PACIFIC JOURNAL OF MATHEMATICS
Vol. 89, No. 2, 1980

SOLUTION OF THE MIDDLE COEFFICIENT PROBLEM FOR CERTAIN CLASSES OF C-POLYNOMIALS

ZALMAN RUBINSTEIN

A well-known conjecture states that for polynomials having all their zeros on the unit circle C half the maximum modulus on C bounds the modulus of all the coefficients. This has been established in all cases except for the middle coefficient of even degree polynomials greater than four. In this note this conjecture is verified for all even degree polynomials having simple zeros in a set of arcs dividing the circle into equal parts and related classes of polynomials. The local extremal polynomials are identified.

1. Introduction. Throughout this note polynomials whose zeros all lie on the unit circumference $C = \{z \mid |z| = 1\}$ will be considered and refered to as C-polynomials. If P is a polynomial $M(P) = \text{Max}_{z \in C} |P(z)|$. Also $P^*(z) = z^n \overline{P(1/\bar{z})}$, where n is the degree of P.

The following conjecture due to P. Erdös was stated in [3], and in corrected form in [4].

Conjecture 1. Let

$$P(z) = a_n z^n + \cdots + a_1 z + a_0 \tag{1}$$

be a C-polynomial. Then $|a_i| \leqq M(P)/2$ for $i = 0, 1, \cdots, n$. This conjecture was established in [5] and [6] for all cases except $n = 2k$ and $i = k$. In [6] another conjecture was raised in this connection.

Conjecture 2. If the zeros of $P(z)$ in (1) all lie on the exterior of C then $|a_i| \leqq M(P)/2$ for $n/2 \leqq i \leqq n$. For comparison we add

Conjecture 3. If the degree of $P(z)$ in (1) is even $n = 2k$ then $|a_k| \leqq M(P)/2$.

The special significance of Conjecture 3 is that it actually is equivalent to Conjectures 1 and 2 but its statement is the most economical. This is summarized in

LEMMA 1. *Conjecture* 3 *implies Conjectures* 1 *and* 2.

Proof. (a) If $P(z)$ is a C-polynomial given by (1) then $P^2(z) = c_{2n}z^n + \cdots + c_n z^n + \cdots + c_0$ is an even degree C-polynomial and $c_n =$

$u(|a_0|^2 + \cdots + |a_n|^2)$ where $|u| = 1$ and $a_k = u\bar{a}_{n-k}$. If Conjecture 3 is true

$$2|c_n| \leqq M(P^2) = M^2(P) .$$

In particular

$$2|a_i|^2 = |a_i|^2 + |a_{n-i}|^2 \leqq \frac{M^2(P)}{2}$$

if $i \neq n/2$, so that $|a_i| \leqq M(P)/2$.

(b) Let $Q(z) = b_n z^n + \cdots + b_0$ be a polynomial of degree n whose zeros all lie on the closed exterior of C. The polynomials $Q(z) + e^{i\theta}z^m Q^*(z)$ are easily seen to be C-polynomials for any non-negative integer m and any real θ. Indeed $|Q_n(z)| \leqq |Q^*(z)|$ for $|z| \geqq 1$ and $|Q(z)| \geqq |Q^*(z)|$ for $|z| \leqq 1$. Thus we construct C-polynomials of degree $(m + n)$ whose kth coefficient is $b_k + e^{i\theta}b_{n-k+m}$. Since $|Q^*(z)| \leqq |Q(z)|$ on $|z| = 1$, for a suitable choice of θ we have

$$|b_k| + |b_j| \leqq \frac{1}{2}(2M(Q)) = M(Q)$$

for $j = n - k,\ n - k + 1, \cdots, n$. If $k \geqq n/2$ we can also choose $j = k$ to obtain $|b_k| \leqq M(Q)/2$. This concludes the proof of the lemma.

We may also remark that the above mentioned conjectures have corresponding counterparts for trigonometric polynomials $T_n(\theta)$ of degree n with only real zeros. It is easily seen that Conjecture 1 can be stated as an integral inequality

$$\left|\int_0^{2\pi} T_n(\theta)d\theta\right| \leqq \pi M(T_n) \tag{2}$$

where $M(T_n(\theta)) = \mathrm{Max}_{\theta \in R}|T_n(\theta)|$.

For these polynomials the inequality

$$\int_0^{2\pi} |T_n(\theta)|d\theta \leqq 4M(T_n)$$

was conjectured by P. Erdös and established in [6].

In this note we shall verify Conjecture 3 for several classes of C-polynomials including the family of C-polynomials whose zeros lie on fixed disjoint open arcs of length $2\pi/n$ on the unit circle one zero on each arc. Although the proof of the main theorem will focus on this family other classes are mentioned following the proof.

2. The main results. Let $n = 2k$. Denote by $S_0, \cdots, S_{2k-1}$ the open disjoint arcs on the unit circle of length $\phi = \pi/k$ whose endpoint are the nth roots of unity. Also let $S_0^\varepsilon, \cdots, S_{2k-1}^\varepsilon$ for $\varepsilon > 0$ sufficiently small denote the closed arcs obtained from $S_0, \cdots, S_{2k-1}$ by deleting two symmetric subarcs of length ε on each end. π_n and $\pi_{n,\varepsilon}$ shall denote the class of C-polynomials of degree n whose zeros $z_j \in S_j$ and $z_j \in S_j^\varepsilon$ respectively ($j = 0, 1, \cdots, n-1$). Furthermore if

$$Q(z) = q_{2k}z^{2k} + \cdots + q_k z^k + \cdots + q_0 \tag{3}$$

then

$$M(Q) = \max_{z \in C} |Q(z)| \quad \text{and} \quad L(Q) = |a_k|/M(Q) .$$

We shall consider the values

$$\alpha = \sup_{Q \in \pi_n} L(Q) \tag{4}$$

and

$$\alpha_\varepsilon = \max_{Q \in \pi_{n\,\varepsilon}} L(Q) .$$

It is clear that monic C-polynomials P_0 and P_ε assuming the extremal values α and α_ε exist. Moreover they can be chosen such that $P_0 \in \bar{\pi}_n$ and $P_\varepsilon \in \pi_{n,\varepsilon}$ where the closure is the uniform closure on compact subsets in the plane.

We shall write

$$P_\varepsilon(z) = a_{2k,\varepsilon}z^{2k} + \cdots + a_{k,\varepsilon}z^k + \cdots + a_{0,\varepsilon} . \tag{5}$$

The zeros of $P_\varepsilon(z)$ all lie in $\bigcup_{j=0}^{2k-1} S_j^\varepsilon$, they are simple and there is exactly one zero in each of the arcs S_j^ε.

In the proof of the main theorem we shall need two auxiliary results.

THEOREM A. [1] *Let Γ be a circle in the complex plane and let γ_i ($i = 1, 2, \cdots, n$) be disjoint open arcs on Γ. Let $z_0 \in \Gamma - \bigcup_{i=1}^n \gamma_i$. Then for any $w_0 \neq 0$, the set of polynomials P of degree n having exactly one zero in each of the arcs γ_i and satisfying $P(z_0) = w_0$ is convex.*

The next result applies to all regular functions.

LEMMA 2. [2] *Let $w(z)$ be regular in the unit disk, with $w(0) = 0$. Then if $|w|$ attains its maximum value on the circle $|z| = r$ at a point ζ, we can write*

$$\zeta w'(\zeta) = kw(\zeta)$$

where $k = k\ (|\zeta|, w)$ is real and $k \geqq 1$.

Now we state the main theorem.

THEOREM. *Let* $Q(z) = q_{2k}z^{2k} + \cdots + q_k z^k + \cdots + q_0$ *be a C-polynomial of degree* $2k$.

(a) *If* $Q \in \pi_{2k,\varepsilon}$ *then*

$$\frac{|q_k|}{M(Q)} \leqq \frac{1}{1 + \sec \varepsilon} \tag{6}$$

and all extremal polynomials are of the same form $cP^*(ze^{i\psi})$ *where*

$$P^*(z) = z^{2k} + 2\cos \varepsilon z^k + 1 . \tag{7}$$

(b) *For all* $Q \in \bar{\pi}_{2k}$, $|q_k|/M(Q) \leqq 1/2$.

Proof. Let $P_\varepsilon(z)$ be an extremal polynomial in the class $\pi_{n,\varepsilon}$, given by (5). For $\phi = \pi/k$ and fixed $\varepsilon > 0$ let $R_j(z) = P_\varepsilon(ze^{ij\phi})$, $j = 0, 1, \cdots, 2k-1$. Then $R_j(z) \in \pi_{n,\varepsilon}$ and hence by Theorem A the polynomials

$$R(z) = \sum_{j=0}^{2k-1} \alpha_j \frac{R_j(z)}{R_j(1)}$$

are also in $\pi_{n,\varepsilon}$ for $\alpha_j \geqq 0$, $\sum_{j=0}^{2k-1} \alpha_j = 1$. Since $R(z)$ is a solution of the extremal problem (4) we have

$$L\left(\sum_{j=0}^{2k-1} \alpha_j \frac{R_j(z)}{R_j(1)}\right) \leqq L(R_0) . \tag{8}$$

(8) can be written as

$$M(R_0)\left|\sum_{j=0}^{2k-1} \frac{\alpha_j e^{i\pi j}}{R_j(1)}\right| \leqq M\left(\sum_{j=0}^{2k-1} \alpha_j \frac{R_j(z)}{R_j(1)}\right) . \tag{9}$$

Since $R_0(z) = P_\varepsilon(z) = a_{2k,\varepsilon}z^{2k} + \cdots + a_{k,\varepsilon}z^k + \cdots a_{0,\varepsilon}$ is a C-polynomial $a_{2k-m,\varepsilon} = u\bar{a}_{m,\varepsilon}$ for $m = 0, 1, \cdots, 2k$ and for some $u = e^{i\theta}$. Therefore $R_0(z)z^{-k}e^{(-i\theta/2)}$ is real on C. In particular the numbers $r_j = R_j(1)e^{-\pi j i}e^{-(i\theta/2)}$ are real for $j = 0, 1, \cdots, 2k-1$. Moreover since R_0 has $2k$ simple zeros on the intervals S_j^ε the numbers r_j have constant sign. Thus (9) can be written in the form

$$M(R_0) \leqq M\left(\sum_{j=0}^{2k-1} (-1)^j b_j R_j\right)$$

for $b_j \geqq 0$, $\sum_{j=0}^{2k-1} b_j = 1$.
Observing that $M(R_j) = M(R_0)$ and letting all the even b_j (or the odd b_j) equal to zero we obtain

$$M(R_0) \leqq M\left(\sum_{l=0}^{k-1} b_{2l}R_{2l}\right) \leqq \sum_{l=0}^{k-1} b_{2l}M(R_{2l}) = M(R_0) . \tag{10}$$

(10) implies the existence of a point z_1 on C such that $R_{2l}(z_1) =$

$M(R_0)e^{i\alpha}$ for $l = 0, \cdots, k-1$. The polynomial R_0 assumes maximum modulus at k symmetrically situated points on C. Moreover it assumes there the same value. Therefore setting $M_0 = M(R_0)$ we have

$$R_0(z) - M_0 e^{i\alpha} = (z^k - \gamma)q(z) \tag{11}$$

where $q(z)$ is a polynomial of degree k and $\gamma \in C$. Let w_l, $l = 0, \cdots, k-1$ denote the kth roots of γ. We have by (11)

$$q(w_l) = \frac{R_0(w_l)}{kw_l^{k-1}} = \frac{1}{k\gamma} w_l R_0'(w_l) . \tag{12}$$

By Lemma 2 applied to the analytic functions $zR_0(z)$ there exists a nonnegative constant c independent of w_l such that

$$w_l R_0'(w_l) = cR_0(w_l) \tag{13}$$

for $l = 0, 1, \cdots, k-1$. Combining (12) and (13) we have

$$q(w_l) = c_1 R_0(w_l) \tag{14}$$

where $c_1 = c/k\gamma$. Hence by (14)

$$q(z) - c_1 R_0(z) = (z^k - \gamma)s(z) \tag{15}$$

where $s(z)$ is a polynomial of degree k.

By (11) and (15) $(z^k - \gamma)$ divides the polynomials $(c_1 R_0 - M_0 c_1 e^{i\alpha})$ and $(q - c_1 R_0)$ and therefore divides the polynomial $(q - M_0 c_1 e^{i\alpha})$. Since q is of degree k

$$q(z) = c_2 z^k + c_3$$

for some constants c_2 and c_3.

Finally by (11)

$$R_0(z) = a_{2k,\varepsilon} z^{2k} + a_{k,\varepsilon} z^k + a_{0,\varepsilon} .$$

It is now easy to evaluate $L(R_0)$. For a second degree polynomial $t(w) = (w - \zeta)(w - \bar{\zeta})$ the maximum of $|t(w)|$ is attained at the points $w = 1$ or $w = -1$ or both. Therefore

$$L(R_0) = \frac{|\mathrm{Re}\,\zeta|}{1 + |\mathrm{Re}\,\zeta|} .$$

$x/(1 + x)$ is increasing for $x > 0$. This establishes (6) for the class $\pi_{2k,\varepsilon}$ for all sufficiently small ε (actually one may restrict ε to $0 < \varepsilon < \pi/2k$).

The preceeding argument also easily implies (7) for the extremals of $\pi_{n,\varepsilon}$ up to the transformations indicated. Every polynomial in π_n is a uniform limit of polynomials in $\pi_{n,\varepsilon}$ as $\varepsilon \to 0$. This completes the proof.

We conclude with a few corollaries.

COROLLARY 1. *If $T_n(\theta)$ is a trigonometric polynomial of degree n whose $2n$ zeros all lie on $2n$ disjoint adjacent closed intervals which can be mapped by a linear transformation onto $2n$ symmetric equal arcs on C, one zero in each interval, then $T_n(\theta)$ satisfies the sharp inequality* (2).

COROLLARY 2. *Conjecture 3 remains true if the zeros of the C-polynomials of degree $2k$ considered lie pairwise on disjoint arcs of length π/k provided no two such pairs lie on disjoint arcs* mod (π/k).

This follows from the fact that only rotations by multiples of π/k were used in the proof of the theorem and the condition above allows the application of Theorem A. As an example consider a C-polynomial of degree $2k$ whose zeros have arguments $(\pi/2k_j \pm \varepsilon_j)$, $j = 1, 3, \cdots, 2k-1$, where $\varepsilon_j (0 < \varepsilon_j < \pi/2k)$ is a monotonic sequence of positive numbers.

We finally remark that the method outlined here can be applied to other extremal problems such as the case where the coefficient of the polynomial is arbitrary.

Added in Proof. Conjecture 3 in form (2) has been recently established by G. K. Kristiansen in the paper "Some inequalities for algebraic and trigonometric polynomials" J. London Math. Soc. (2), **20** (1979), 300–314. The estimate of the main theorem of this paper is independent of the above mentioned result.

REFERENCES

1. H. J. Fell, *On the zeros of convex combinations of polynomials*, Pacific. J. Math., (to appear).

2. I. S. Jack, *Functions stralike and convex of order* α, J. London Math. Soc., (2) (1971), 469–474.

3. W. K. Hayman, *Research problems in function theory*, Athlone Press, London, 1967.

4. ———, *Research Problems in Function Theory*, Symposium on Complex Analysis Canterbury, (1973), 143–154.

5. P. J. O'hara and R. S. Rodriguez, *Some properties of self-inversive polynomials*, Proc. Amer. Math. Soc., **44** (1974), 331–335.

6. E. B. Saff and T. Sheil-Small, *Coefficient and integral mean estimates for algebraic and trigonometric polynomials with restricted zeros*, J. London Math. Soc., (2) **9** (1974), 16–22.

Received June 5, 1979.

THE UNIVERSITY OF MICHIGAN
ANN ARBOR, MI 48109
AND
UNIVERSITY OF HAIFA
HAIFA, ISRAEL