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LOCAL Λ SETS FOR PROFINITE GROUPS

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Let E be a subset of the dual $\hat{G}$ of a profinite group G. It is shown that if E is a local Λ set then the degrees of the elements of E must be bounded. It follows that $\hat{G}$ contains an infinite Sidon set if and only if $\hat{G}$ has infinitely many elements of the same degree. This characterisation is the same as one previously obtained for compact Lie groups.

Preliminaries. Let G be a compact group with normalized Haar measure λ_G. For $p \in [1, \infty[$ the Banach space of pth power integrable complex-valued functions on G is denoted $(L^p(G), \|\cdot\|_p)$. The *dual object* $\hat{G}$ of G is taken to be a maximal set of pairwise inequivalent continuous irreducible unitary representations of G. For each $\sigma \in \hat{G}$ let d_σ be the *degree* or dimension of the representation space of σ and let χ_σ denote its *trace*. The *Fourier transform* of $f \in L^1(G)$ is the matrix-valued function $\hat{f}$ on $\hat{G}$ defined by

$$\hat{f}(\sigma) = \int_G f(x)\sigma(x^{-1})d\lambda_G(x) \qquad (\sigma \in \hat{G}) .$$

If E is a subset of $\hat{G}$ let $S_E(G)$ denote the set of all trigonometric polynomials on G whose Fourier transforms are supported by just one element of E. For $p \in]1, \infty[$ call E a *local Λ_p set* if there exists a positive constant κ such that

$$\|f\|_p \leqq \kappa\|f\|_1$$

for all $f \in S_E(G)$. Call E a *local central Λ_p set* if there exists a positive constant κ such that

$$\|\chi_\sigma\|_p \leqq \kappa\|\chi_\sigma\|_1$$

for all $\sigma \in E$. Further, E is a *local Λ set* if there exists a positive constant κ such that

$$\|f\|_p \leqq \kappa p^{1/2}\|f\|_2$$

for all $f \in S_E(G)$ and all $p \in]2, \infty[$. A local Λ set is local Λ_p for every $p \in]1, \infty[$. See §37 of [4] for a general introduction to the theory of lacunary sets.

If G is *profinite* and $\{N_\alpha\}_{\alpha \in A}$ is a neighborhood base at the identity consisting of open normal subgroups of G then each $\sigma \in \hat{G}$ has kernel containing some N_α by Lemma (28.17) of [4]. Thus we may write

$$\hat{G} = \bigcup_{\alpha \in A} (G/N_\alpha)^\wedge$$

if we identify a representation of a quotient of G with a representation of G. We say G is *tall* if for each positive integer n there are only finitely many elements of $\hat{G}$ of degree n. Structural characterisations of tall profinite groups are given in [7]. We will show that a profinite group G admits an infinite (local) Sidon set if and only if G is not tall.

The main theorem.

LEMMA 1. *Let H be an open subgroup of a compact group G having index $[G\colon H] = t$ and let $\{x_1 = 1, x_2, \cdots, x_t\}$ be a set of left coset representatives for H. Then we have*

$$\int_G f(x) d\lambda_G(x) = t^{-1} \sum_{i=1}^{t} \int_H f(x_i h) d\lambda_H(h) \tag{1}$$

for every continuous complex-valued function f on G.

Proof. It is easily verified that the right hand side of (1) defines a positive left invariant normalized measure on G.

LEMMA 2. *Let G and H be as in Lemma 1. If $\sigma \in \hat{G}$ and $|\chi_\sigma(h)| = d_\sigma$ for all $h \in H$ then*

$$\|\chi_\sigma\|_p \geqq d_\sigma / t^{1/p}$$

for all $p \in [1, \infty[$.

Proof. By Lemma 1 we have

$$\begin{aligned} \|\chi_\sigma\|_p^p &= t^{-1} \sum_{i=1}^{t} \int_H |\chi_\sigma(x_i h)|^p d\lambda_H(h) \\ &\geqq t^{-1} \int_H |\chi_\sigma(h)|^p d\lambda_H(h) \\ &= t^{-1} d_\sigma^p \end{aligned}$$

from which the lemma follows at once.

LEMMA 3. *Let G and H be as in Lemma 1 and let f be a continuous complex-valued function on G which vanishes outside H. Define a continuous function g on H by setting $g(h) = f(h)$ for all $h \in H$. Then for $p \in [1, \infty[$ we have*

$$\|f\|_p = t^{-1/p} \|g\|_p \, .$$

Proof. This follows immediately from Lemma 1.

LEMMA 4. *Let G be a compact group and let $E \subset \hat{G}$ be a Λ_p set for some $p \in]1, \infty[$. Suppose that for each $\sigma \in E$ there is an open subgroup H_σ of G of index t_σ and a representation $\tau \in \hat{H}_\sigma$ such that σ is equivalent to the induced representation τ^G. Then we have*

$$\sup\{t_\sigma : \sigma \in E\} < \infty .$$

Proof. For each $\sigma \in E$ define a continuous function f_σ on G by setting

$$f_\sigma(x) = \begin{cases} \chi_\tau(x) & \text{for } x \in H_\sigma \\ 0 & \text{for } x \in G - H_\sigma . \end{cases}$$

Now for each $\rho \in \hat{G}$ we have

$$\rho|_{H_\sigma} \cong \bigoplus_{\upsilon \in \hat{H}_\sigma} n_\rho(\upsilon) \cdot \upsilon$$

where $n_\rho(\upsilon)$ denotes the multiplicity of υ in the representation of H_σ obtained by restricting the domain of ρ. Since we have

$$\hat{f}_\sigma(\rho) = t_\sigma^{-1} \int_{H_\sigma} \chi_\tau(h) \rho(h^{-1}) d\lambda_{H_\sigma}(h)$$

by Lemma 1, the orthogonality relations for H_σ then show that $\hat{f}_\sigma(\rho)$ vanishes for all $\rho \in \hat{G}$ for which $n_\rho(\tau) = 0$. By Frobenius reciprocity, these are all ρ except $\sigma \cong \tau^G$ and so we have that $f_\sigma \in S_E(G)$. Using Lemma 3 and a standard inequality for L^p spaces (see (13.17) of [5]) we have

$$\begin{aligned} \|f_\sigma\|_p &= t_\sigma^{-1/p} \|\chi_\tau\|_p \\ &\geqq t_\sigma^{-1/p} \|\chi_\tau\|_1 \\ &= t_\sigma^{1-1/p} \|f_\sigma\|_1 . \end{aligned}$$

Now if E is a local Λ_p set then there is a positive constant κ such that

$$\|f_\sigma\|_p \leqq \kappa \|f_\sigma\|_1 \quad \text{for all} \quad \sigma \in E$$

so the above calculation shows that

$$t_\sigma^{1-1/p} \leqq \kappa \quad \text{for all} \quad \sigma \in E$$

and this can only happen if

$$\sup\{t_\sigma : \sigma \in E\} < \infty .$$

LEMMA 5. (*Jordan, Blichfeldt*). *Let G be a finite complex linear* *group of degree n. Then G has an abelian normal subgroup A such that*

$$[G: A] < 6^{4n^2/\log n} .$$

Proof. See p. 177 of [3] and observe that

$$n!\, 6^{\pi(n+1)+1} < 6^{4n^2/\log n}$$

where $\pi(m)$ denotes the number of primes not exceeding m.

THEOREM. *Let G be a profinite group and let $E \subset \hat{G}$ be a local Λ set. Then we have*

$$\sup\{d_\sigma : \sigma \in E\} < \infty .$$

Proof. For each $\sigma \in E$ we may apply Lemma 5 to the finite group $G/\ker \sigma$ to obtain an open normal subgroup A_σ of G such that $A_\sigma \supset \ker \sigma$, $A_\sigma/\ker \sigma$ is abelian and

$$[G: A_\sigma] < 6^{4d_\sigma^2/\log d_\sigma} .$$

By Clifford's theorem (see §14 of [3]), for each σ there is an irreducible 1-dimensional representation ξ_σ of A_σ and positive integers e_σ and t_σ such that

$$\sigma|_{A_\sigma} \cong e_\sigma \cdot \{\xi_\sigma^{x_1} \oplus \cdots \oplus \xi_\sigma^{x_{t_\sigma}}\}$$

where $\{x_1 = 1, x_2, \cdots, x_{t_\sigma}\}$ is a set of left coset representatives of the inertia group S_σ given by

$$S_\sigma = \{x \in G: \xi_\sigma^x = \xi_\sigma\}$$

with $[G: S_\sigma] = t_\sigma$. Also for each $\sigma \in E$ we have $\sigma \cong \tau_\sigma^G$ where τ_σ is an irreducible representation of S_σ satisfying $\tau_\sigma|_{A_\sigma} = e_\sigma \cdot \xi_\sigma$. Since E is local A_p for every $p \in]1, \infty[$, we have by Lemma 4 that

$$B = \{\sup t_\sigma : \sigma \in E\} < \infty .$$

Also, since ξ_σ is 1-dimensional, we have for all $x \in A_\sigma$ that

$$|\chi_{\tau_\sigma}(x)| = e_\sigma \cdot |\xi_\sigma(x)| = e_\sigma = d_{\tau_\sigma} .$$

Thus, applying Lemma 2, we get for $p \in]1, \infty[$ that

$$||\chi_{\tau_\sigma}||_p \geqq d_{\tau_\sigma}/[S_\sigma : A_\sigma]^{1/p} .$$

Now define a continuous function f_σ on G by setting

$$f_\sigma(x) = \begin{cases} t_\sigma^{1/2}\chi_{\tau_\sigma}(x) & \text{for } x \in S_\sigma \\ 0 & \text{for } x \in G - S_\sigma . \end{cases}$$

Arguing precisely as in the proof of Lemma 4 we have that $f_\sigma \in S_E(G)$ and, by Lemma 3, we have for $p \in [2, \infty[$ that

$$(2) \qquad ||f_\sigma||_p = t_\sigma^{1/2-1/p}||\chi_{\tau_\sigma}||_p \geqq ||\chi_{\tau_\sigma}||_p .$$

In particular, we have

$$||f_\sigma||_2 = ||\chi_{\tau_\sigma}||_2 = 1 .$$

Taking $p = 4d_\sigma^2/\log d_\sigma$ and observing that

$$d_\sigma = t_\sigma d_{\tau_\sigma} \leqq B\cdot d_{\tau_\sigma}$$

we have from (1) and (2) that

$$\begin{aligned} ||f_\sigma||_{4d_\sigma^2/\log d_\sigma} &\geqq d_{\tau_\sigma}/[S_\sigma\colon A_\sigma]^{\log d_\sigma/4d_\sigma^2} \\ &\geqq B^{-1}d_\sigma/[G\colon A_\sigma]^{\log d_\sigma/4d_\sigma^2} \\ &\geqq d_\sigma/6B . \end{aligned}$$

Now, since E is local Λ, there is a constant κ such that for each $\sigma \in E$ and all $p \in]2, \infty[$ we have

$$||f_\sigma||_p \leqq \kappa p^{1/2}||f_\sigma||_2 = \kappa p^{1/2} .$$

Again taking $p = 4d_\sigma^2/\log d_\sigma$, we then see that

$$d_\sigma/6B \leqq \kappa(4d_\sigma^2/\log d_\sigma)^{1/2}$$

and so we have

$$\log d_\sigma \leqq 144B^2\kappa^2 \quad \text{for all} \quad \sigma \in E .$$

It follows that

$$\sup\{d_\sigma\colon \sigma \in E\} < \infty .$$

COROLLARY. *Let G be a profinite group. The following statements are equivalent*:

(i) *G is tall*;
(ii) *$\hat{G}$ contains no infinite local Λ sets*;
(iii) *$\hat{G}$ contains no infinite local Sidon sets*;
(iv) *$\hat{G}$ contains no infinite Sidon sets.*

Proof. The implication (i) $\Rightarrow$ (ii) follows immediately from the theorem while the implications (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv) are well known (see § 37 of [4]). Finally, the implication (iv) $\Rightarrow$ (i) is contained in Corollary 2.5 of [6].

Complements. A result similar to ours for compact Lie groups may be found in Cecchini [1]. An immediate consequence of our theorem is that if the dual $\hat{G}$ of a profinite group G is a local Λ set then the degrees of the elements of $\hat{G}$ must be bounded. Parker [11] has proved the same conclusion under the weaker assumption that $\hat{G}$ is a local central Λ_4 set. If we restrict G to be a *pro-nilpotent* group (i.e., a projective limit of finite nilpotent groups) then a good deal more can be said with the aid of the following lemma.

LEMMA. *Let G be a finite nilpotent group and let $\sigma \in \hat{G}$. Then we have*

$$||\chi_\sigma||_4^4 > \log d_\sigma .$$

Proof. We show by induction on d_σ that the tensor product representation $\sigma \otimes \sigma$ splits into more than $\log d_\sigma$ irreducible components (not necessarily pairwise inequivalent). The assertion of the lemma then follows immediately. The lemma clearly holds when $d_\sigma = 1$. Now suppose that $d_\sigma > 1$. By Corollary 15.6 of [3] there is a 1-dimensional representation ρ of a subgroup H of G such that $\sigma \cong \rho^G$. Let M be a maximal subgroup of G containing H. Then M is normal in G with prime index q and $\tau = \rho^M$ is an irreducible representation of M satisfying $\sigma \cong \tau^G$. Let $\{x_1 = 1, x_2, \cdots, x_q\}$ be a set of coset representations for M. By Mackey's tensor product theorem (see Theorem 44.3 of [2]) we have

$$\begin{aligned} \sigma \otimes \sigma &\cong \tau^G \otimes \tau^G \\ &\cong (\tau \otimes \tau)^G \oplus \left[\bigoplus_{i=2}^{q} (\tau^{x_i} \otimes \tau)^G \right] . \end{aligned}$$

By induction $\tau \otimes \tau$, and therefore $(\tau \otimes \tau)^G$, splits into more than $\log d_\tau$ components. Thus, if m is the number of irreducible components of $\sigma \otimes \sigma$ counted according to multiplicity, then

$$\begin{aligned} m &> \log d_\tau + q - 1 \\ &> \log d_\tau + \log q \\ &= \log d_\sigma . \end{aligned}$$

PROPOSITION. *Let G be a pro-nilpotent group and let $E \subset \hat{G}$ be either a local central Λ_4 set for a local Λ_p set or some $p \in]1, \infty[$. Then we have*

$$\sup\{d_\sigma : \sigma \in E\} < \infty .$$

Proof. By our opening remarks every continuous irreducible representation of G is essentially a representation of a finite nilpotent quotient of G. Thus, if E is a local central Λ_4 set, then the preceding lemma shows that $\sup\{d_\sigma: \sigma \in E\}$ is finite. If E is a local Λ_p set then, since each $\sigma \in \hat{G}$ is induced from a 1-dimensional representation of an open subgroup of index d_σ, Lemma 4 shows that $\sup\{d_\sigma: \sigma \in E\}$ is finite.

EXAMPLE. Let $G = \prod_{n=6}^{\infty} A_n$ where for each n A_n is the alternating group on n letters. By Theorem 2.5 of [7] G is tall so $\hat{G}$ contains no infinite local Λ sets by our theorem. However $\hat{G}$ does contain an infinite local central Λ_4 set. For each A_n has an irreducible representation σ_n of degree $n-1$ obtained by restricting to A_n the irreducible representation of S_n (the symmetric group on n letters) afforded by the partition $[n-1, 1]$ of n. From p. 766 of [9] we have that $\sigma_n \otimes \sigma_n$ splits into 4 irreducible components. Thus, if π_n is the projection of G onto A_n, then $E = \{\sigma_n \circ \pi_n: n = 6, 7, \cdots\}$ is an infinite local central Λ_4 set for G. In addition, Corollary 4.2 of [10] shows that E is a central Sidon set. Thus G is a profinite group which admits infinite central Sidon sets but no infinite Sidon set. In view of Theorem 9 of [13] and §§ 3, 4 of [6] it is unlikely that such examples exist when G is connected.

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