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REGULARITY OF CAPILLARY SURFACES OVER DOMAINS WITH CORNERS

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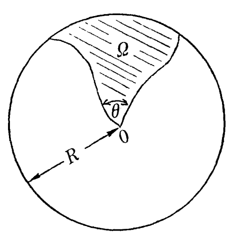
Using the usual mathematical model (capillary surface equation with contact angle boundary condition) we discuss regularity of the equilibrium free surface of a fluid in a cylindrical container in case the container cross-section has corners.

It is shown that good regularity holds at a corner if the "corner angle" θ satisfies $0<\theta<\pi$ and $\theta+2\beta>\pi$, where $0<\beta\leq\pi/2$ is the contact angle between the fluid surface and the container wall.

It is known that no regularity holds in case $\theta+2\beta<\pi$, hence only the borderline case $\theta+2\beta=\pi$ remains open.

We here want to examine the regularity of solutions of capillary surface type equations (subject to contact angle boundary conditions) on domain $\Omega\subset\boldsymbol{R}^2$ in a neighbourhood of a point of $\partial\Omega$ where there is a corner.

To be specific let Ω (as depicted in the diagram) be a region contained in $D_R=\{x\in\boldsymbol{R}^2: |x|<R\}$ ($R>0$ given) such that $\partial\Omega$ consists of a circular segment of ∂D_R together with two compact Jordan arcs γ_1, γ_2 such that $\gamma_1\cap\gamma_2=\{0\}$. γ_1, γ_2 are supposed to be $C^{1,\alpha}$ for some $0<\alpha<1$, and to meet at 0 with angle (measured in Ω) θ, $0<\theta<\pi$. We also suppose (without loss of generality, since we can always take a smaller R) that γ_i intersects ∂D_ρ in a single point for each $i=1,2$, $0<\rho<R$.



Then we look at (weak) $C^{1,\alpha}(\bar{\Omega}\sim\{0\})$, solutions of the equation

$$\sum_{i=1}^{2} D_i\left(\frac{D_i u}{\sqrt{1+|Du|^2}}\right)=H(x,u)\quad\text{on}\quad\Omega\,,\tag{0.1}$$

where H is a locally bounded measurable function on $\bar{\Omega}\times\boldsymbol{R}$.

It is assumed that a contact angle boundary condition holds; to be precise, we suppose

$$\nu(X) \cdot \mu(X) = \cos \beta \tag{0.2}$$

at each point $X = (x, u(x))$ with $x \in (\gamma_1 \cup \gamma_2) \sim \{0\}$. Here and subsequently $\nu(X)$ denotes the upward unit normal of the graph M of u at X (although we will assume that ν is defined on all of $(\bar{\Omega} \sim \{0\}) \times \boldsymbol{R}$ by $\nu(x, t) \equiv (-Du(x), 1)/\sqrt{1 + |Du|^2}$ for $(x, t) \in (\bar{\Omega} \sim \{0\}) \times \boldsymbol{R}$; thus ν is constant on vertical lines), and $\mu(X)$ denotes the inward pointing unit normal of the boundary cylider $((\gamma_1 \cup \gamma_2) \sim \{0\}) \times \boldsymbol{R}$. Notice that of course (0.2) can be expressed as $\partial u/\partial \eta/\sqrt{1 + |Du|^2} = \cos \beta$, where $\partial u/\partial \eta$ denotes the directional derivative of u in the direction of the outward unit normal to $\partial \Omega \sim \partial D_R$.

As is well-known, in case $H(x, u) \equiv \kappa u + \lambda$ (κ, λ constants) the equation (0.1) with boundary condition (0.2) is the usual model for the equilibrium free surface of a fluid in a cylindrical container, with side walls including $(\gamma_1 \cup \gamma_2) \times \boldsymbol{R}$, subject to the influence of a uniform gravitational field acting in the vertical direction. (The case $\kappa = 0$ corresponds to zero gravity, while $\kappa > 0$, $\kappa < 0$ correspond to gravitational fields acting vertically downwards and upwards respectively.)

The "contact angle" β of (0.2) is supposed to be a constant, with

$$0 < \beta < \pi \, , \tag{0.3}$$

but we could, without significant changes to the proofs, allow the case when β is a Hölder continuous function satisfying (0.3) at each point of $\gamma_1 \cup \gamma_2$.

The angle θ (measured in Ω) between the arcs γ_1, γ_2 at 0 is assumed to satisfy

$$0 < \theta < \pi \, , \qquad \theta > \pi - 2\tilde{\beta} \tag{0.4}$$

where $\tilde{\beta} = \beta$ if $0 < \beta \leqq \pi/2$ and $\tilde{\beta} = \pi - \beta$ in case $\pi/2 < \beta < \pi$. That some condition on the relation between θ and β is necessary in order to deduce any regularity of u near 0 is evident from the results of Concus and Finn [4], who show that, in case

$$\limsup_{t \to -\infty} \sup_{x \in \Omega} H(x, t) = -\infty \quad \text{and} \quad \liminf_{t \to +\infty} \inf_{x \in \bar{\Omega}} H(x, t) = +\infty \, , \tag{0.5}$$

u is bounded near 0 if and only if $\theta \geqq \pi - 2\tilde{\beta}$.

The main result to be proved here is given in the following theorem. Notice that we need to assume *a-priori* that u is bounded in Ω.

THEOREM 1. *Suppose* $u \in C^{1,\alpha}(\bar{\Omega} \sim \{0\}) \cap L^\infty(\Omega)$ *satisfies* (0.1), (0.2), *and suppose that* (0.3) *and* (0.4) *also hold.*

Then $\lim_{x \to 0, x \in \bar{\Omega}} u(x)$ *and* $\lim_{x \to 0, x \in \bar{\Omega}} Du(x)$ *both exist (with values in*

$\boldsymbol{R}$ and $\boldsymbol{R}^2$ respectively); thus u extends to a $C^1(\bar{\Omega})$ function.

In view of the result of Concus and Finn referred to above, we are able to state the following corollary of the theorem.

COROLLARY 1. *Suppose $u \in C^{1,\alpha}(\bar{\Omega} \sim \{0\})$ satisfies* (0.1), (0.2) *and suppose* (0.3), (0.4), (0.5) *also hold.*

Then the conclusion of Theorem 1 *remains valid.*

The general idea of the proof of Theorem 1 is first to show that there is a point $(0, z_0) \in \{0\} \times \boldsymbol{R}$ at which the graph M of u has a nonvertical tangent plane $z = z_0 + \sum_{i=1}^{2} a_i x_i$ (a_1, a_2 constants), in the sense that $|u(x_1, x_2) - z_0 - \sum_{i=1}^{2} a_i x_i| = o(\sqrt{x_1^2 + x_2^2})$ as $\sqrt{x_1^2 + x_2^2} \to 0$. This is achieved in §§1-3, using some geometric measure theoretic arguments (involving interior regularity and first variation theory). A key point here is a positive lower bound for the two dimensional density of $M = \text{graph}\, u$ at any point of $\bar{M} \cap \{0\} \times \boldsymbol{R}$. (See inequality (1.12) of §1.) In particular there are no "cusp-like" singularities. The angle condition (0.4) is needed to prove this lower density bound; (0.4) is *not* needed for any of the other results in this paper.

Having established the existence of a nonvertical tangent plane at $(0, z_0)$ one then uses (in §4) the interior regularity theory and the boundary regularity results of Jean Taylor [10], away from $\{0\} \times \boldsymbol{R}$ (i.e., away from the singular part of the boundary cylinder), to conclude the existence of a limit for $Du(x)$ as $x \to 0$.

We should remark that while this paper is concerned only with nonparametric capillary surfaces in cylindrical containers, it is evident that regularity results for parametric solutions in general polyhedral-type containers satisfying suitable edge and vertex angle conditions can be obtained by appropriate modification of the method described here.

1. Preliminary area bounds. In this section, and subsequently, Ω and u are as described above, with $\sup_\Omega |u| \leqq L < \infty$ (L a given fixed constant); ν and μ are also as described in the introduction, and we use the following additional notation:

$$D_\rho = \{x \in \boldsymbol{R}^2 : |x| < \rho\} \quad (\rho > 0)\,;$$

$$B_\rho(Y) = \{X \in \boldsymbol{R}^3 : |X - Y| < \rho\} \quad (\rho > 0 \text{ and } Y \in \boldsymbol{R}^3)^1\,;$$

$$\mu^{(1)} = \lim_{\substack{X \to 0 \\ X \in \gamma_1 \times \boldsymbol{R}}} \mu(X)\,, \qquad \mu^{(2)} = \lim_{\substack{X \to 0 \\ X \in \gamma_2 \times \boldsymbol{R}}} \mu(X)\,;$$

[1] As a rule we will represent points in $\boldsymbol{R}^3$ by upper-case letters $X, Y, \cdots$ and points in $\boldsymbol{R}^2$ by lower-case letters $x, y, \cdots$.

$$M = \text{graph}\, u = \{X = (x, u(x)) : x \in \bar{\Omega} \sim \{0\}\} ;$$

$$\partial M = \{X = (x, u(x)) : x \in \partial\Omega \sim (\{0\} \cup \partial D_R)\} ;$$

$$\mathfrak{H}^1 = \text{1-dimensional Hausdorff measure in } \boldsymbol{R}^2 \text{ or } \boldsymbol{R}^3 ;$$

$$\mathfrak{H}^2 = \text{2-dimensional Hausdorff measure in } \boldsymbol{R}^3 ;$$

J will denote any constant such that

$$|H(x, u(x))| \leqq J \quad \text{for all} \quad x \in \Omega \sim \{0\} .$$

Our first task in this section will be to establish upper bounds on the area of M. In fact we will show

$$\mathfrak{H}^2(M \cap (D_\rho \times \boldsymbol{R})) \leqq c\rho , \qquad 0 < \rho < R , \tag{1.1}$$

where c is a constant depending only on J, L and R.

To see this we first multiply the equation (0.1) by a function $\phi \in C^1(\bar{\Omega} \sim \{0\})$ and integrate over the subdomain $U \equiv (D_\rho \sim D_\sigma) \cap \Omega$, where $0 < \sigma < \rho \leqq R$. This gives

$$-\int_U \frac{Du \cdot D\phi}{\sqrt{1 + |Du|^2}} dx = \int_{\partial U} \phi \frac{Du \cdot \eta}{\sqrt{1 + |Du|^2}} dx + \int_U H(x, u)\phi dx , \tag{1.2}$$

where η denotes the inward unit normal of ∂U. We then take $\phi \equiv u$ and let $\sigma \to 0$. One readily checks that (1.2) then yields (1.1).

We are also here going to need the classical first variation formula for M. This says

$$\int_M \delta^M \cdot \phi d\mathfrak{H}^2 = -\int_M \phi \cdot \boldsymbol{H} d\mathfrak{H}^2 - \int_{\partial M} \phi \cdot \eta d\mathfrak{H}^1 , \tag{1.3}$$

where the notation is as follows:

η denotes the unit normal to ∂M which is tangent to M and which points *into* $\Omega \times \boldsymbol{R}$;

$\boldsymbol{H}$ = mean curvature vector of $M = H(X)\nu(X)$ at each point of M by virtue of (0.1);

$\phi = (\phi_1, \phi_2, \phi_3)$ is any $C^1(\bar{\Omega} \times \boldsymbol{R})$ vector field which vanishes in a neighborhood of $(\{0\} \times \boldsymbol{R}) \cup (\partial D_R \times \boldsymbol{R})$; $\delta^M \cdot \phi = \sum_{i=1}^3 \delta_i^M \phi_i$, where $\delta M = (\delta_1^M, \delta_2^M, \delta_3^M)$ is the gradient operator relative to M, defined by

$$\delta_i^M h(X) = \sum_{j=1}^3 (\delta_{ij} - \nu_i(X)\nu_j(X)) D_j h(X) , \qquad X \in M ,$$

whenever $h \in C^1(\bar{\Omega} \times \boldsymbol{R})$. (Thus $\delta^M h$ is the orthogonal projection of the ordinary gradient $Dh(X)$ onto the tangent space of M at X.)

Using this formula, we can bound the length of ∂M by the following argument.

Let r be the radial distance function defined by $r(x, t) = |x|$, $x, t \in \boldsymbol{R}^2 \times \boldsymbol{R}$, let ϕ be any C^1 vector field on $\bar{\Omega} \times \boldsymbol{R} \sim \{0\} \times \boldsymbol{R}$ with $\sup r|D\phi| < \infty$ and support $|\phi| \subset D_R \times \boldsymbol{R}$, and for $0 < 4\sigma < \rho < R$ let $\psi_\sigma \in C^1(\boldsymbol{R}^3)$ be such that $\psi_\sigma(x, t) = \gamma(|x|)$ for $(x, t) \in \boldsymbol{R}^2 \times \boldsymbol{R}$, where $\gamma \in C^1(\boldsymbol{R})$ satisfies the conditions:

$$\begin{cases} \gamma = 0 \quad \text{on} \quad [0, \sigma]\,, \qquad \gamma \equiv 1 \quad \text{on} \quad [\rho - \sigma, R] \\ \gamma' = \rho^{-1} \quad \text{on} \quad [2\sigma, \rho - 2\sigma]\,, \qquad 0 \leqq \gamma' \leqq \rho^{-1} \quad \text{on} \quad [0, R]\,. \end{cases}$$

(Thus $\gamma(t) \to \min\{t/\rho, 1\}$ uniformly as $\sigma \to 0$ for $t \in [0, R]$.)

Then, upon substituting $\psi_\sigma \phi$ in place of ϕ in (1.3) and letting $\sigma \to 0$, we deduce

$$\begin{aligned} \rho^{-1}\int_{M \cap (D_\rho \times \boldsymbol{R})} \phi \cdot \delta^M r \, d\mathfrak{H}^2 + \int_{\partial M} \min\{r/\rho, 1\}\phi \cdot \eta \, d\mathfrak{H}^1 \\ = -\int_M \min\{r/\rho, 1\}(\delta^M \cdot \phi + H\nu \cdot \phi) d\mathfrak{H}^2\,. \end{aligned} \tag{1.4}$$

Now

$$\eta = \frac{\mu - (\nu \cdot \mu)\nu}{|\mu - (\nu \cdot \mu)\nu|} = \frac{\mu - \cos\beta\nu}{|\mu - \cos\beta\nu|} \quad \text{on} \quad \partial M \tag{1.5}$$

by virtue of (0.2). Thus if γ is the unit vector bisecting the angle θ formed by the tangents to γ_1, γ_2 at 0, we have

$$\eta \cdot \gamma \geqq \frac{\mu \cdot \gamma - |\cos\beta|}{|\mu - \cos\beta\nu|} \geqq \frac{1}{2}\left(\sin\frac{\theta}{2} - |\cos\beta|\right) > 0 \tag{1.6}$$

on $\partial M \cap (D_{\rho_0} \times \boldsymbol{R})$ for sufficiently small $\rho_0 > 0$. (That $\sin\theta/2 - |\cos\beta| > 0$ is just a restatement of (0.4).)

By (1.1) we thus deduce from (1.4) (after taking $\phi =$ scalar function $\times \gamma$ and letting $\rho \downarrow 0$) that

$$\mathfrak{H}^1(\partial M \cap (D_R \times \boldsymbol{R})) < \infty\,. \tag{1.7}$$

In terms of the varifold $V = \boldsymbol{v}(M)$ associated with M([1, 3.5]), this, along with (0.1) and (1.1), tells us that

$$\|\delta V\|((D_R \sim \{0\}) \times \boldsymbol{R}) < \infty\,, \tag{1.8}$$

where δV denotes the first variation of V and $\|\delta V\|$ is its total variation ([1, 4.1, 4.2]). We can therefore use [2, 3.1 (7)] to deduce

$$\rho^{-1}\int_{M \cap (D_\rho \times \boldsymbol{R})} |\delta^M r - Dr|^2 d\mathfrak{H}^2 \longrightarrow 0 \quad \text{as} \quad \rho \longrightarrow 0\,. \tag{1.9}$$

In view of (1.1) (1.9) and Schwartz inequality, we see from (1.4) that

$$\rho^{-1}\int_{M\cap(D_\rho\times R)}\psi\gamma\cdot Dr\,d\mathfrak{H}^2+\int_{\partial M}\psi\gamma\cdot\eta\,d\mathfrak{H}^1 \tag{1.10}$$
$$\leqq(1+J)\int_M(\psi+|\delta^M\psi|)d\mathfrak{H}^2+o(1)$$

as $\rho\to 0$, where γ is the constant vector of (1.6), and suppose $\psi\subset D_{\rho_0}\times \boldsymbol{R}$.

Since $\gamma\cdot Dr\geqq\cos\theta/2>0$, and since (1.6) holds, we then have

$$\limsup_{\rho\downarrow 0}\rho^{-1}\int_{M\cap(D_\rho\times R)}\psi d\mathfrak{H}^2+\int_{\partial M}\psi d\mathfrak{H}^2$$
$$\leqq c(1+J)\int_M(\psi+|\delta^M\psi|)d\mathfrak{H}^2$$

whenever support $\psi\subset D_{\rho_0}\times\boldsymbol{R}$, where c depends on θ and β. In terms of the varifold $V=\boldsymbol{v}(M)$ this says

$$||\delta V||(\psi)\leqq c(1+J)\int(\psi+|\delta^M\psi|)d||V|| \tag{1.11}$$

by [2, 3.1(2)].

With the help of the isoperimetric inequality [1, 7.1] and a minor variation of the iteration argument of [1, 7.5(6)] (taking $f=1$ there), we then deduce

$$\mathfrak{H}^2(M\cap B_\rho(Y))\geqq c\rho^2(1+\rho_0)^{-2}\,,\qquad 0<\rho<\rho_0-\sigma\,, \tag{1.12}$$
$$Y\in\bar{M}\cap(D_\sigma\times\boldsymbol{R})$$

for some positive constant c depending only on J and the constant c in (1.11). We deduce particularly that the bound (1.12) holds also for $Y\in\bar{M}\cap(\{0\}\times\boldsymbol{R})$. For convenience of notation we will henceforth suppose $0\in\bar{M}\cap(\{0\}\times\boldsymbol{R})$ (this can be arranged by replacing u by $u-z_0$ for suitable z_0), and hence (1.12) holds with $Y=0$.

Notice that (1.12) says in particular that M cannot have a "cusp-like" singularity at a point of $\{0\}\times\boldsymbol{R}$. If the condition (0.4) is violated however, it appears intuitively evident that there exists graphs M of bounded mean curvature which *do* exhibit such singularities.

2. Monotonicity and consequences. In this section we first want to establish a certain monotonicity property. (See (2.6) below.) It seems likely that this can be proved by modifying the relevant argument of Jean Taylor [10]. It will be convenient here however to use standard varifold theory [1, §§3, 4, 5.1–5.4]; the reader will see that only a few of the more elementary aspects of [1] are used in this section, and as in §1 only the stationary character of M, rather than a minimizing property, is needed.

To begin, suppose ϕ is a C^1 vectorfield in $\boldsymbol{R}^3$ with the properties

$$\text{(2.1)} \qquad \phi \textit{ is parallel to } (0, 0, 1) \textit{ on } \{0\} \times \boldsymbol{R},\ \phi \textit{ is tangent to } (\partial\Omega \sim \partial D_R) \times \boldsymbol{R} \textit{ on } (\partial\Omega \sim \partial D_R) \times \boldsymbol{R}.$$

Let $F = \{(x, t): x \in \gamma_1 \cup \gamma_2 \sim \{0\},\ t \leqq u(x)\}$ and for $0 < \sigma < R$ let $F_\sigma = F \cap \{(x, t): \sigma \leqq |x| \leqq R - \sigma\}$. The classical divergence theorem (e.g., [7, 5.6.9]), which we apply to F_σ and let $\sigma \to 0$, gives

$$\text{(2.2)} \qquad \delta W(\psi\phi) = -\int_{\partial M} \psi\phi \cdot \gamma d\mathfrak{H}^1$$

whenever ψ is a $C_c^1(D_R \times \boldsymbol{R})$ function. Here W denotes the two dimensional varifold $\boldsymbol{v}(F)$ associated with F, and γ denotes the unit normal of ∂M which is tangent to F and which points into F.

Since $\cos\beta\gamma \cdot \phi = \eta \cdot \phi$ (η as in (1.3)) whenever ϕ is as in (2.1), we can then multiply by-$\cos\beta$ in (2.2) and add the result to (1.3) $\left(\text{which says } \delta V(\psi\phi) = -\int_{\partial M} \psi\eta \cdot \phi d\mathfrak{H}^1 - \int_M \psi H\phi \cdot \nu d\mathfrak{H}^2\right)$, thus obtaining

$$\text{(2.3)} \qquad (\delta V - \cos\beta\delta W)(\psi\phi) = -\int_M H\psi\phi \cdot \nu d\mathfrak{H}^2$$

whenever ϕ is as in (2.1). Similarly if we take $\widetilde{W} = \boldsymbol{v}(\widetilde{F})$, $\widetilde{F} = \{(x, t): x \in \gamma_1 \cup \gamma_2 \sim \{0\},\ t \geqq u(x)\}$, we deduce

$$\text{(2.3)}' \qquad (\delta V + \cos\beta\delta \widetilde{W})(\psi\phi) = -\int_M H\psi\phi \cdot \nu d\mathfrak{H}^2 .$$

Since γ_1, γ_2 are $C^{1,\alpha}$ curves, one can readily check that there is a C^1 vector field ϕ as in (2.1) such that

$$\text{(2.4)} \qquad \begin{aligned} &\sup_{X \in D_R \times \boldsymbol{R}} |X|^{-1-\alpha} |X - \phi(X)| < \infty , \\ &\sup_{X \in D_R \times \boldsymbol{R}} |X|^{-\alpha} |D(X - \phi(X))| < \infty . \end{aligned}$$

Next, let $Z = V - \cos\beta W$ in case $\cos\beta < 0$ and $Z = V + \cos\beta\widetilde{W}$ in case $\cos\beta > 0$. By (2.3), (2.3)′ and (2.4) we then have

$$\text{(2.5)} \qquad |\delta Z(\gamma(|X|))X| \leqq c \int (|X|^\alpha \gamma(|X|) + |X|^{1+\alpha}\gamma'(|X|)) d\|Z\| ,$$

where c depends only on J, for any $C_c^1((-R, R))$ function γ. In view of this, a minor modification of the argument of [1, 5.1] or [8, §3] shows that, for a suitable constant c,

$$\text{(2.6)} \qquad \exp(c\rho^\alpha)\frac{\|Z\|(B_\rho(0))}{\rho^2} \quad \text{is increasing in} \quad \rho,\ 0 < \rho < R .$$

Furthermore, by (1.12), (2.2), (2.6) and [1, 4.12] we deduce that there is nonzero stationary varifold C in the varifold tangent of Z at 0. Thus, writing μ_r to represent the homothetic transformation $X \mapsto rX$ $(r > 0)$, we can find a sequence $r_k \to \infty$ so that $V_\infty = \lim_{k\to\infty} \mu_{r_k \#} V$, $W_\infty = \lim_{k\to\infty} \mu_{r_k \#} W$, and $\tilde{W}_\infty = \lim_{k\to\infty} \mu_{r_k \#} \tilde{W}$ all exist and so that $C = V_\infty - \cos\beta W_\infty$ or $C = V_\infty + \cos\beta \tilde{W}_\infty$ according as $\cos\beta$ is negative or positive. Evidentally $\mu_{r\#}\|C\| = \|C\|$ (by (2.6)).

An immediate consequence of (1.12) is that, for each $\rho > 0$, there is a sequence $\varepsilon_k \to 0$ such that

$$(2.7) \qquad B_\rho(0) \cap M_k \subset \{Y \in B_\rho(0)\colon \operatorname{dist}(Y, \operatorname{spt}\|V_\infty\|) < \varepsilon_k\} .$$

Here $M_k = \mu_{r_k}(M)$ and $\operatorname{spt}\|V_\infty\|$ denotes the support of the measure $\|V_\infty\|$ ($\|V_\infty\|$ = weight of V_∞ [1, 3.1]).

Indeed, if (2.7) were false, there would exist $\varepsilon > 0$, a subsequence $\{k'\} \subset \{k\}$ and a sequence $\{X_{k'}\}$ with $X_{k'} \in M_{k'} \cap A_\varepsilon$ for k', where for each $n > 0$ we let

$$A_\eta = \{Y \in \bar{B}_\rho(0)\colon \operatorname{dist}(Y, \operatorname{spt}\|V_\infty\|) \geqq \eta\} .$$

Applying the inequality (1.12) to $M_{k'}$ (notice that (1.12) holds with the same constant c if M is replaced by M_k, because $M_k = \mu_{r_k}(M)$), we deduce

$$\mathfrak{H}^2(M_k \cap A_{\varepsilon/2}) \geqq \mathfrak{H}^2(M_k \cap B_{\varepsilon/2}(X_k)) \geqq c\varepsilon^2/4 ,$$

thus contradicting the fact that

$$\limsup_{k\to\infty} \mathfrak{H}^2(M_k \cap A_{\varepsilon/2}) \leqq \|V_\infty\|(A_{\varepsilon/2})(=0)$$

(which holds because $v(M_k) \to V_\infty$).

3. Tangent plane for M at 0.

From the interior nonparametric regularity theory [9, §3] (alternatively from the parametric theory of [1, §8] or [3] or [6]), we deduce that there exist λ, $\delta \in (0, 1)$ and a constant $c > 0$, all depending only on ρJ, such that, whenever $Y \in M$ and $B_\rho(Y) \cap (\partial\Omega \times \boldsymbol{R}) = \emptyset$

$$(3.1) \qquad B_{\lambda\rho}(Y) \cap M \text{ is connected}, \quad |\nu(X) - \nu(\bar{X})| \leqq c(\rho^{-1}|X - \bar{X}|)^\delta ,$$

for $X, \bar{X} \in B_{\lambda\rho}(Y) \cap M$.

Let $\{r_k\}$ be the sequence used to construct the varifold C in §2, let $\Omega_k = \{r_k x\colon x \in \Omega\}$, $M_k = \mu_{r_k}(M)$ (=graph u_k, where u_k is defined by $u_k(x) = r_k u(r_k^{-1}x)$, $x \in \Omega_k$), and let V_∞, W_∞, $\tilde{W}_\infty$ be as in §2. Also, let Ω_∞ be the domain enclosed by the rays which are tangent to γ_1, γ_2 at 0, so that the Lebesgue measure of $[(\Omega_\infty \sim \Omega_k) \cup (\Omega_k \sim \Omega_\infty)] \cap D_\rho$ converges to zero as $k \to \infty$ for each $\rho > 0$.

In view of (3.1) and in view of the fact that (by (0.1)) M_k)) M_k has mean curvature bounded by J/r_k, we deduce that

$$V_\infty \llcorner (\Omega_\infty \times \boldsymbol{R}) = \boldsymbol{v}(M_\infty) ,$$

where $M_\infty(=\lim M_k$ taken in $\Omega_\infty \times \boldsymbol{R}$ in the varifold sense) is either empty or a smooth minimal (not necessarily connected) submanifold of $\Omega_\infty \times \boldsymbol{R}$ with

$$(3.2) \qquad \mathfrak{H}^2(M_\infty \cap B_\rho(0)) < \infty \quad \text{for each} \quad \rho > 0 \text{ (by (2.6))}$$

and with $\mu_r(M_\infty) = M_\infty$ for each $r > 0$. This last property just says that M_∞ is a cone, which is true by (2.6) and [1, 5.2(2)(a)].

One now readily checks (from the fact that M_∞ is a C^2 cone with zero mean curvature) that

$$(3.3) \qquad M_\infty = \bigcup_{j=1}^{N} \pi_j \cap (\Omega_\infty \times \boldsymbol{R}) ,$$

where π_j are planes through the origin and $\pi_i \cap \pi_j \cap \Omega_\infty \times \boldsymbol{R} = \emptyset$ for $i \neq j$. We must consider the possibility that $N = \infty$ here, but in any case by (3.2) we see immediately that at most a finite subcollection of $\{\pi_1, \pi_2, \cdots\}$ intersects a given compact subset of $\Omega_\infty \times \boldsymbol{R}$. Evidently, since M_∞ is the limit (taken in $\Omega_\infty \times \boldsymbol{R}$ in the varifold sense) of the sequence M_k of graphs, we easily deduce from (3.3) that *either*

Case 1. $N = 1$ and $M_\infty = \pi_1 \cap (\Omega_\infty \times \boldsymbol{R})$ for some plane π_1 such that $\pi_1 \cap (\{0\} \times \boldsymbol{R}) = \{0\}$; *or*

Case 2. $N < \infty$ and $M_\infty = \bigcup_{j=1}^{N} \pi_j \cap (\Omega_\infty \times \boldsymbol{R})$, where $\pi_1, \pi_2, \cdots, \pi_N$ are planes with the line $\{0\} \times \boldsymbol{R}$ in common. (Notice that to get $N < \infty$ here, it is necessary to use (3.2).)

To proceed further, we need to consider the variational problem satisfied by M. For any bounded Borel set $A \subset \boldsymbol{R}^3$ and any open $W \subset \Omega \times \boldsymbol{R}$ we let

$$E(W, A) = \mathfrak{H}^2(\partial W \cap \Omega \times \boldsymbol{R} \cap A) - \cos\beta\mathfrak{H}^2(\partial W \cap \partial\Omega \times \boldsymbol{R} \cap A) + \int_{A \cap W} K(X)dX ,$$

where K is defined on $\Omega \times \boldsymbol{R}$ by $K(x, t) = H(x, u(x))$, $(x, t) \in \Omega \times \boldsymbol{R}$, so that K is constant on vertical lines.

We claim that $U = \{(x, t) \in \Omega \times \boldsymbol{R}: t < u(x)\}$ minimizes E in the sense that

$$(3.4) \qquad E(U, B_\rho(0)) \leqq E(W, B_\rho(0))$$

whenever W satisfies

$$(3.5)\qquad W \subset \Omega \times \boldsymbol{R}\,, \qquad \mathfrak{H}^2(\partial W \cap B_\rho(0)) < \infty\,, \\ ((W \sim U) \cup (U \sim W)) \cap B_\rho(0) \subset\subset B_\rho(0)\,.$$

To see this, first note that the equation (0.1) can be written $\operatorname{div} \nu = K$ on $\Omega \times \boldsymbol{R}$, where K is as above. An alternative way of writing this is

$$(3.6)\qquad d(^*\nu) = K dx_1 \wedge dx_2 \wedge dx_3 \quad \text{on} \quad \Omega \times \boldsymbol{R}\,,$$

where $^*\nu$ denotes the 2-form $\nu_1 dx_2 \wedge dx_3 - \nu_2 dx_1 \wedge dx_3 + \nu_3 dx_1 \wedge dx_2$. Let $[W]$, $[U]$ denote the 3-currents obtained by integrating 3-forms over W and U respectively; $\partial[W]$, $\partial[U]$ are rectifiable in $B_\rho(0)$ by (1.1), (3.5) and [5, 4.5.6(1)].

Next let ψ_σ be a nonnegative $C^1(\boldsymbol{R}^3)$ function with $\psi_\sigma \equiv 1$ or $B_\rho(0) \sim (D_\sigma \times \boldsymbol{R})$, $\psi_\sigma \equiv 0$ or $D_{\sigma/2} \times \boldsymbol{R}$ and $\sup_{\boldsymbol{R}^3} |D\psi_\sigma| \leqq 3/\sigma$, and use the identity

$$\partial([W] - [U])(\psi_\sigma.(^*\nu)) = ([W] - [U])(d(\psi_\sigma.(^*\nu)))\,.$$

Letting $\sigma \downarrow 0$ and using [5, 4.5.6(4)] to evaluate the left side of this identity, we deduce

$$\int_{U \cap B_\rho(0)} K(X)dX + \int_{\partial U \cap B_\rho(0)} \nu \cdot \eta_U d\mathfrak{H}^2 \\ = \int_{W \cap B_\rho(0)} K(X)dX + \int_{\partial W \cap B_\rho(0)} \nu \cdot \eta_W d\mathfrak{H}^2\,,$$

where η_V, η_W denote the exterior normals of U and W respectively. (See [5, 4.5.5] for the definition of η_W; notice that unless W is a reasonably nice set, we may have $\eta_W = 0$ on a set of positive $\mathfrak{H}^2$ measure in $\partial W \cap B_\rho(0)$.)

Since $\eta_U = \nu$ on $\partial U \cap (\omega \times \boldsymbol{R})$ and

$$\eta_W = \mu \quad \mathfrak{H}^2\text{-a.e. on} \quad \partial W \cap (\partial\Omega \times \boldsymbol{R}) \cap \{X \in B_\rho(0) \colon \eta_W(X) \neq 0\}\,,$$

we then have (3.4), as required, by virtue of (0.2).

Now define, for any open $W \subset \Omega_k \times \boldsymbol{R}$ and any bounded Borel set $A \subset \boldsymbol{R}^3$,

$$E_k(W, A) = \mathfrak{H}^2(\partial W \cap (\Omega_k \times \boldsymbol{R}) \cap A) - \cos\beta \mathfrak{H}^2(\partial W \cap (\partial\Omega_k \times \boldsymbol{R}) \cap A) \\ + r_k^{-1} \int_{W \cap A} K(r_k^{-1}X)dX\,.$$

(We also include $k = \infty$ in this definition, in which case the last term is to be interpreted as zero.) Since $E_k(\boldsymbol{\mu}_{r_k} W, \boldsymbol{\mu}_{r_k} A) = r_k^2 E(W, A)$ whenever W is as in (3.5), it is evident from (3.4) that for $k = 1, 2, \cdots$ we have

$$(3.4)' \qquad E_k(U_k, B_\rho(0)) \leqq E_k(W, B_\rho(0))(U_k = \mu_{r_k}(U)) ,$$

whenever W is an open set such that

$$(3.5)' \qquad \begin{gathered} W \subset \Omega_k \times \boldsymbol{R} , \qquad \mathfrak{H}^2(\partial W \cap B_\rho(0)) < \infty , \\ ((U_k \sim W) \cup (W \sim U_k)) \cap B_\rho(0) \subset\subset B_\rho(0) . \end{gathered}$$

We can now show that $M_\infty \neq \phi$. In fact we will show that

$$(3.7) \qquad V_\infty \llcorner (\partial\Omega_\infty \times \boldsymbol{R}) = 0 ,$$

which is a stronger statement because $V_\infty \neq 0$ by (1.12).

To prove (3.7) first note that since $V_\infty = \lim_{k\to\infty} \mu_{r_k\#} V$, by virtue of (1.11) and (2.6) we can apply [1, 5.4] to deduce that $\Theta^2(\|V_\infty\|, (Y \geqq 1$ for $\|V_\infty\|$ – a.e. Y. If (3.7) fails we can therefore take a point $Y \in \partial\Omega_\infty \times \boldsymbol{R} \sim ((\{0\} \times \boldsymbol{R}) \cup (\bigcup_{j=1}^N \pi_j))$ such that $\Theta^2(\|V_\infty\|, Y) \geqq 1$.

Hence for each $\varepsilon > 0$ we can find $\rho > 0$ such that

$$(3.8) \qquad \begin{gathered} B_{2\rho}(Y) \cap \left((\{0\} \times \boldsymbol{R}) \cup \left(\bigcup_{j=1}^N \pi_j\right)\right) = \phi , \\ \frac{\mathfrak{H}^2(B_{\rho/2}(Y) \cap M_k)}{\pi(\rho/2)^2} \geqq 1 - \varepsilon \end{gathered}$$

for all sufficiently large k, and (by virtue of (2.7))

$$(3.9) \qquad M_k \cap B_\rho(Y) \subset \{X \in \Omega_k \times \boldsymbol{R}: \operatorname{dist}(X, \partial\Omega_k \times \boldsymbol{R}) < \sigma_k\rho\} ,$$

where $\sigma_k \to 0$ as $k \to \infty$.

Next, let $\{f_k\}$ be a sequence of C^∞ mappings of $\boldsymbol{R}^3$ into $\boldsymbol{R}^3$ with the properties:

$$f_k(\bar{\Omega}_k \times \boldsymbol{R}) \subset \bar{\Omega}_k \times \boldsymbol{R} , \qquad f_k(B_{\rho/2}(Y)) \subset B_{\rho/2}(Y) , \qquad f_k(X) = X ,$$

$$X \in (\boldsymbol{R}^3 \sim B_\rho(Y)) \cup (\partial\Omega_k \times \boldsymbol{R}) , \quad f_k(B_\rho(Y) \sim B_{\rho/2}(Y)) \subset B_\rho(Y) \sim B_{\rho/2}(Y)$$

$$f_k\{X \in B_{\rho/2}(Y): \operatorname{dist}(X, \partial\Omega_k \times \boldsymbol{R}) < \sigma_k\} \subset B_{\rho/2}(Y) \cap \partial\Omega_k \times \boldsymbol{R}$$

$$\sup_{X \in \boldsymbol{R}^3} \|Df_k(X)\| \leqq 1 + c\sigma_k , \quad c \text{ independent of } k .$$

(It is left to the reader to check that such a sequence exists.)

For each k we now let $U_k = \mu_{r_k}(U)$, $\tilde{U}_k =$ interior $f_k(U_k)$, and we let E_k be as in (3.4)′. From construction of the f_k, we know that for $k = 1, 2, \cdots$,

$$(3.10) \qquad E_k(\tilde{U}_k, B_{\rho/2}(Y)) = 0 ,$$

$$E_k(\tilde{U}_k, B_\rho(Y) \sim B_{\rho/2}(Y)) \leqq (1 + c\sigma_k)^2 E_k(U_k, B_\rho(Y) \sim B_{\rho/2}(Y)) ,$$

and, by virtue of (3.8),

$$(3.11) \qquad E_k(U_k, B_{\rho/2}(Y)) - (1 - \varepsilon - |\cos\beta|)\pi(\rho/2)^2 + \tilde{\sigma}_k \geqq 0 ,$$

where $\tilde{\sigma}_k \to 0$ as $k \to \infty$. Combining (3.10), (3.11), we deduce that (for $\varepsilon < 1 - |\cos \beta|$ and k sufficiently large)

$$E_k(\tilde{U}_k, B_\rho(Y)) < E_k(U_k, B_\rho(Y)) ,$$

and hence, since $f_k(X) \equiv X$ for all $X \in \boldsymbol{R}^3 \sim B_\rho(Y)$,

$$E_k(\tilde{U}_k, B_\sigma(0)) < E_k(U_k, B_\sigma(0)) \qquad (\sigma > \rho + |Y|) ,$$

thus contradicting (3.4)′ for all sufficiently large k. Thus (3.7) is proved; hence

$$M_\infty \neq \phi \quad \text{and} \quad V_\infty = \boldsymbol{v}(M_\infty) . \tag{3.12}$$

By virtue of (3.1) and the definition of U_k it now readily follows that there is an open $U_\infty \subset \Omega_\infty \times \boldsymbol{R}$ such that $\partial U_\infty \cap (\Omega_\infty \times \boldsymbol{R}) = M_\infty$ and $(U_\infty \sim U_k) \cup (U_k \sim U_\infty)$ has measure locally converging to zero. Furthermore by (3.1), (3.3), (3.4)′, (2.7), (3.7) and the fact that $\mu_{r_k\#} V \to V_\infty$, we easily deduce

$$E_\infty(U_\infty, B_\rho(0)) \leqq E_\infty(W, B_\rho(0)) \tag{3.13}$$

for every open W satisfying

$$\begin{gathered} W \subset \Omega_\infty + \boldsymbol{R} , \qquad \mathfrak{H}^2(\partial W \cap B_\rho(0)) < \infty , \\ ((W \sim U_\infty) \cup (U_\infty \sim W)) \cap B_\rho(0) \subset\subset B_\rho(0) . \end{gathered} \tag{3.14}$$

Here we use the notation that

$$E_\infty(W, A) = \mathfrak{H}^2(\partial W \cap (\Omega_\infty \times \boldsymbol{R}) \cap A) - \cos \beta \mathfrak{H}^2(\partial W \cap (\partial\Omega_\infty \times \boldsymbol{R}) \cap A)$$

for any W as in (3.14) and any bounded Borel set A.

Now we want to show Case 2 is impossible. To see this, note first that in Case 2 $U_\infty = U_\infty^{(1)} \times \boldsymbol{R}$ for some open $U_\infty^{(1)} \subset \Omega_\infty$ with $\partial U_\infty^{(1)}$ a finite union of rays emanating from the origin. Define

$$E_\infty^{(1)}(W) = \mathfrak{H}^1(\partial W \cap \Omega_\infty \cap D_1) - \cos \beta \mathfrak{H}^1(\partial W \cap \partial\Omega_\infty \cap D_1)$$

for any open W satisfying

$$\begin{gathered} W \subset \Omega_\infty , \qquad \mathfrak{H}^1(\partial W \cap D_1) < \infty , \\ ((W \sim U_\infty^{(1)}) \cup (U_\infty^{(1)} \sim W)) \cap D_1 \subset\subset D_1 , \end{gathered} \tag{3.15}$$

and note that it follows from (3.13) that

$$E_\infty^{(1)}(U_\infty^{(1)}) \leqq E_\infty^{(1)}(W) \tag{3.16}$$

for any W as in (3.15).

Since $\Omega_\infty \sim \bar{U}_\infty^{(1)}$ clearly satisfies a variational principle similar to that satisfied by $U_\infty^{(1)}$ but with $\pi - \beta$ in place of β, in case $N > 1$

we can suppose without loss of generality that there is a component W^* of $U_\infty^{(1)}$ with $\bar{W}^* \cap \partial\Omega_\infty = \{0\}$. But then

$$E_\infty^{(1)}((U_\infty^{(1)} \sim W^*) \cup \tilde{W}^*) < E_\infty^{(1)}(U_\infty^{(1)}) ,$$

where $\tilde{W}^*$ is obtained by "smoothing out" the vertex of W^* at 0. Since this contradicts (3.16), we deduce $N = 1$.

To show that we also get a contradiction in Case 2 if $N = 1$, we note that if β_0 is the angle formed by $U_\infty^{(1)}$ at 0, and if $\beta_0 < \beta$, then we have

$$E_\infty^{(1)}(W^*) < E_\infty^{(1)}(U_\infty^{(1)}) \tag{3.17}$$

if W^* is constructed as follows:

Let $p \in \partial D_{1/2} \cap (\partial U_\infty^{(1)} \sim \partial\Omega_\infty)$ and let q be the point on $\partial U_\infty^{(1)} \cap \partial\Omega_\infty$ at distance ε from 0. We then let $W^* = U_\infty^{(1)} \sim H$, where H is the closed 1/2-plane with $0 \in H \sim \partial H$ and $\{p, q\} \subset \partial H$. For ε small enough one then easily checks that (3.17) holds. Thus we deduce

$$\beta_0 \geqq \beta \; . \tag{3.18}$$

However, again using the fact that $\Omega_\infty \sim \bar{U}_\infty^{(1)}$ satisfies a similar variational problem with $\pi - \beta$ in place of β, we can deduce by the same argument that

$$\theta - \beta_0 \geqq \pi - \beta \; . \tag{3.18$'$}$$

Adding (3.18) and (3.18)′ we have $\theta \geqq \pi$, thus contradicting (0.4).

Thus Case 2 is impossible, and we are left with Case 1. Notice that the plane π_1 in Case 1 is uniquely determined by β and Ω_∞. In fact a standard (nonparametric) argument (based on the fact that (3.13) holds) shows that π_1 must make an angle (measured in U_∞) of β with each component of $(\partial\Omega_\infty \times \boldsymbol{R}) \sim (\{0\} \times \boldsymbol{R})$. Thus π_1 is characterized by saying that π_1 has a unit normal ν^0 with the properties

$$\nu^0 \cdot (0, 0, 1) > 0 \; , \quad \nu^0 \cdot \mu^{(1)} = \cos\beta = \nu^0 \cdot \mu^{(2)} (\mu^{(i)} = \lim_{\substack{X \to 0 \\ X \in \gamma_i}} \mu(X)) \; . \tag{3.19}$$

(This characterizes π_1 completely because $\mu^{(1)}$ and $\mu^{(2)}$ are linearly independent.)

Thus we have shown that $M_\infty = \pi_1 \cap (\Omega_\infty \times \boldsymbol{R})$ with π_1 having unit normal ν^0 as in (3.19), *independent of the particular sequence* $\{r_k\}$ *chosen to construct* M_∞. It follows that $\{\mu_{r_k \#} V\}$ converges to the same limit $v(\pi_1 \cap (\Omega_\infty \times \boldsymbol{R}))$ for every sequence $r_k \to \infty$. In particular we may take $r_k = 2^k$. One easily checks that (2.7) then implies

$$\lim_{\substack{x\to 0\\ x\in\Omega}} \frac{\left|u(x) - \sum_{i=1}^{2} (\nu_i^0/\nu_3^0)x_i\right|}{|x|} = 0\,, \tag{3.20}$$

where ν_1^0, ν_2^0, ν_3^0 are the components of the vector ν^0 normal to π_1. In particular, we deduce $\lim_{x\to 0, x\in\Omega} u(x)$ exists, thus completing the proof of the first assertion of Theorem 1.

4. Conclusion of proof. Here let $M_k = \mu_{2^k} M$, $u_k(x) = 2^k u(2^{-k}x)$, $x \in \Omega_k$, where $\Omega_k = \mu_{2^k}(\Omega)$. (Thus M_k, Ω_k are as in the previous section, with $r_k = 2^k$.)

We know from (3.20) that

$$\left|u_k(x) - \sum_{i=1}^{2} (\nu_i^0/\nu_3^0)x_i\right| \longrightarrow 0 \quad \text{as} \quad k \longrightarrow \infty \tag{4.1}$$

uniformly for $1 \leqq |x| \leqq 2$.

On the other hand (3.1), applied to M_k, gives us λ, $\delta \in (0, 1)$ and $c > 0$ so that

$$|\nu^{(k)}(X) - \nu^{(k)}(Y)| \leqq c\left(\frac{|X - Y|}{\sigma}\right)^{\delta} \tag{4.2}$$

whenever $X = (x, u_k(x))$, $Y = (y, u_k(y))$ are such that $|X - Y| < \lambda\sigma$ and $x, y \in \{z \in \Omega_k\colon \operatorname{dist}(z, \partial\Omega_\infty) > \sigma\}$. Here $\nu^{(k)}$ denotes the upward unit normal of graph u_k, and $\sigma > 0$ is arbitrary.

By combining (4.1), (4.2) we then easily deduce that $Du_k(x) \to (\nu_3^0)^{-1}(\nu_1^0, \nu_2^0)$ as $k \to \infty$, the convergence being uniform for $x \in \{y \in \boldsymbol{R}^2\colon 1 \leqq |y| \leqq 2,\ \operatorname{dist}(y, \partial\Omega_\infty) > \sigma\}$ ($\sigma > 0$ arbitrary).

Writing this last conclusion in terms of u, we have

$$\lim_{\substack{x\to 0\\ x\in S_\sigma}} Du(x) = (\nu_3^0)^{-1}(\nu_1^0, \nu_2^0)\,, \tag{4.3}$$

where $S_\sigma = \{x \in \Omega\colon \operatorname{dist}(x/|x|, \partial\Omega_\infty) > \sigma\}$.

On the other hand, if we use the boundary regularity theory of J. Taylor [10], we deduce by (4.1) that (4.2) actually holds for any $X = (x, u_k(x))$, $Y = (y, {}_k(y))$ with $|X - Y| < \sigma$ and $x, y \in \{z \in \Omega_k\colon 1 \leqq |z| \leqq 2,\ \operatorname{dist}(z, \Omega_\infty) < \sigma\}$, provided σ is sufficiently small (independent of k). Combining this fact with (4.1) and reasoning as before, we deduce

$$\lim_{\substack{x\to 0\\ x\in T_\sigma}} Du(x) = (\nu_3^0)^{-1}(\nu_1^0, \nu_2^0) \tag{4.4}$$

where $T_\sigma = \{x \in \Omega\colon \operatorname{dist}(x/|x|, \partial\Omega_\infty) < \sigma\}$.

Theorem 1 is now established by combining (4.3) and (4.4).

REFERENCES

1. W. K. Allard, *On the first variation of a Varifold,* Annals of Math., **95** (1972), 417-491.

2. ———, *On the first variation of a Varifold: Boundary behavior,* Annals of Math., **101** (1975), 418-446.

3. F. J. Almgren, Jr., *Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints,* Mem. Amer. Math. Soc., **4**, No. 165, (1976).

4. P. Concus and R. Finn, *On capillary free surfaces in a gravitational field,* Acta Math., **132** (1974), 207-223.

5. H. Federer, *Geometric Measure Theory,* Springer-Verlag-Berlin-Heidelberg-New York, 1969.

6. E. Giusti, *Minimal surfaces and functions of bounded variation,* Notes on Pure Mathematics, Australian National University, (1977).

7. W. Klingenberg, *A Course in Differential Geometry,* Springer-Verlag, Berlin-Heidelberg-New York, 1978.

8. J. Michael and L. Simon, *Sobolev and mean-value inequalities on generalized submanifolds of* $\mathbf{R}^n$, Comm. Pure App. Math., **26** (1973), 361-379.

9. L. Simon, *A Holder estimate for quasiconformal maps between surfaces in Euclidean space,* Acta Math., **139** (1977), 19-51.

10. J. Taylor, *Boundary regularity for solutions to various capillarity and free boundary problems* Comm. in P.D.E. **2** (1977), 323-357.

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