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CYCLIC VECTORS FOR $L^p(G)$

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If G is a first countable locally compact group, then $L^p(G)$ has a cyclic vector with compact support, where $1 \leq p < \infty$.

In [3] Greenleaf and Moskowitz proved the existence of cyclic vectors for the left and right regular representation of $L^2(G)$, where G is a first countable, locally compact group, see also [4] and [5]. We generalize this result to $L^p(G)$ $(1 \leqq p < \infty)$ and certain other $L^1(G)$-modules.

THEOREM. *Let G be a locally compact group.*

(i) *If G is first countable, then there exists a continuous function u on G with compact support such that the left invariant hull of u is dense in $L^p(G)$ for $1 \leqq p < \infty$. The right hull of u (for the corresponding right action of G on $L^p(G)$) is also dense in $L^p(G)$.*

(ii) *Conversely, if $1 \leqq p < \infty$ and $L^p(G)$ has a cyclic vector, then G is first countable.*

For the proof of the theorem we need two lemmas:

LEMMA 1. *Assume that H is a closed subgroup of G which is isomorphic to $\boldsymbol{R}$. If the nonzero measure μ is concentrated on a compact subset of H, then $\{f*\mu: f \in \mathscr{K}(G)\}$ is dense in $L^p(G)$ for $1 < p < \infty$.*

Proof of Lemma 1. Define q by $1/q + 1/p = 1$. If the space defined above is not dense in $L^p(G)$, there exists a nonzero continuous function $g \in L^q(G)$ such that $\langle f*\mu, g\rangle = 0$ for all $f \in \mathscr{K}(G)$, the space of continuous functions with compact support (if g is not continuous, replace g by $h*g \neq 0$, $h \in \mathscr{K}(G)$). Put $g^\vee(x) = g(x^{-1})(x \in G)$, then $\mu*g^\vee = 0$ on G. Put $\mu_1 = \Delta_G(\cdot)^{-1/q}\cdot\mu$ and for $y \in G$, $x \in H$, set $g_y(x) = g(y^{-1}x)\Delta_G(x)^{+1/q}$ (Δ_G denotes the modular function on G). By Weil's formula ([7], pp. 42-45) $g_y \in L^q(H)$ holds for a.e. $y \in G$. A short calculation shows that

$$\mu_1*g_y^\vee(x) = \mu*g^\vee(xy)\Delta_G(x)^{-1/q} \quad \text{for} \quad x \in H .$$

Since g is continuous we conclude that $\mu_1*g_y^\vee = 0$ on H. μ_1 is concentrated on a compact subset of $H = \boldsymbol{R}$ and nonzero. The Fourier transform $\hat{\mu}_1$ is an analytic function. It follows that it has at most countably many zeros. By [1] the set $\{f*\mu_1: f \in \mathscr{K}(H)\}$ is dense in

$L^p(H)$ for $1 < p < \infty$. If $g_y \in L^q(H)$, it follows from this that $g_y = 0$. Again by Weil's formula we conclude that $g = 0$.

In a similar way we obtain:

LEMMA 2. *Let H be a closed subgroup of G, μ a bounded measure on H. If μ generates a dense left (right) ideal in $L^1(H)$ then it generates a dense left (right) ideal in $L^1(G)$. If H is compact, the same holds for $L^p(G)(1<p<\infty)$.*

Proof of the theorem. (i) We use Yamabe's theorem to find an open subgroup G_1 of G, and a compact subgroup N of G_1, normal in G_1, such that G_1/N is a connected Lie group. Now we use the description of the Haar measure given in [2]. There exist closed subgroups $H_1, \cdots, H_n$ of G_1, each of them being isomorphic to $\boldsymbol{R}$, and a compact subgroup $K \supseteq N$ such that $G_1 = H_1 \cdots H_n K$ and this is a topological decomposition of G_1. The Haar measure on G_1 is simply the product of the Haar measures on $H_1, \cdots, H_n$ and K. Now let f be a continuous function on $\boldsymbol{R}$ with compact support and nowhere vanishing Fourier transform. Let μ_i be the measure on $H_i = \boldsymbol{R}$ defined by $f (i = 1, \cdots, n)$. Since G is metrizable, the same holds for K and it follows that the dual of K is countable. Let g be a continuous function on K such that $U(g)$ is invertible for any continuous, irreducible representation of K. Let μ_{n+1} be the measure defined by g. It follows from Lemmas 1 and 2 that $\{h*\mu_1*\cdots*\mu_{n+1}: h \in K(G)\}$ is dense in $L^p(G)$ for $1 \leqq p < \infty$. The measure $\mu_1*\cdots*\mu_{n+1}$ is absolutely continuous on G_1, its derivative with respect to Haar measure is $u(x_1 \cdots x_{n+1}) = f(x_1) \cdots f(x_n) g(x_{n+1}) (x_i \in H_i \; i = 1, \cdots, n, x_{n+1} \in K)$. It follows that u has the properties stated in the theorem. The proof for the right invariant hull is similar.

(ii) This part is entirely analogous to the case of $L^2(G)$ which was proved in [4] Theorem 2.1.

DEFINITION (see [6]). A symmetric Segal algebra $S(G)$ on G is a dense, left and right invariant linear subspace of $L^1(G)$, such that $S(G)$ is a Banach space with respect to a norm $\| \;\|_S$, $\|f\|_1 \leqq \|f\|_S$, for $f \in S(G)$, $y \to L_y$ and $y \to R_y$ are strongly continuous representations of G by isometries on $S(G)$, [6], Ch. 6, §2.1, 2.2. (it follows in particular that $S(G)$ is a left and right $L^1(G)$ module and that the action of $L^1(G)$ is contractive).

COROLLARY. *If $S(G)$ is a symmetric Segal algebra on G and the function u of the theorem belongs to $S(G)$, the left and right invariant hulls of u are both dense in $S(G)$.*

Proof of the corollary. Take $g \in S(G)$. Since right translation is continuous on $S(G)$, there exists $h \in K(G)$ such that $\|g*h - g\|_S < \varepsilon$. By the theorem there exists $k \in K(G)$ such that $\|k*u - h\|_1 < \varepsilon$. It follows that

$$\|g*k*u - g\|_S \leqq \|g*k*u - g*h\|_S + \|g*h - g\|_S < \varepsilon(\|g\|_S + 1) .$$

The proof for the right invariant hull of u is similar.

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