PACIFIC JOURNAL OF MATHEMATICS
Vol. 89, No. 2, 1980

HOMOTOPY CLASSIFICATION OF LENS SPACES FOR ONE-RELATOR GROUPS WITH TORSION

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Let Ξ be a one-relator group with presentation $\mathscr{R} = (x_1, \cdots, x_n: R^p)$ where R is not a proper power and $p \geqq 2$. Then given any integer q, relatively prime to p, we can construct the Lens space $\mathscr{L}(p, q)$ for Ξ from the cellular model $C(\mathscr{R})$ of the presentation $\mathscr{R}$ by attaching a 3-cell via the attaching map $R^q - 1$, which generates the ideal $Z\Xi(R-1) \approx \pi_2(C(\mathscr{R}))$. In this paper we classify these Lens spaces up to homotopy type. We also discuss the non-cancellation aspect of these Lens spaces.

Introduction. In this paper we are interested in Lens spaces for one-relator groups with torsion. Given relatively prime integers p and q, with $p \geqq 2$, we have the ordinary Lens space $L(p, q)$ with fundamental group finite cyclic of order p. The 2-skeleton of $L(p, q)$ is the cellular model $C(\mathscr{R})$ of the presentation

$$\mathscr{R} = (x: x^p)$$

and $L(q, p)$ is obtained from its 2-skeleton by attaching a 3-cell via the attaching map $x^q - 1$, which generates the ideal $ZZ_p(x-1) \approx \pi_2(C(\mathscr{R}))$. The cellular chain complex of the universal covering $\tilde{L}(p, q)$ of $L(p, q)$ is given by

$$C_*(L(p, q)): ZZ_p \xrightarrow{x^q - 1} ZZ_p \xrightarrow{1 + x + \cdots + x^{p-1}} ZZ_p \xrightarrow{x-1} ZZ_p$$

where x is a generator of the cyclic group Z_p. J. H. C. Whitehead [11] has shown that $L(p, q)$ and $L(p, r)$ have the same homotopy type if and only if qr or $-qr$ is a quadratic residue mod p. We consider the following analogue: Let Ξ be a one-relator group with presentation

$$(1) \qquad \mathscr{R} = (x_1, \cdots, x_n: R^p)$$

where R is not a proper power and $p \geqq 2$. Then given any integer q, relatively prime to p, we can construct the Lens space $\mathscr{L}(p, q)$ obtained from the cellular model $C(\mathscr{R})$ of the presentation R by attaching a 3-cell via the attaching map $R^q - 1$, which generates the ideal $Z\Xi(R-1) \approx \pi_2(C(\mathscr{R}))$. Clearly the fundamental group of $\mathscr{L}(p, q)$ is isomorphic to Ξ. The cellular chain complex for the universal covering $\tilde{\mathscr{L}}(p, q)$ of the Lens space $\mathscr{L}(p, q)$ is given by

$$C_*(\tilde{\mathscr{L}}(p, q)): Z\Xi \xrightarrow{R^q - 1} Z\Xi \xrightarrow{\partial_2} (Z\Xi)^n \xrightarrow{\partial_1} Z\Xi$$

where $\partial_1 = (x_1 - 1, \cdots, x_n - 1)$ and ∂_2 is Jacobian matrix of the presentation $\mathscr{R}$ described in the free differential calculus of R. H. Fox [2].

DEFINITION. An element R of free group F generated by $x_1, \cdots, x_n$ is *primitive* if it is a member of a free basis for F.

THEOREM 1. *Let Ξ have presentation* (1) *with single relator R^p a power of a primitive element R of F. Then two Lens spaces $\mathscr{L}(p, q)$ and $\mathscr{L}(p, r)$ for Ξ have the same homotopy type if and only if qr or $-qr$ is a quadratic residue* mod p.

Thus for these "power of a primitive" one-relator groups, the homotopy classification for the Lens spaces $\mathscr{L}(p, q)$ for Ξ is identical to that for the ordinary Lens spaces $L(p, q)$ for the cyclic group Z_p. We are able to solve the classification problem for the Lens spaces $\mathscr{L}(p, q)$ for the remaining one-relator group modulo this conjecture:

CONJECTURE. Let Ξ be a one-relator group given by the presentation (1). If R is not primitive in the free group F generated by $x_1, \cdots, x_n$, then each automorphism $\alpha \in \operatorname{Aut} \Xi$ is induced by an automorphism of the free group F. (See S. Pride [7].)

THEOREM 2. *Let Ξ have presentation* (1) *with single relator R^p a power of a nonprimitive element R of the free group F. If the above conjecture holds for Ξ, then two Lens spaces $\mathscr{L}(p, q)$ and $\mathscr{L}(p, r)$ for Ξ have the same homotopy type if and only if $q \equiv \pm r$* mod p.

The outline of this paper is as follows. In §1, we give the proofs of Theorems 1 and 2. In §2, we discuss the automorphisms of one-relator groups and give several examples for which the above conjecture holds. Finally, in §3, we discuss the non-cancellation aspect of these Lens spaces.

§0. **Notation.** For simplicity, we employ the same notation for elements of F and Ξ. We let $Z\Xi$ denote the integral group ring of Ξ. All $Z\Xi$-modules are left $Z\Xi$-modules. Any element $w \in Z\Xi$ defines a left $Z\Xi$-module homomorphism $w: Z\Xi \to Z\Xi$ given by the right multiplication. For $w \in \Xi$ and a positive integer s, we let $\langle w, s\rangle = 1 + w + \cdots + w^{s-1}$ and $\langle w, -s\rangle = -w^{-s}\langle w, s\rangle$ in $Z\Xi$. We have the following $\langle\ \rangle$-identities

$$(w - 1)\langle w, s\rangle = w^s - 1\,, \qquad \langle w, s\rangle + w^s\langle w, t\rangle = \langle w, s + t\rangle\,,$$
$$\langle w, s\rangle\langle w^s, t\rangle = \langle w, st\rangle\,.$$

If $w \in \Xi$ has order p, then $\langle w, s\rangle\langle w, p\rangle = s\langle w, p\rangle$.

All the spaces in this paper are connected CW-complexes with some zero cell chosen as basepoint which is preserved by all maps and homotopies.

1. Proofs. Throughout this paper Ξ will be a one-relator group given by presentation (1). We denote by S the element $\langle R, p\rangle = 1 + R + \cdots + R^{p-1}$ of the integral group ring $Z\Xi$.

The following is a Ξ-resolution of the trivial Ξ-module Z (see R. Lyndon [4]):

$$\cdots \xrightarrow{S} Z\Xi \xrightarrow{R-1} Z\Xi \xrightarrow{S} Z\Xi \xrightarrow{R-1} Z\Xi$$
$$\xrightarrow{\partial_2} (Z\Xi)^n \xrightarrow{\partial_1} Z\Xi \xrightarrow{\varepsilon} Z \longrightarrow 0$$

where $\varepsilon\colon Z\Xi \to Z$ is the augmentation homomorphism,

$$\partial_1 = (x_1 - 1, \cdots, x_n - 1)\ , \quad \text{and} \quad \partial_2 = S(\partial R/\partial x_1, \cdots, \partial R/\partial x_n)\ .$$

To the Lens space $\mathscr{L}(p, q)$ for Ξ, we can associate its algebraic 3-type $T(\mathscr{L}(p, q)) = (\Xi, Z\Xi S, k)$ where $\pi_1\mathscr{L}(p, q) = \Xi$, $\pi_3(\mathscr{L}(p, q)) = Z\Xi S$ and $k \in H^4(\Xi, Z\Xi S)$ is the obstruction invariant of MacLane-Whitehead [5]. An isomorphism

$$(\Phi, \Psi)\colon T(\mathscr{L}(p, q)) \longrightarrow T(\mathscr{L}(p, r)) = (\Xi, Z\Xi S, k')$$

between the algebraic 3-types consists of a group isomorphism Φ: $\Xi \to \Xi$ and a $Z\Xi$-module isomorphism $\Psi\colon Z\Xi S \to {}_\Phi Z\Xi S$ for which $\Phi^*(k') = \Psi_*(k)$ in the diagram:

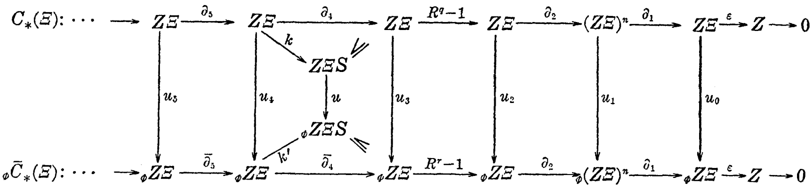
$$H^4(\Xi, Z\Xi S) \xrightarrow{*\Psi} H^4(\Xi, {}_\Phi Z\Xi S) \xleftarrow{\Phi^*} H^4(\Xi, Z\Xi S)\ .$$

Note that we view ${}_\Phi Z\Xi S$ as a $Z\Xi$-module this way: $w \cdot \alpha = \Phi(w)\alpha$ where $w \in Z\Xi$ and $\alpha \in Z\Xi S$. It is known that two Lens spaces $\mathscr{L}(p, q)$ and $\mathscr{L}(p, r)$ are homotopically equivalent if and only if $T(\mathscr{L}(p, q))$ is isomorphic to $T(\mathscr{L}(p, r))$. (See Theorems 4 and 5 of MacLane-Whitehead [5] and Theorem 15 of Whitehead [12]).

The cellular chain complex $C_*(\tilde{\mathscr{L}}(p, q))$ provides us with a partial free resolution $\varepsilon\colon C_*(\tilde{\mathscr{L}}(p, q)) \to Z$ which we can extend to get a free resolution $\varepsilon\colon C_*(\Xi) \to Z$ of the trivial module Z over $Z\Xi$ (see [3]). Likewise we also obtain the free resolution $\varepsilon\colon \bar{C}_*(\Xi) \to Z$ of the trivial module Z over $Z\Xi$ by extending the partial free resolution $\varepsilon\colon C_*(\tilde{\mathscr{L}}(p, r)) \to Z$. Let u be any chain map

$$u\colon C_*(\Xi) \longrightarrow \bar{C}_*(\Xi)$$

extending the identity map on Z. Thus we have the commutative diagram:



where u also denotes the restriction of the chain map u_3 to $Z\Xi S$. The commutativity relation $u_3\partial_4 = \bar{\partial}_4 u_4$ implies that $u_*(k) = \Phi^*(k')$. (See [1].) Therefore $\mathscr{L}(p, q)$ and $\mathscr{L}(p, r)$ are homotopically equivalent if and only if for some chain map u, $u = u_3 | Z\Xi S \colon Z\Xi S \to {}_{\Phi}Z\Xi S$ is a $Z\Xi$-module isomorphism; for then (Φ, u) constitutes an isomorphism of their algebraic 3-types.

We recall now the following two results of R. H. Fox [2] which are crucial to the following lemma.

I. Fundamental formula of free calculus.

Let F denote the free group generated by $x_1, \cdots, x_n$ and let $v \in ZF$. Then

$$\sum_{i=1}^{n} \frac{\partial v}{\partial x_j}(x_j - 1) = v - \varepsilon(v)$$

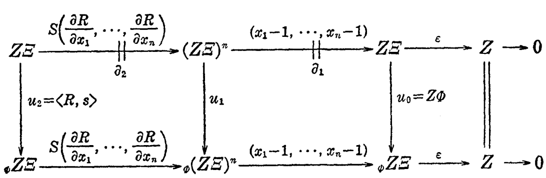
where $\varepsilon \colon ZF \to Z$ is the augmentation map.

II. Chain rule of differentiation.

Let λ be a homomorphism from a free group Y into a free group X. Then, for any $v \in ZY$,

$$\frac{\partial}{\partial x_j}\lambda(v) = \sum_k \lambda\left(\frac{\partial v}{\partial y_k}\right)\frac{\partial}{\partial x_j}\lambda(y_k) .$$

Lemma 1. *Let $\Phi \colon \Xi \to \Xi$ be a group homomorphism such that $\Phi(R) = R^s$, $(s, p) = 1$. Then the following commutes:*



where

$$u_1 = \begin{pmatrix} \dfrac{\partial\Phi(x_1)}{\partial x_1} & \cdots & \dfrac{\partial\Phi(x_1)}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial\Phi(x_n)}{\partial x_1} & \cdots & \dfrac{\partial\Phi(x_n)}{\partial x_n} \end{pmatrix}_{n \times n} .$$

Proof. To see that $u_0\partial_1 = \partial_1 u_1$, we consider the ith column of the matrix $\partial_1 u_1$:

$$\frac{\partial}{\partial x_1}\Phi(x_i)(x_1 - 1) + \cdots + \frac{\partial}{\partial x_n}\Phi(x_i)(x_n - 1)$$

$= \Phi(x_i) - 1$ [by the fundamental formula of free calculus], which is the ith column of the matrix $u_0\partial_1$. That $u_1\partial_2 = \partial_2 u_2$ follows from the relations

$$\begin{aligned}
u_1\partial_2 &= \Phi(S)\Big[\Phi\Big(\frac{\partial R}{\partial x_1}\Big)\Big(\frac{\partial}{\partial x_1}\Phi(x_1), \cdots, \frac{\partial}{\partial x_n}\Phi(x_1)\Big) + \cdots \\
&\qquad \cdots + \Phi\Big(\frac{\partial R}{\partial x_n}\Big)\Big(\frac{\partial}{\partial x_1}\Phi(x_n), \cdots, \frac{\partial}{\partial x_n}\Phi(x_n)\Big)\Big] \\
&= S\Big(\Phi\Big(\frac{\partial R}{\partial x_1}\Big)\frac{\partial}{\partial x_1}\Phi(x_1) + \cdots + \Phi\Big(\frac{\partial R}{\partial x_n}\Big)\frac{\partial}{\partial x_1}\Phi(x_n), \cdots \\
&\qquad \cdots, \Phi\Big(\frac{\partial R}{\partial x_1}\Big)\frac{\partial}{\partial x_n}\Phi(x_1) + \cdots + \Phi\Big(\frac{\partial R}{\partial x_n}\Big)\frac{\partial}{\partial x_n}\Phi(x_n)\Big) \\
&= S\Big(\frac{\partial}{\partial x_1}\Phi(R), \cdots, \frac{\partial}{\partial x_n}\Phi(R)\Big) \quad \text{[by chain rule]} \\
&= S\Big(\frac{1}{\partial x_1}R^s, \cdots, \frac{\partial}{\partial x_n}R^s\Big) \\
&= \langle R, s\rangle S\Big(\frac{\partial R}{\partial x_1}, \cdots, \frac{\partial R}{\partial x_n}\Big) \\
&= \partial_2 u_2 .
\end{aligned}$$

This completes the proof. □

The above lemma can be used to simplify the proof of Theorem 1 of [3].

Next we show that all inner automorphisms of Ξ are contained in the image of the evaluation homomorphism $\sharp\colon \mathscr{E}(\mathscr{L}(p, q)) \to \operatorname{Aut}\Xi$ where $\mathscr{E}(\mathscr{L}(p, q))$ is the self-equivalence group of the Lens space $\mathscr{L}(p, q)$ for Ξ.

LEMMA 2. *Let Φ be an inner automorphism of the one-relator group Ξ given by presentation* (1). *Then $\Phi \in \sharp(\mathscr{E}(\mathscr{L}(p, q)))$.*

Proof. We may assume that $\Phi(g) = x_1 g x_1^{-1}$ for all g in Ξ. Then we have the diagram:

$$\begin{array}{ccccccccccccc}
C_*(\tilde{\mathscr{L}}(p, q))\colon 0 & \to & Z\Xi S & \longrightarrow & Z\Xi & \xrightarrow{R^q - 1} & Z\Xi & \xrightarrow{\left(\frac{\partial R^p}{\partial x_i}\right)} & (Z\Xi)^n & \xrightarrow{(x_i - 1)} & Z\Xi & \xrightarrow{\varepsilon} & Z \to 0 \\
 & & \downarrow x_1 & & u_3 \downarrow = x_1 & & u_2 \downarrow = x_1 & & \downarrow u_1 & & \downarrow Z\Phi & & \| \\
{}_\Phi C_*(\tilde{\mathscr{L}}(p, q))\colon 0 & \to & {}_\Phi Z\Xi S & \longrightarrow & {}_\Phi Z\Xi & \xrightarrow{R^q - 1} & {}_\Phi Z\Xi & \xrightarrow{\left(\frac{\partial R^p}{\partial x_i}\right)} & {}_\Phi(Z\Xi)^n & \xrightarrow{(x_i - 1)} & {}_\Phi Z\Xi & \xrightarrow{\varepsilon} & Z \to 0
\end{array}$$

where u_1 is as in Lemma 1. Again, using the chain rule, one can show that u_1 and u_2 make the diagram commute. Because the multiplication $x_1\colon Z\Xi S \to {}_\Phi Z\Xi S$ by the group element x_1 is clearly an isomorphism, there is a self-homotopy equivalence $f\colon \mathscr{L}(p, q) \to \mathscr{L}(p, q)$ inducing the inner automorphism $f_\# = \Phi$ on the fundamental group (see the preliminary remarks at the begining of this section). $\square$

PROPOSITION 1. *Let $\Phi\colon \Xi \to \Xi$ be an automorphism. Then there is a positive integer s such that $\Phi(R) = gR^s g^{-1}$ where $g \in \Xi$ and $(p, s) = 1$. Moreover there exists a homotopy equivalence*

$$\mathscr{L}(p, q) \longrightarrow \mathscr{L}(p, qs^2)$$

inducing Φ on the fundamental groups.

Proof. By Theorem 4.13 on page 269 of Magnus, Karrass, and Solitar [6], there is a positive integer s such that $\Phi(R) = gR^s g^{-1}$; moreover, $(p, s) = 1$ since $\Phi(R)$ has order p. Since Lemma 2 handles the problem of conjugation, we may assume that $\Phi(R) = R^s$ and so we have $\Phi(S) = S$. Thus in view of Lemma 1, we have the commutative diagram

$$\begin{array}{lccccccccccccc}
C_*(\mathscr{L}(p, q))\colon & 0 \to & Z\Xi S & \to & Z\Xi & \xrightarrow{R^q - 1} & Z\Xi & \xrightarrow{\partial_2} & (Z\Xi)^n & \xrightarrow{\partial_1} & Z\Xi & \xrightarrow{\varepsilon} & Z & \to 0 \\
 & & & & & & \downarrow{\scriptstyle u_2 = \langle R, s\rangle} & & \downarrow{\scriptstyle u_1} & & \downarrow{\scriptstyle u_0} & & \| & \\
{}_\Phi C_*(\mathscr{L}(p, qs^2))\colon & 0 \to & {}_\Phi Z\Xi S & \to & {}_\Phi Z\Xi & \xrightarrow{R^{qs^2} - 1} & {}_\Phi Z\Xi & \xrightarrow{\partial_2} & {}_\Phi(Z\Xi)^n & \xrightarrow{\partial_1} & {}_\Phi Z\Xi & \xrightarrow{\varepsilon} & Z & \to 0
\end{array} .$$

Because $(qs^2, p) = 1$, there exist integers a and b such that $aqs^2 + bp = 1$. We let $u_3 = \langle R, qs\rangle\langle R, s\rangle\langle R^{qs^2}, a\rangle$ which makes the square commute since

$$\begin{aligned}
\Phi(R^q - 1)\langle R, s\rangle &= (R^{qs} - 1)\langle R, s\rangle \\
&= \langle R, qs\rangle\langle R, s\rangle(R - 1)
\end{aligned}$$

and

$$\begin{aligned}
&\langle R, qs\rangle\langle R, s\rangle\langle R^{qs^2}, a\rangle(R^{qs^2} - 1) \\
&\quad = \langle R, qs\rangle\langle R, s\rangle\langle R^{qs^2}, a\rangle\langle R, qs^2\rangle(R - 1) \\
&\quad = \langle R, qs\rangle\langle R, s\rangle\langle R, aqs^2\rangle(R - 1) \\
&\quad = \langle R, qs\rangle\langle R, s\rangle(R - 1) .
\end{aligned}$$

Let $b\colon Z\Xi \to {}_\Phi Z\Xi S$ be the $Z\Xi$-module homomorphism given by $1 \to bS$. Then $u_3 + b$ restricts to $Z\Xi S$ to give the $Z\Xi$-module homomorphism $u\colon Z\Xi S \to {}_\Phi Z\Xi S$ which maps $1 \cdot S \to (aqs^2 + bp)S = 1 \cdot S$; so u is an isomorphism. Thus $u_0 = Z\Phi$, u_1, u_2, and $u_3 + b$ constitute a chain map $C_*(\tilde{\mathscr{L}}(p, q)) \to {}_\Phi C_*(\tilde{\mathscr{L}}(p, qs^2))$ which restricts to give an isomorphism on the third homotopy module $Z\Xi S$. Hence there is a homotopy equivalence $f: \mathscr{L}(p, q) \to \mathscr{L}(p, qs^2)$ inducing Φ on the fundamental group. □

PROPOSITION 2. *Two Lens spaces $\mathscr{L}(p, q)$ and $\mathscr{L}(p, r)$ for Ξ are homotopically equivalent via the identity homomorphism on the fundamental groups if and only if $q \equiv \pm r \bmod p$.*

Proof. First we assume that $\mathscr{L}(p, q)$ and $\mathscr{L}(p, r)$ are homotopically equivalent via the identity homomorphism on the fundamental groups. Because $(r, p) = 1$, we may choose an integer r' such that $rr' \equiv 1 \bmod p$. Then we have the diagram

$$\begin{array}{lccccccccccccc}
C_*(\mathscr{L}(p, q)): 0 & \longrightarrow & Z\Xi S & \longrightarrow & Z\Xi & \xrightarrow{R^q - 1} & Z\Xi & \xrightarrow{\partial_2} & (Z\Xi)^n & \xrightarrow{\partial_1} & Z\Xi & \xrightarrow{\varepsilon} & Z & \longrightarrow 0 \\
 & & & & \Big\downarrow {\scriptstyle u_3 = \langle R, q\rangle\langle R^r, r'\rangle} & & \Big\| & & \Big\| & & \Big\| & & \Big\| & \\
C_*(\mathscr{L}(p, r)): 0 & \longrightarrow & Z\Xi S & \longrightarrow & Z\Xi & \xrightarrow{R^r - 1} & Z\Xi & \xrightarrow{\partial_2} & (Z\Xi)^n & \xrightarrow{\partial_1} & Z\Xi & \xrightarrow{\varepsilon} & Z & \longrightarrow 0
\end{array}$$

which commutes. Because $\mathscr{L}(p, q)$ and $\mathscr{L}(p, r)$ have the same homotopy type, there is a $Z\Xi$-homomorphism $\gamma: Z\Xi \to Z\Xi S$ such that $u_3 + \gamma: Z\Xi \to Z\Xi$ restricts to $Z\Xi S$ to give an isomorphism $u: Z\Xi S \to Z\Xi S$. γ is given by the formula (3.5) in Eilenberg-Maclane [1]. Now we have commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z\Xi S & \longrightarrow & Z\Xi & \longrightarrow & Z\Xi(R-1) & \longrightarrow & 0 \\
 & & \Big\downarrow {\scriptstyle u} & & \Big\downarrow {\scriptstyle u_3 + \gamma} & & \Big\| & & \\
0 & \longrightarrow & Z\Xi S & \longrightarrow & Z\Xi & \longrightarrow & Z\Xi(R-1) & \longrightarrow & 0 \, .
\end{array}$$

Therefore by Five lemma, $u_3 + \gamma$ is an isomorphism. But then $\varepsilon(u_3 + \gamma) = qr' + p\varepsilon(v) = \pm 1$ for some $v \in Z\Xi$. Therefore $qr' \equiv \pm 1 \bmod p$, i.e., $q \equiv \pm r \bmod p$.

Conversely, let us assume that $q \equiv \pm r \bmod p$. We may choose an integer k such that $kp = q \mp r$. Also, because $(r, p) = 1$, there exist integers r' and p' such that $rr' + pp' = 1$. As above, we may take the chain map $u_3 = \langle R, q\rangle\langle R^r, r'\rangle$, and let $-kr' \pm p': Z\Xi \to Z\Xi S$ be the $Z\Xi$-module homomorphism given by $1 \to (-kr' \pm p')S$. Then $u_3 + (-kr' \pm p')$ restricts to $Z\Xi S$ to give $Z\Xi$-module homomorphism which takes $1 \cdot S \to (qr' - kr'p \pm p'p)S = \pm 1S$, so that this restriction is an isomorphism. Therefore by the preliminary remarks at the start of this section, $\mathscr{L}(p, q)$ and $\mathscr{L}(p, r)$ have the same homotoy type. □

In our attempts towards proving the general classification we are faced with the following problem:

Given any integer s, relatively prime to p, does there exist an automorphism $\Phi \in \operatorname{Aut} \Xi$ with $\Phi(R) = R^s$?

In §2, we show that if the single-relator R^p is power of a primitive, the answer to the above question is in the affimative. This allows us to "shuffle" the k-invariants of the Lens space $\mathscr{L}(p, q)$ for Ξ, and the classification for these Lens spaces is identical to that for the ordinary Lens spaces $L(p, q)$ (Theorem 1). On the other hand, if the single-relator R^p is not power of a primitive, Pride conjectures that every $\Phi \in \operatorname{Aut} \Xi$ is free in which case we show that $\Phi(R) = gR^{\pm 1}g^{-1}$ for some g in Ξ. Thus we do not have the freedom of "shuffiing" the k-invariants; and so the solution to the general problem of homotopy classification for these Lens spaces is finer than that for the ordinary Lens spaces. Indeed it is identical to one where we insist on the identity homomorphism on the fundamental group (see Proposition 2 and Theorem 2).

PROPOSITION 3. *If there is an automorphism $\Phi \in \operatorname{Aut} \Xi$ with $\Phi(R) = gR^s g^{-1}$ where $(s, p) = 1$ and $g \in \Xi$, then there is a homotopy equivalence between the Lens spaces $\mathscr{L}(p, q)$ and $\mathscr{L}(p, r)$ for Ξ inducing Φ on the fundamental group if and only if $qs^2 \equiv \pm r \bmod p$.*

Proof. By Proposition 1, there is a homotopy equivalence between $\mathscr{L}(p, q)$ and $\mathscr{L}(p, r)$ inducing Φ on the fundamental group if and only if there is a homotopy equivalence between $\mathscr{L}(p, qs^2)$ and $\mathscr{L}(p, r)$ inducing the identity on the fundumental group and, by Proposition 2, this is the case if and only if $qs^2 \equiv \pm r \bmod p$. We are done. □

THEOREM 1. *Let Ξ have presentation* (1) *with the single-relator R^p a power of a primitive element R of F. Then two Lens spaces $\mathscr{L}(p, q)$ and $\mathscr{L}(p, r)$ for Ξ have the same homotopy type if and only if qr or $-qr$ is a quadratic residue* mod p.

Proof. We first note that since R is primitive, given any s, relatively prime to p, there is an automorphism $\Phi \in \operatorname{Aut} \Xi$ such that $\Phi(R) = R^s$ (see §2). Thus, by Proposition 3, $\mathscr{L}(p, q)$ and $\mathscr{L}(p, r)$ have the same homotopy type if and only if for some s, $(s, p) = 1$, we have that $qs^2 \equiv \pm r \bmod p$, i.e., $(qs)^2 \equiv \pm qr \bmod p$, and this is the case if and only if qr or $-qr$ is a quadratic residue $\bmod p$. □

THEOREM 2. *Let Ξ have presentation* (1) *with single relator R^p a power of a nonprimitive element R of the free group F. When conjecture stated in the introduction holds for Ξ, then two Lens spaces $\mathscr{L}(p, q)$ and $\mathscr{L}(p, r)$ have the same homotopy type if and* *only if* $q \equiv \pm r \bmod p$.

Proof. Because R is a nonprimitive element of the free group F, the conjecture gives that for any automorphism $\Phi \in \operatorname{Aut} \Phi$, $\Phi(R) = gR^{\pm 1}g^{-1}$, $g \in \Xi$ (see §2). The theorem now is immediate from Proposition 3. □

2. Automorphisms of one-relator groups.

DEFINITION. An automorphism $\alpha: \Xi \to \Xi$ is *free* on a presentation $\mathscr{R}$ for Ξ if it can be induced by an isomorphism on the free group on the generators of $\mathscr{R}$.

LEMMA 11. *Let* $\alpha: \Xi \to \Xi$ *be a free automorphism on the presentation* $\mathscr{R}$ *for* Ξ. *Then* $\alpha(R) = gR^{\pm 1}g^{-1}$ *for some* g *in* Ξ.

Proof. By Theorem N5 of [6], page 172 (see also A. J. Sieradski [10, page 91]), we have that $\alpha(R^p)$ is freely equal (as a word in x_i) to $gR^{\pm p}g^{-1}$ for some g in Ξ. This implies that $\alpha(R)^p$ is freely equal $(gR^{\pm 1}g^{-1})^p$. But then by Exercise 2, page 41 of [6], $\alpha(R) = gR^{\pm 1}g^{-1}$. □

Thus if $\alpha: \Xi \to \Xi$ is an automorphism for Ξ with $\alpha(R) = R^s$ where $s \neq \pm 1$, then α is not free.

EXAMPLES. (1) G. Rosenberger in [9] has shown that every automorphism α of the group $H_{g,n}$ given by the presentation

$$\left(a_1, \cdots, a_g, t_1, u_1, \cdots, t_n u_n : \left(a_1^{n_1} \cdots a_g^{n_g} \prod_{i=1}^{n} [t_i, u_i]\right)^p\right)$$

($p \geqq 2$, $n_j \geqq 2$, for $j = 1, \cdots, g$) is free.

(2) Let $C_{n,p}$ be the group presented by

$$\mathscr{P} = (a_1, \cdots, a_{n-1}, c : c^p) .$$

For any integer s such that $(s, p) = 1$, there is an automorphism α of $c_{n,p}$ with $\alpha_s(c) = c^s$. This is simply given by taking $a_i \to a_i$, $i = 1, \cdots, n-1$, and $c \to c^s$. In fact the following example generalizes this situation.

(3) Let Ξ be a group given by presentation (1). If R is primitive in the free group F generated by $x_1, \cdots, x_n$, then for all s, $(s, p) = 1$, there is an automorphism α of Ξ such that $\alpha(R) = R^s$. This follows because R primitive implies that Ξ is isomorphic to $C_{n,p}$, and therefore $\operatorname{Aut} \Xi$ is isomorphic to $\operatorname{Aut} C_{n,p}$. Now it is easy

to get a required α.

(4) Let π be the group presented by

$$(x_1, x_2 \colon Q^q)$$

where Q is not a proper power and $q > 1$. Suppose also that Q is not primitive in the free group F generated by x_1, x_2. Then for any α in $\operatorname{Aut}\pi$, $\alpha(Q) = gQ^{\pm 1}g^{-1}$ for some g in π. This follows from a result of S. Pride [8] which states that π has only one Nielsen equivalence class.

3. Non-cancellation penomenon. In this section we show that although two Lens spaces $\mathscr{L}(p, q)$ and $\mathscr{L}(p, r)$ for Ξ may have distinct homotopy type, $\mathscr{L}(p, q) \vee S^3$ and $\mathscr{L}(p, r) \vee S^3$ are always homotopically equivalent.

THEOREM. *For every pair of Lens spaces $\mathscr{L}(p, q)$ and $\mathscr{L}(p, r)$ for Ξ, there is a homotopy equivalence*

$$\mathscr{L}(p, q) \vee S^3 \longrightarrow \mathscr{L}(p, r) \vee S^3$$

inducing the identity on the fundamental group.

Proof. We will show that the theorem holds for the pair $\mathscr{L}(p, q)$ and $\mathscr{L}(p, 1)$. Because q and p are relatively prime, there exists positive integers q' and r' such that $qq' + pp' = 1$. We have the commutative diagram

$$\begin{array}{lcccccccccc}
C_*(\tilde{\mathscr{L}}(p, q) \vee S^3)\colon Z\Xi \oplus Z\Xi & \xrightarrow{(R^q-1,\,0)} & Z\Xi & \xrightarrow{\left(\frac{\partial R^p}{\partial x_i}\right)} & (Z\Xi)^n & \xrightarrow{(x_i-1)} & Z\Xi & \xrightarrow{\varepsilon} & Z & \to & 0 \\
\downarrow u & & \| & & \| & & \| & & \| & & \\
C_*(\tilde{\mathscr{L}}(p, 1) \vee S^3)\colon Z\Xi \oplus Z\Xi & \xrightarrow{(R-1,\,0)} & Z\Xi & \xrightarrow{\left(\frac{\partial R^p}{\partial x_i}\right)} & (Z\Xi)^n & \xrightarrow{(x_i-1)} & Z\Xi & \xrightarrow{\varepsilon} & Z & \to & 0
\end{array}$$

where $u = \begin{pmatrix} \langle R, q\rangle & p' \\ -\langle R, p\rangle & \langle R^q, q'\rangle \end{pmatrix}$. Clearly u is an isomorphism with inverse $u^{-1} = \begin{pmatrix} \langle R^q, q'\rangle & -p' \\ \langle R, p\rangle & \langle R, q\rangle \end{pmatrix}$. Hence we can construct a homotopy equivalence between $\mathscr{L}(p, q)$ and $\mathscr{L}(p, 1)$ inducing the identity on the fundamental group. □

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Received July 19, 1978 and in revised form July 6, 1979. Part of this work appeared in the thesis submitted by the author in partial fulfillment of the requirements for the Ph. D. at the University of Oregon. The author wishes to thank his advisor Professor Allan J. Sieradski and Professor Micheal N. Dyer for their guidance and valuable advice.

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