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IMBEDDING SMOOTH INVOLUTIONS IN TRIVIAL BUNDLES

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Let $\widetilde{M}^n$ denote a closed manifold with smooth involution, and $\widetilde{M}^n \to M^n$ the corresponding principal Z_2-bundle ξ classified by $w_1(\xi)$ in $H^1(M^n; Z_2)$. The existence of a nested sequence $\widetilde{N}^{n-k+1} \subset \widetilde{N}^{n-k+2} \subset \cdots \subset \widetilde{N}^{n-1} \subset \widetilde{N}^n = \widetilde{M}^n$ of characteristic submanifolds (corresponding to principal Z_2-bundles $\xi_{n-k+1} \subset \xi_{n-k+2} \subset \cdots \subset \xi_{n-1} \subset \xi_n = \xi$) satisfying $(w_1(\xi_i))^{n-k+i} = 0$, for $1 \leq k \leq n$ and all i with $n-k+1 \leq i \leq n$, provides a necessary and sufficient condition for imbedding ξ in vector bundle $M^n \times R^k \to M^n$ (preserving fibers).

For closed n-manifolds, the condition $(w_1(\xi))^k = 0$ allows ξ to be imbedded in the vector bundle $M^n \times R^k \to M^n$ "up to free Z_2-cobordism."

1. Introduction. In the following, all manifolds will be smooth and compact, and maps are understood to be smooth where appropriate. Let $\pi: \widetilde{M}^n \to M^n$ denote a principal Z_2-bundle ξ, with $\widetilde{M}^n$ and M^n being n-manifolds. An *imbedding* of ξ into the trivial fiber bundle $M^n \times F$ over M^n is a smooth map $h: \widetilde{M}^n \to M^n \times F$ which maps $\widetilde{M}^n$ diffeomorphically onto its image $h(\widetilde{M}^n)$ in $M^n \times F$, such that the following diagram is commutative,

$$\begin{array}{ccc} \widetilde{M}^n & \xrightarrow{h} & M^n \times F \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\text{proj}} & \\ & M^n\,. & \end{array}$$

Recently, V. L. Hansen [3] has shown that for connected topological spaces X with nondegenerate base point and with the homotopy type of a CW complex of dimension $m \geqq 1$, any finite covering map $\pi: E \to X$ can be imbedded into the trivial real $(m+1)$-plane bundle over X. Our notion of imbedding is the same as his, except we will work in the smooth category.

One naturally asks the following question. To what extent does the vanishing of some iterated cup product of the characteristic class $w_1(\xi)$ of the involution $\widetilde{M}^n$ determine an imbedding of ξ into the trivial vector bundle $M^n \times R^k$ over M^n? The purpose of this paper is to investigate this question.

The vanishing of w_1^2 has already appeared in the literature in other contexts, for example ([2], [8], [9]). We will be dealing, however, with general two-fold covers rather than just orientation covers.

A good reference then is [1], page 60.

2. Imbeddings. We begin with a normalization lemma.

LEMMA 2.1. *If the principal $\boldsymbol{Z}_2$-bundle $\pi\colon \tilde{M} \to M$ imbeds in the trivial vector bundle $M \times \boldsymbol{R}^k \to M$, then there is an imbedding map g sending $\tilde{M}$ to $M \times S^{k-1}$, such that fibers of $\tilde{M} \to M$ are mapped antipodally into S^{k-1}.*

Proof. Let h be an imbedding of $\tilde{M}$ into $M \times R^k$. For each x in M, let y and y' be the two distinct elements in the fiber $\pi^{-1}(x)$. Define $g\colon \tilde{M} \to M \times S^{k-1}$ by

$$g(y) = \left(x, \frac{h_2(y) - h_2(y')}{\|h_2(y) - h_2(y')\|}\right)$$

where $h = (h_1, h_2)$ and $\|\ \ \|$ is the standard norm in $\boldsymbol{R}^k$. The map g is as required.

In light of the previous lemma, one has the following result.

PROPOSITION 2.2. *The principal $\boldsymbol{Z}_2$*-bundle *$\xi\colon \tilde{M} \to M$ imbeds in $(M \times R^k \to M)$ if and only if ξ is classified by the diagram below with $\tilde{f}$ $\boldsymbol{Z}_2$-equivariant.*

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{f}} & S^{k-1} \subset S^{\infty} \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & \boldsymbol{R}p^{k-1} \subset \boldsymbol{R}p^{\infty} \end{array} .$$

COROLLARY 2.3. *If the principal $\boldsymbol{Z}_2$-bundle ξ $\tilde{M} \to M$ imbeds in $(M \times \boldsymbol{R}^k \to M)$, then the k-fold cup product $(w_1(\xi))^k$ of the characteristic class $w_1(\xi)$ of the involution $\tilde{M}$ is zero.*

Proof. One has the following diagram classifying ξ.

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{f}} & S^{k-1} \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & \boldsymbol{R}p^{k-1} \end{array} .$$

Let c be the generator of $H^1(\boldsymbol{R}p^{k-1}; \boldsymbol{Z}_2)$. Then

$$(w_1(\xi))^k = (f^*(c))^k = f^*(c^k) = f^*(0) = 0 .$$

For $k = 1$, there is the following result.

LEMMA 2.4. *Let* ξ *denote the principal* $\boldsymbol{Z}_2$*-bundle* $\tilde{M} \to M$. *Then* $w_1(\xi) = 0$ *if and only if* ξ *imbeds in* $(M \times \boldsymbol{R}^1 \to M)$.

Proof. If ξ imbeds in $M \times \boldsymbol{R}^1$, then $w_1(\xi) = 0$, by Corollary 2.3. On the other hand if $w_1(\xi) = 0$, the line bundle γ associated to ξ is orientable, and so the structure group $\boldsymbol{Z}_2 = O(1)$ of γ reduces to $SO(1) = \{1\}$, the trivial group. Thus γ is a trivial $\boldsymbol{R}$-bundle, since it is classified in $BSO(1) =$ point. The natural inclusion of ξ in γ shows that ξ imbeds in $M \times \boldsymbol{R}^1$.

Now let ξ denote the principal $\boldsymbol{Z}_2$-bundle $\tilde{M}^n \to M^n$, and let η denote the principal $\boldsymbol{Z}_2$-bundle $\tilde{N}^{n-1} \to N^{n-1}$, where $\tilde{N}^{n-1}$ is a characteristic submanifold for $\tilde{M}^n$. We will need the following lemma.

LEMMA 2.5. *An imbedding of* η *in the trivial vector bundle* $N^{n-1} \times \boldsymbol{R}^{k-1} \to N^{n-1}$ *gives rise to an imbedding of* ξ *in the trivial vector bundle* $M^n \times \boldsymbol{R}^k \to M^n$.

Proof. Since $\tilde{N}$ is a characteristic submanifold for $\tilde{M}$, write $\tilde{M}$ as

$$\tilde{P}_- \cup (\tilde{N} \times [-1, 1]) \cup \tilde{P}_+$$

with the involution on $\tilde{M}$ interchanging the two copies $\tilde{P}_-$ and $\tilde{P}_+$ of $\tilde{P}$. Also, write M as

$$P \cup Q$$

with Q being the disk-bundle of N in M.

Given the imbedding $h\colon \tilde{N} \to N \times \boldsymbol{R}^{k-1}$ covering N, extend h in the obvious fashion to an imbedding

$$g\colon \tilde{N} \times [-1, 1] \longrightarrow Q \times \boldsymbol{R}^{k-1} \times [-1, 1] \subseteqq Q \times \boldsymbol{R}^k \quad \text{covering} \quad Q\,.$$

Lemma 2.1 says that g may be taken as an imbedding of $\tilde{N} \times [-1, 1]$ into $Q \times S^{k-1}$; moreover, g may be chosen to take $\tilde{N} \times \{-1, 1\}$ into $\partial Q \times S^0$ for some inclusion of S^0 in S^{k-1}. Since the disjoint union $\tilde{P}_- + \tilde{P}_+$ is a trivial two-fold cover of P, the map g extends to an imbedding of $\tilde{M}$ into $M \times S^{k-1}$ covering M. Hence ξ imbeds in the trivial vector bundle $M \times \boldsymbol{R}^k$.

We are now ready to state the first theorem.

THEOREM 2.6. *Let* ξ *denote the principal* $\boldsymbol{Z}_2$*-bundle* $\tilde{M}^n \to M^n$ *of closed manifold classified by* $w_1(\xi)$ *in* $H^1(M^n; \boldsymbol{Z}_2)$. *The existence of a nested sequence* $\tilde{N}^{n-k+1} \subset \tilde{N}^{n-k+2} \subset \cdots \subset \tilde{N}^{n-1} \subset \tilde{N}^n = \tilde{M}^n$ *of characteristic submanifolds (corresponding to principal* $\boldsymbol{Z}_2$*-bundles* $\xi_{n-k+1} \subset$

$\xi_{n-k+2} \subset \cdots \subset \xi_{n-1} \subset \xi_n = \xi$) *satisfying* $(w_1(\xi_i))^{k-n+i} = 0$, *for* $1 \leqq k \leqq n$ *and all* i *with* $n - k + 1 \leqq i \leqq n$, *provides a necessary and sufficient condition for imbedding* ξ *in* $M \times \boldsymbol{R}^k$.

Proof. Assume that ξ imbeds in $M \times \boldsymbol{R}^k$. By Proposition 2.2, ξ is classified by the diagram

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{f}} & S^{k-1} \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & \boldsymbol{Rp}^{k-1} \end{array} .$$

Thus $(w_1(\xi))^k = 0$. Deform f to be transverse regular on $\boldsymbol{Rp}^{k-2}$. Then $f^{-1}(\boldsymbol{Rp}^{k-2})$ is a 1-codimensional submanifold N^{n-1} of M^n with the pull-back covering ξ_{n-1}: $\tilde{N}^{n-1} \to N^{n-1}$ satisfying $(w_1(\xi_{n-1}))^{k-1} = 0$ and $\tilde{N}^{n-1}$ a characteristic submanifold for $\tilde{M}^n$. Continuing to deform f, one gets $(w_1(\xi_i))^{k-n+i} = 0$ for the rest of the ξ_i and the nested sequence of characteristic submanifolds. Thus necessity is proved.

To prove sufficiency induct on $k \geqq 1$. For $k = 1$, see Lemma 2.4. As an inductive step, ξ_{n-1} imbeds in $N^{n-1} \times \boldsymbol{R}^{k-1}$, and along with Lemma 2.5 sufficiency is proved.

3. Imbedding up to $\boldsymbol{Z}_2$-cobordism. Let $\mathcal{N}_*(\boldsymbol{Z}_2)$ denote the bordism module of fixed point free involutions as defined in [1, p. 59]. It is well-known that $\mathcal{N}_*(\boldsymbol{Z}_2)$ is a free module over the unoriented bordism ring $\mathcal{N}_*$ with base the class of the antipodal map on the j-sphere $[A, S^j]$, for $j = 0, 1, 2, \cdots$. In $\mathcal{N}_n(\boldsymbol{Z}_2)$, the bordism class of the involution $\tilde{M}^n$ is determined uniquely by the integers mod 2 $\langle w_{i(1)} \cdots w_{i(k)}(w_1(\xi))^m, [M^n]\rangle$ where ξ: $\tilde{M}^n \to M^n$ is the 2-fold cover, the $w_{i(j)}$ are the Stiefel-Whitney classes of M^n, $[M^n]$ is the fundamental class of M^n, and $i(1) + \cdots + i(k) + m = n$. See [1, p. 60].

THEOREM 3.1. *Let* ξ: $\tilde{M}^n \to M^n$ *be a principal* $\boldsymbol{Z}_2$*-bundle of closed manifolds and let* $k \geqq 1$. *If* $(w_1(\xi))^k = 0$, *then there is a representative* $\tilde{H}^n$ *of the class of* $\tilde{M}^n$ *in* $\mathcal{N}_n(\boldsymbol{Z}_2)$ *such that* η: $\tilde{H}^n \to H^n$ *imbeds in* $H^n \times \boldsymbol{R}^k$.

Proof. Notice first of all that if $k \geqq n + 1$ or $k = 1$, then previous results allow one to take $\tilde{H}^n = \tilde{M}^n$. Suppose, therefore, that $1 < k \leqq n$. Write $\tilde{M}^n$ up to fixed point free $\boldsymbol{Z}_2$-cobordism as the disjoint union

$$\sum_{j=0}^{n} (A, S^j) \times V^{n-j}$$

where (A, S^j) denotes the antipodal map on the j-sphere and V^{n-j}

denotes a closed $(n-j)$-manifold with trivial action. Let $u \in H^1(\ ;\boldsymbol{Z}_2)$ generally denote the characteristic class of an involution. Note that for $j<r$, $(A, S^j) \times V^{n-j}$ has $u^r = 0$.

Assume $(w_1(\xi))^n = 0$, then

$$0 = \langle (w_1(\xi))^n, [M]\rangle = \left\langle u^n, \left[\sum_{j=0}^{n} \boldsymbol{R}p^j \times V^{n-j}\right]\right\rangle = \sum_{j=0}^{n} \langle u^n, [\boldsymbol{R}p^j \times V^{n-j}]\rangle$$
$$= \langle u^n, [\boldsymbol{R}p^n \times V^0]\rangle .$$

Thus V^0 must represent the trivial element in $\mathscr{N}_0$. Now write $\widetilde{M}^n$ up to fixed point free $\boldsymbol{Z}_2$-cobordism as $\sum_{j=0}^{n-1} (A, S^j) \times V^{n-j}$. Moreover, since $\mathscr{N}_1 = 0$, one may as well write $\sum_{j=0}^{n-2} (A, S^j) \times V^{n-j}$.

Suppose inductively that the vanishing of $(w_1(\xi))^n, (w_1(\xi))^{n-1}, \cdots, (w_1(\xi))^{r+1}$ allows one to remove $(A, S^n) \times V^0, (A, S^{n-1}) \times V^1, \cdots, (A, S^{r+1}) \times V^{n-(r+1)}$ from the bordism representation. Let $(w_1(\xi))^r = 0$, and let p^{n-r} generally denote a homogeneous polynomial of degree $(n-r)$ in the Stiefel-Whitney classes of a manifold. Then $(w_1(\xi))^r = 0$ implies that

$$0 = \langle w_1(\xi)^r p^{n-r}, [M]\rangle = \left\langle u^r p^{n-r}, \sum_{j=0}^{r} [\boldsymbol{R}p^j \times V^{n-j}]\right\rangle$$
$$= \sum_{j=0}^{r} \langle u^r p^{n-r}, [\boldsymbol{R}p^j \times V^{n-j}]\rangle = \langle u^r p^{n-r}, [\boldsymbol{R}p^r \times V^{n-r}]\rangle .$$

Since u is "dual" to $\boldsymbol{R}p^{r-1} \times V^{n-r}$, (see [7, p. 155]),

$$0 = \langle u^r p^{n-r}, [\boldsymbol{R}p^r \times V^{n-r}]\rangle = \langle u^{r-1} p^{n-r}, [\boldsymbol{R}p^{r-1} \times V^{n-r}]\rangle .$$

Successive use of this duality shows $\langle p^{n-r}, [V^{n-r}]\rangle = 0$ for all p^{n-r} on the Stiefel-Whitney classes of V^{n-r}. Thus V^{n-r} represents a trivial element in $\mathscr{N}_{n-r}$.

Hence $\widetilde{M}^n$ represents the same element in $\mathscr{N}_*(\boldsymbol{Z}_2)$ as $\widetilde{H}^n$ which equals $\sum_{j=0}^{k-1} (A, S^j) \times V^{n-j}$. Clearly, $\eta\colon \widetilde{H}^n \to H^n$ imbeds $H^n \times \boldsymbol{R}^k$.

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