problem* (i) *and* (ii) *in Ω_ε cannot be extended continuously to the vertex of the re-entrant corner of angle θ.*

Although this theorem shows that u_ε need not extend nicely to the vertex, simple experiments with glass slides placed vertically in water indicate that the capillary surface itself still extends in a regular fashion to its boundary.

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