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ON HOLOMORPHIC APPROXIMATION IN WEAKLY PSEUDOCONVEX DOMAINS

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A uniform estimate for solutions to the equation $\bar{\partial}u=\alpha$ in a weakly pseudoconvex domain is obtained, provided that the form α vanishes near the set of degeneracy of the Levi form. Under the additional hypothesis that the closure of the domain is holomorphically convex, analogous estimates are obtained for solutions defined in a full neighborhood of the closure. Applications are given to Mergelyan type approximation problems in a weakly pseudoconvex domain D. In particular, it is shown that any function in $A(D)$ can be uniformly approximated by functions in $A(D)$ which extend holomorphically across all strongly pseudoconvex boundary points. When $\bar{D}$ is holomorphically convex, it is shown that the Mergelyan problem can be localized to a small neighborhood of the set on which the Levi form degenerates.

Introduction. A bounded domain D in C^n has the *Mergelyan Property* if continuous functions on $\bar{D}$ which are holomorphic in the interior of $\bar{D}$ can be approximated uniformly on $\bar{D}$ by functions holomorphic in a neighborhood of $\bar{D}$. For $n > 1$, the first non-trivial domains for which the Mergelyan property was verified were the strictly pseudoconvex ones (Henkin [7], Kerzman [9], Lieb [12]). For a more general class of pseudoconvex domains, Range [14] considered approximation by functions which extend holomorphically across strictly pseudoconvex boundary points. Our interest in the problem was rekindled by an example due to Diederich and Fornaess [3] of a smooth pseudoconvex domain D for which approximation by functions holomorphic in a full neighborhood of $\bar{D}$ is impossible. The failure of the Mergelyan property in this example is intimately connected with the absence of a Stein neighborhood basis for $\bar{D}$. Thus a principal objective of this paper is the removal of the hypothesis of existence of Stein neighborhoods which was required in [14].

The key technical tool in [14] is an estimate for solutions to $\bar{\partial}u = \alpha$ when the support of α is disjoint from the non-strictly pseudoconvex boundary. In the absence of a Stein neighborhood base, such estimates were first obtained by Beatrous [1] by using Kohn's global regularity theorem for $\bar{\partial}$ in the construction of a global Ramirez-Grauert-Lieb type kernel which solves the $\bar{\partial}$-problem with the above mentioned support condition. In the present paper, we apply directly the Bochner-Martinelli-Koppelman formula for (0, 1)-forms [11] and Kohn's theorem in order to pass from local to global in the relevant $\bar{\partial}$-problem in [14].

It has been conjectured that any smooth, bounded domain whose closure is holomorphically convex should have the Mergelyan property. In this direction, we show that holomorphic convexity of $\bar{D}$ implies the Mergelyan property if one can approximate near the non strictly pseudoconvex points of ∂D (Theorem 2.3). Again, the essential tool is an estimate for $\bar{\partial}$ with a support condition. The Mergelyan type theorems in [5] and [15] are easy consequences of this result.

1. A $\bar{\partial}$-problem with support condition. Let D be a domain in C^n. A point $p \in \partial D$ is called a *strictly pseudoconvex boundary point* if ∂D is of class C^k, $k \geqq 2$, in a neighborhood of p and strictly Levi pseudoconvex at p (cf. [6] p. 262). The collection of all such points is denoted by $S(\partial D)$. The set $S(\partial D)$ is relatively open in ∂D, and if D is a bounded domain with a C^2 boundary $S(\partial D)$ is non-empty. The set $NS(\partial D) = \partial D, S(\partial D)$ is the set of non strictly pseudoconvex boundary points.

For K compact in C^n, $\mathcal{O}(K)$ denotes the algebra of functions which are holomorphic on some neighborhood of K. The set K is said to be *holomorphically convex* if every nonzero algebra homomorphism from $\mathcal{O}(K)$ into C is given by point evaluation at some point in K. If K has a Stein neighborhood basis then K is holomorphically convex; the converse is in general false.

Given D and an open set U we set $B^{\infty}_{0,1}(D, U) = \{\alpha \in C^{\infty}_{0,1}(D): \bar{\partial}\alpha = 0, \operatorname{supp} \alpha \subset U, \text{ and } \|\alpha\|_D < \infty\}$. Here $\| \ \|_D$ denotes the sup norm on D.

THEOREM 1.1. *Suppose D is pseudoconvex and $U_1 \subset\subset U_2$ are open with $U_1 \cap \partial D \subset\subset U_2 \cap \partial D \subset\subset S(\partial D)$.*

(a) *If ∂D is of class C^{∞} then there are a constant C and a linear operator*

$$T: B^{\infty}_{0,1}(D, U_1) \longrightarrow C(\bar{D}) \cap C^{\infty}(D)$$

which satisfy

(i) $\bar{\partial}(T\alpha) = \alpha$;

(ii) $\|T\alpha\|_D \leqq C\|\alpha\|_D$;

(iii) $\|T\alpha\|_{D,U_2} \leqq C\|\alpha\|_{L^1_{0,1}(D)}$.

The constant C can be chosen to be independent of small perturbations of $U_1 \cap \partial D$.

(b) *If ∂D is of class C^2 and $\bar{D}$ is holomorphically convex then there are a neighborhood basis $\{D_\varepsilon, 0 < \varepsilon < \varepsilon_0\}$ of $\bar{D}$, a constant C,* *and linear operators*

$$T_\varepsilon\colon B^\infty_{0,1}(D_\varepsilon,\ U_1) \longrightarrow L^\infty(D_\varepsilon) \cap C^\infty(D_\varepsilon)$$

which satisfy

(iv) $\bar\partial(T_\varepsilon\alpha) = \alpha$;

(v) $\|T_\varepsilon\alpha\|_{D_\varepsilon} \leqq C\|\alpha\|_{D_\varepsilon}$

for any ε *with* $0 < \varepsilon < \varepsilon_0$ *and any* $\alpha \in B^\infty_{0\,1}(D_\varepsilon,\ U_1)$.

REMARK 1.2. If $\bar D$ has a Stein neighborhood basis, part (a) is contained in Theorem 2.1 in [14]. We will neither prove nor use (iii) in this paper; however, it is needed to extend the results of [14] to the domains considered here.

REMARK 1.3. We have stated only the simplest estimates. Any estimate valid for the local solution operator used in the proof will carry over to the global operators. For the construction of local solution operators in the case of a C^2 boundary, see [13] or [16].

Proof. (I) Choose another open set U with $U_1 \subset\subset U \subset\subset U_2$. By applying successively one of the local solution operators for $\bar\partial$ and the bumping technique (cf. [9] and [14]) one obtains a small perturbation $\tilde D$ of D with $\tilde D,\ U_2 = D\backslash U_2$ and $U \cap \bar D \subset \tilde D$ which has the following property: For any $\alpha \in B^\infty_{c,1}(D,\ U_1)$ there are a form $\tilde\alpha \in B^\infty_{c,1}(\tilde D,\ U) \cap C^\infty_{0,1}(\bar D)$ and a function $u_1 \in C(\bar D) \cap C^\infty(D)$ with supp $u_1 \subset U$ such that $\alpha = \tilde\alpha + \bar\partial u_1$. Moreover, $\tilde\alpha$ and u_1 depend linearly on α, and the following estimates are satisfied:

$$\begin{cases} \|\tilde\alpha\|_{\tilde D} \leqq C_1\|\alpha\|_D \\ \|u_1\|_D \leqq C_1\|\alpha\|_D\,. \end{cases} \tag{1.4}$$

The domain $\tilde D$ and the constant C_1 can be chosen to be independent of small perturbations of $S(\partial D) \cap U_1$.

(II) For part (a) we proceed as follows. By a small perturbation of $S(\partial D)$, choose another pseudoconvex domain D^* such that $D^*\backslash U_2 = D\backslash U_2$, $U \cap \bar D \subset D^*$, and $U \cap \bar D^* \subset \tilde D$. Observe that these properties of D^* persist if D is replaced by a sufficiently small perturbation of D within U_1. Apply the Bochner-Martinelli-Koppelman formula (see for example [13]) to write $\tilde\alpha = \beta + \bar\partial u_2$ where

$$\beta = \int_{\partial\tilde D} \tilde\alpha \wedge K_1\,, \qquad u_2 = \int_{\tilde D} \tilde\alpha \wedge K_0\,,$$

and K_0, K_1 are the Bochner-Martinelli-kernels of the appropriate bidegree. Clearly u_2 is continuous on $\bar D$ and $\|u_2\|_{\tilde D} \leqq C_2\|\tilde\alpha\|_{\tilde D}$; also $\bar\partial\beta = 0$ and, since $\tilde\alpha = 0$ on $\partial\tilde D\backslash U$, it follows by differentiation under the integral sign that $\beta \in C^{\infty}_{0,1}(\bar{D}^*)$ and

$$||\beta||_{C^k_{0,1}(\bar{D}^*)} \leqq \gamma_k ||\tilde{\alpha}||_{\tilde{D}}$$

for $k = 0, 1, 2, \cdots$. By Theorem 3.19 of [10] and Sobolev's Lemma there is, for sufficiently large k, a linear operator

$$T_k: C^{\infty}_{0\,1}(\bar{D}^*) \cap \ker \bar{\partial} \longrightarrow C^1(\bar{D}^*) \cap C^{\infty}(D^*)$$

which inverts $\bar{\partial}$ and satisfies the estimate

$$||T_k f||_{C^1(\bar{D}^*)} \leqq \gamma'_k ||f||_{C^k_{0\,1}(\bar{D}^*)} .$$

It follows that $T\alpha = T_k\beta + u_2 + u_1$ satisfies all the required conditions.

(III) For part (b) we need the following well known characterization of compact holomorphically convex sets, the proof of which can be found in [2]. For any open set U in C^n we let $E(U)$ denote its envelope of holomorphy, and we denote the associated spread map by $\prod_U$.

LEMMA 1.5. *A compact set K in C^n is holomorphically convex if and only if for every neighborhood basis $\{U_i\}_{i \in I}$ for K we have*

$$K = \bigcup_{i \in I} \prod_{U_i}(E(U_i)) .$$

Choose a function $\rho \in C^2(C^n)$ such that $D = \{z \in C^n: \rho < 0\}$, $d\rho \neq 0$ on ∂D, and ρ is strictly plurisubharmonic in a neighborhood of $U_2 \cap \partial D$. Set $D_\varepsilon = \{z \in C^n: \rho(z) < \varepsilon\}$. For $0 < \varepsilon < \varepsilon_0$, with ε_0 sufficiently small, D_ε has C^2 boundary and ∂D_ε is strictly pseudoconvex at points in $U_2 \cap \partial D_\varepsilon$, so part (I) of the proof can be applied to D_ε. Given $\alpha \in B^{\infty}_{0,1}(D_\varepsilon, U_1)$, one obtains $\tilde{\alpha} \in B^{\infty}_{0\,1}(\tilde{D}, U)$ and $u_1 \in C^{\infty}(D_\varepsilon)$ as before, where the constants in (1.4) and the domain $\tilde{D}$ can be chosen independently of ε for $0 < \varepsilon < \varepsilon_0$. Since $\tilde{\alpha} = 0$ on $\tilde{D} \backslash U$, $\tilde{\alpha}$ can be trivially extended to a neighborhood V of $\tilde{D}$ with $V \cap U \subset \tilde{D} \cap U$. Using Lemma 1.5 one finds a neighborhood Ω of $\bar{D}$ with $\Omega \subset \prod_\Omega(E(\Omega)) \subset V$. By Theorem 3.4.10 of [8] one can solve $\bar{\partial} u_2 = \prod^*_\Omega \tilde{\alpha}$ on the Stein manifold $E(\Omega)$ with L^2-estimates with respect to a suitable Riemannian metric. If ε_0 is such that $D_{\varepsilon_0} \subset\subset \Omega$, it follows by interior elliptic estimates that for $0 < \varepsilon < \varepsilon_0$

$$||u_2||_{D_\varepsilon} \leqq C_3 ||\tilde{\alpha}||_V .$$

Thus $T_\varepsilon \alpha = (u_1 + u_2)|_{D_\varepsilon}$ is a solution to $\bar{\partial} u = \alpha$ which satisfies the desired estimate.

2. Approximation theorems. We consider first a bounded pseudoconvex domain with a smooth boundary. We denote the continuous boundary value algebra $C(\bar{D}) \cap \mathcal{O}(D)$ by $A(D)$.

LEMMA 2.1. *Let E be a closed subset of $S(\partial D)$. Then every $f \in A(D)$ can be approximated uniformly by functions in $A(D)$ which extend holomorphically across E.*

If $\bar{D}$ has a Stein neighborhood basis, the lemma is a special case of Theorem 3.2 in [14]. The proof in [14] is technically more complicated as it covers more general situations. We indicate the direct argument for the case considered here.

Proof (Sketch). Fix an open neighborhood U of E with $U \cap \partial D \subset S(\partial D)$. Cover the compact set E by finitely many balls $B_1, \cdots, B_r$ such that $\bar{B}_i \subset U$ and f can be approximated uniformly on $B_i \cap \bar{D}$ by translates $\{f_i^\delta, \delta > 0\}$ of f. To the collection $\{f, f_1^\delta, \cdots, f_r^\delta\}$ one applies the usual Cousin I type argument as in [9] or [12]. The crucial step involves solving $\bar{\partial} u = \alpha^\delta$ where, in our case, α^δ is a (0, 1)-form on a small perturbation D^δ of D inside U with supp $\alpha^\delta \subset U$ and $\|\alpha^\delta\|_D \to 0$ as $\delta \to 0$. Theorem 1.1.a applies, so the proof can be completed as in [9] or [12].

THEOREM 2.2. *Every $f \in A(D)$ can be approximated uniformly on $\bar{D}$ by functions which extend holomorphically across $S(\partial D)$.*

Proof. Write $S(\partial D) = \bigcup_{j=1}^\infty E_j$, where $E_1 \subset E_2 \subset \cdots$ and E_j is closed. A sequence $\{f_j\}$ of approximating functions is constructed inductively by applying Lemma 2.1 to a sequence $\{D_j\}$ of small perturbations of D satisfying $D_1 = D$ and $D_{j+1} \supset D_j \cup E_j$. The details are left to the reader.

We now consider bounded domains with C^2 boundary and holomorphically convex closure. The following theorem, which first appeared in [1], localizes the problem of verifying the Mergelyan property to a small neighborhood of $NS(\partial D)$.

THEOREM 2.3. *Suppose D is a bounded domain with a C^2 boundary such that $\bar{D}$ is holomorphically convex. A function $f \in A(D)$ can be approximated uniformly on $\bar{D}$ by functions in $\mathcal{O}(\bar{D})$ if and only if there is a neighborhood W of $NS(\partial D)$ such that $f|_{\overline{W \cap D}}$ can be approximated uniformly on $\overline{W \cap D}$ by functions in $\mathcal{O}(\overline{W \cap D})$.*

The proof of the nontrivial implication is based on the obvious modification of the classical argument (cf. [9], [12]). The relevant $\bar{\partial}$-problem is solved by Theorem 1.1.b.

Obviously the above condition is satisfied for all $f \in A(D)$ if $NS(\partial D)$ is finite, so the result of [15] is an immediate corollary. More generally, one obtains the following result of Fornaess and Nagel [5].

COROLLARY 2.4. *Let D be a bounded, pseudoconvex domain with a C^2 boundary. For $z \in \partial D$, let $\boldsymbol{n}(z)$ denote the outer unit normal to ∂D at z. Suppose there are a neighborhood W of $NS(\partial D)$ and a holomorphic vector field $F: W \to \boldsymbol{C}^n$ with* $\mathrm{Re}\,\langle \boldsymbol{n}(z), F(z)\rangle > 0$ *for $z \in W \cap \partial D$. Then D has the Mergelyan property.*

Proof. It follows easily from standard arguments that under the above hypothesis $\bar{D}$ has a Stein neighborhood basis. Moreover, after shrinking W if necessary, it follows that for any $f \in A(D)$ the functions $f_\delta(z) = f(z - \delta F(z))$, are holomorphic on $\overline{W \cap D}$ for sufficiently small $\delta > 0$. Clearly $f_\delta \to f$ uniformly as $\delta \to 0$, so by Theorem 2.3 f can be approximated uniformly on D by functions in $\mathscr{O}(\bar{D})$.

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