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ON THE DEGENERACY OF A SPECTRAL SEQUENCE ASSOCIATED TO NORMAL GROSSINGS

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Let W be a complex analytic manifold and V a divisor with normal crossings, and consider the Leray spectral sequence associated to the inclusion map of $W-V$ into W. We give two homological reformulations for any of the $d_r^{p,q}$ to be the zero map for $r \geqq 2$. These conditions are shown to be satisfied if W is compact Kähler, but it is easy to give examples when it does not degenerate at E_3 if W is only a differentiable manifold. The nondegeneracy at E_3 for arbitrary V in a compact Kähler manifold is interpreted in terms of reiterated residues.

1. Introduction.

1.1. Let $j: W - V \subset W$ be the inclusion. When W is a projective, algebraic manifold and V is a divisor with normal crossings, then Deligne [2] in the course of defining his mixed Hodge structure in $W - V$ shows that the Leray spectral sequence associated to j degenerates at E_3, i.e., $E_3^{p,q} \cong E_\infty^{p,q}$ for all p, q. Griffiths-Schmid [7] give another proof of the degeneracy at E_3, when W is a compact Kähler manifold and V again has normal crossings.

In this article, we let W be a complex analytic manifold and V a divisor with normal crossings. In Theorem 2.3 we then give two homological reformulations for any of the $d_r^{p,q}$ to be the zero map. It is easy to give differentiable examples where these topological conditions are not satisfied. However, we know of no examples of non-Kähler compact complex manifolds where these conditions fail. The degeneracy at E_3 might be true for surfaces but we feel it is not true in dimensions greater than two.

If W is compact Kähler, then these homological conditions are satisfied for divisors with normal crossings. However, if one allows arbitrary singularities on V, then one can give examples when the homological conditions of (2.3) are not satisfied; so that the general position is needed, although the abutment of the spectral sequence is the same when one resolves the singularities to normal crossings. This phenomenon is interpreted in terms of iterated residues in (3.2.1).

As far as notation goes, $H^*(X)$ or $H_*(X)$ will always mean coefficients in some fixed field. If W is noncompact, the homology or cohomology can either be with compact or closed support. By closed support in $W - V$, we shall mean support closed in $W - V$ *and* closed in W, cf., Fotiadi, et. al [3, Part III].

2. A homological reformulation for degeneracy.

2.1. Suppose W is a complex manifold of complex dimension n and V is a divisor with normal crossings. We suppose V_i, $i \in I$ are the nonsingular components of V and we set $V_{i_1 \cdots i_k} = V_{i_1} \cap \cdots \cap V_{i_k}$, which is a submanifold of complex dimension $n-k$ or the empty set. We let M_k denote the k-tuple points of V (i.e., $M_k = \cup(V_{i_1 \cdots i_k} - \bigcup_{i_{k+1}} V_{i_1 \cdots i_k i_{k+1}})$) and $\bar{M}_k$ denotes the closure of M_k. Set $M_0 = W - V$.

Then in Gordon [4, p. 133] a subgroup of $H_p(\bar{M}_q)$ is defined, called the *tubular cycles*, and denoted by $H_p(\bar{M}_q)_\Delta$, such that one has the tube over cycle map $\tau_q\colon H_p(\bar{M}_q)_\Delta \to H_{p+1}(M_{q-1})$. One also has the Gysin mapping $\tilde{\tau}_q\colon H_p(M_q) \to H_{p+1}(M_{q-1})$, cf., [4, p. 134].

2.1.1. DEFINITION.

$$H_p(V)_\Delta = H_p(\bar{M}_1)_\Delta \bigoplus_{q=2}^{p} \tilde{\tau}_2 \cdots \tilde{\tau}_{q-1}\tau_q H_{p-q+1}(\bar{M}_q)_\Delta .$$

If we let τ denote τ_1 on the first summand and $\tilde{\tau}_1$ on the last p summands of (2.1.1), then Gordon [4, p. 143] shows:

2.1.2. The following is exact:

$$H_{p+2}(W) \xrightarrow{I} H_p(V)_\Delta \xrightarrow{\tau} H_{p+1}(W - V)$$

where I is transverse intersection with V.

2.2. We are now going to define the notion of absolutely relative class of degree k.

Suppose V_i, $i \in I'$ are complex manifolds in general position and suppose we have $\gamma_p \in H_p(\cup V_i)$. Let $\gamma_p \cap V_i = \gamma_{p,i} \in H_p(V_i, \bigcup_{j,j\in I'} V_{ij})$ and $j_i\colon V_i \subset (V_i, \bigcup_{j,j\in I'} V_{ij})$ be the inclusion. (Recall, $V_{ij} = V_i \cap V_j$.) Finally, let $\mathrm{Hom}\colon H_p(V_i, \bigcup_j V_{ij}) \xrightarrow{\sim} H^p(V_i, \bigcup_j V_{ij})$ be vector space duality. The pairing can be thought of as being given by deRham's theorem via integration, where $H^*(V_i, \bigcup_j V_{ij})$ can be represented by forms which vanish on $\bigcup_j V_{ij}$, cf., Leray [9, Chapter 3]. If the V_i are noncompact, then $H_*^c(V_i, \bigcup_j V_{ij}) \xrightarrow{\sim} H_F^*(V_i, \bigcup_j V_{ij})$ where c and F denote compact and closed support respectively.

2.2.1. DEFINITION. We say $\gamma_p \in H_p(\bigcup_{i\in I'} V_i)$ is an *absolutely relative class* of $\bigcup_{i\in I'} V_i$ if

(i) $0 \neq \gamma_{p,i} \in H_p(V_i, \bigcup_j V_{ij})$ for all $i \in I'$.

(ii) $(j_i)^* \mathrm{Hom}(\gamma_{p,i}) = 0$ for all $i \in I'$.

Thus, suppose $V = V_1 \cup V_2$ are two Riemann surfaces with two double points P and Q in common. Then a real line joining P to

Q in V_i for $i = 1$ and 2 would represent an absolutely relative class if it did not wrap around a handle of V_i in going from P to Q.

Suppose γ_p is an absolutely relative class of $\bigcup_{i \in I'} V_i$. Then we have,

$$0 \neq \partial_* \gamma_{p,i} = \sum_{j \in I'} \gamma_{p-1,ij} \in H_{p-1}(\bigcup_j V_{ij}) .$$

2.2.2. DEFINITION. γ_p is an *absolutely relative class of degree one* of $\cup V_i$ if γ_p is an absolutely relative class such that $\partial_* \gamma_{p-1,ij} = 0$ for all $i, j \in I'$. An absolutely relative class is of *degree* k if $\forall i \in I'$, $\sum_{j \in I'} \gamma_{p-1,ij}$ is an absolutely relative class of degree $k-1$ in $\bigcup_j V_{ij}$.

2.2.3. DEFINITION. γ_p is an *absolutely relative class of degree zero* of $\bigcup_{I'} V_i$ if

(i) $0 \neq \gamma_{p,i} \in H_p(V_i, \bigcup_j V_{ij})$ for all $i \in I'$.

(ii) $\partial_* \gamma_{p,i} = 0$ for all $i \in I'$.

Essentially, to say a class is absolutely relative of degree k means if one takes the Maier-Vietoris sequence of $\bigcup_{i \in I} V_i$ and breaks it up into a diagram of horizontal and vertical maps, as e.g., for $|I'| = 3$ in Diagram 1, then γ is of degree k if one can take k non-zero boundary operators on γ before it is zero.

$$\begin{array}{ccccc}
H_{p-1}(V_{23}) & & & & H_{p-2}(V_{123}) \\
\uparrow {\scriptstyle *\partial} & & & & \uparrow {\scriptstyle \partial_*} \\
H_p(V_1) \oplus H_p(V_2 \cup V_3) & \longrightarrow & H_p(V_1 \cup V_2 \cup V_3) & \xrightarrow{\partial_*} & H_{p-1}(V_{12} \cup V_{13}) \\
\uparrow & & & & \uparrow \\
H_p(V_2) \oplus H_p(V_3) & & & & H_{p-1}(V_{12}) \oplus H_{p-1}(V_{13})
\end{array}$$

DIAGRAM 1

Suppose γ is a p-dimensional absolutely relative class of degree k. Then after tracing k boundary operators, we have

$$\bigoplus_{i_1 < \cdots < i_k} \gamma_{i_1} \cdots {}_{i_k} \in \bigoplus H_{p-k}(V_{i_1} \cdots {}_{i_k}) .$$

Then if P denotes Poincaré duality followed by Hom, we can form $P(\gamma_{i_1 \cdots i_k}) \in H_{2n-k+p}(V_{i_1} \cdots {}_{i_k})$ where $n - 1 = \dim_C V_i$.

2.2.4. DEFINITION. γ, an absolutely relative class of degree k, is called *tubular* if

$$\bigoplus_{i_1 < \cdots < i_k} P(\gamma_{i_1 \cdots i_k}) \in H_{2n-k+p}(\bar{M}_k)_\Delta .$$

By definition, $H_q(\bar{M}_k)_\Delta \subset H_q(\bar{M}_k)$, but in fact one easily shows that $H_q(\bar{M}_k)_\Delta \subset \bigoplus_{i_1<\cdots<i_k} H_q(V_{i_1\cdots i_k})$. The mapping is given by sending γ_q onto $\gamma_q \cap V_{i_1\cdots i_k}$ which is of dimension q by (a) of the definition of tubular cycles in [4, p. 133]. It is an injection by arguments similar to those on [4, p. 135]. Hence Definition 2.2.4 is well-defined.

2.3. THEOREM. *Suppose W is a complex manifold of dimension n and $V = \bigcup_{i\in I} V_i$ is a divisor with normal crossings. Let $j\colon W - V \subset W$ be the inclusion. The then following are equivalent:*

2.3.1. In compact (respectively, closed) support, the Leray spectral sequence of j has $d^{p,q}_{k+2}$ acting as the zero map.

2.3.2. $\tilde{\tau}_{q-k}\colon \tilde{\tau}_{q-k-1} \cdots \tilde{\tau}_q H_p(\bar{M}_q)_\Delta \to H_{p+k-1}(M_{q-k-1})$ is an injection in compact (respectively closed) support. For $k = 0$, $\tilde{\tau}_q = \tau_q$, and by definition, for $s \leqq 0$, $\tilde{\tau}_s$ is always an injection.

2.3.2. Suppose γ is an absolutely relative tubular class of degree k of $\bar{M}_{q-k} (= \cup V_{i_1\cdots i_{q-k}})$ of real dimension $2n - 2q - p + k$ in closed (respectively compact) support. Let $f_{q-k}\colon \bar{M}_{q-k} \subset \bar{M}_{q-k-1}$ be the inclusion map. Then $(f_{q-k})_* \gamma = 0$. By definition for $s \leqq 0$, f_s is always the zero map.

Proof of Theorem 2.3. The proof that (2.3.1) is equivalent to (2.3.2) follows because $d_2^{p,q}$ is the Gysin mapping, cf., Gordon [6, Proposition 3.3.2]. But since we are working with coefficients in a field, the vector space dual to $d_2^{p,q}$ acting on homology is just the transverse intersection mapping I. But by (2.1.2), Image $I = \operatorname{Ker} \tau$.

The equivalence of (2.3.2) and (2.3.3) is just Poincaré-Lefschetz duality, where we change supports, since if $V_{i_1\cdots i_k}$ is noncompact, then $H^c_p(V_{i_1\cdots i_k}, M_{k+1} \cap V_{i_1\cdots i_k}) \xrightarrow{\sim} H^F_{2n-p-2k}(V_{i_1\cdots i_k} - M_{k+1} \cap V_{i_1\cdots i_k})$, where by closed support in $V_{i_1\cdots i_k} - M_{k+1} \cap V_{i_1\cdots i_k}$ we mean closed in $V_{i_1\cdots i_k} - M_{k+1} \cap V_{i_1\cdots i_k}$ and closed in $V_{i_1\cdots i_k}$, cf., Fotiadi, et. al. [3, Part III].

For example, suppose $|I| = 2$, then we have the following diagram of exact rows:

$$\begin{array}{ccccc} H^F_{2n-p-2}(V_i) & \longrightarrow & H^F_{2n-p-2}(V_i,\ V_{12}) & \xrightarrow{(\partial_i)_*} & H^F_{2n-p-3}(V_{12}) \\ \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\ H^c_p(V_i) & \longleftarrow & H^c_p(V_i - V_{12}) & \xleftarrow{\tau_2} & H^c_{p-1}(V_{12}) \end{array}$$

DIAGRAM 2

where the first row is the exact sequence of the pair $(V_i,\ V_{12})$ and the second row follows from the Gysin sequence of the same pair.

The vertical isomorphisms are Poincaré-Lefschetz duality followed by Hom.

Hence $\ker \tau_2 \cong \operatorname{coker} (\partial_i)_*$, so that a basis of $H_{p-1}(V_{12})$ can be chosen such that $\tau_2(\gamma) \neq 0$ in $V_i - V_{12}$ iff $P^{-1}(\gamma) = (\partial_i)_*\gamma_i$ for $i = 1$ and 2.

Then by a similar argument, if we set $\gamma_1 - \gamma_2 = \gamma' \in H_{2n-p-2}(V)$, then $\tilde{\tau}_1\tilde{\tau}_2\gamma \neq 0$ in $W - V$ iff $\gamma' \in \text{Image}\,\{\partial_*: H_{2n-p-1}(W, V) \to H_{2n-p-2}(V)\}$. But γ' is a tubular absolutely relative class of degree one.

A similar argument works for arbitrary I.

2.4. COROLLARY. *Suppose W is a compact Kähler manifold and V is a divisor with normal crossings. If γ is an absolutely relative class of degree $k \geqq 1$ in V, then $i_*\gamma = 0$ in W where $i: V \subset W$ is the inclusion. Thus,* (2.3) *gives that the spectral sequence collapses at E_3.*

Corollary 2.4 follows immediately from the discussion in Griffiths-Schmid [7, Chapter 4]. The key step is the *principle of the two types.* If we have a form ω of type $(q, p - q)$ which is exact, then $\omega = d\eta_1 = d\eta_2$ where η_1 is of type $(q - 1, p - q)$ and η_2 is of type $(q, p - q - 1)$ [7, Lemma 2.13].

Using the principle of the two types, the idea of the proof of (2.4) is as follows: Suppose $V = V_1 \cup V_2$. If γ is an absolutely relative class of V, then $\gamma = \gamma_1 + \gamma_2$ for $\gamma_i \in H_p(V_i)$ and $0 \neq \partial_*\gamma_1 = -\partial_*\gamma_2 = \gamma_{12} \in H_{p-1}(V_{12})$. If $i_*\gamma \neq 0$ for $i: V \subset W$, then there is a harmonic form ω which represents $\text{Hom}\, i_*\gamma$. We can assume ω is of pure type $(q, p - q)$ by looking at each component one by one. Then if $i_i: V_i \subset W$, then $(i_i)''\omega$ is exact, thus by the principle of the two types, $(i_i)''\omega = d\eta_1 = d\eta_2$. But then by Leray [9, Chapter 3], $\eta_i | V_{12}$ represents $\text{Hom}(\gamma_{12})$. Thus $\text{Hom}(\gamma_{12})$ can be represented by two different types, $(q - 1, p - q)$ and $(q, p - q - 1)$, which is impossible on compact Kähler manifolds. This is also what happens in the lemma of the two filtrations of Deligne [2, 1.3].

2.5. We note that (2.3) is a topological and not an analytic fact. That is, suppose ω is a orientable, C^∞ manifold and $V = \bigcup_i V_i$ is a collection of C^∞ submanifolds in general position. Then (2.1) can be modified so that (2.1.2) is still true, cf., Gordon [5], and (2.2) still makes sense. Then (2.3) will still be true with appropriate modification in the indices of (2.3.2) and (2.3.3) which will only depend on the codimension of the V_i.

In particular, suppose W is a compact Kähler manifold and $V = \bigcup_i V_i$ are complex submanifolds in general position. Let $N = \max_i n_i$ where n_i is the complex codimension of V_i. Then (2.4) actually shows that the Leray spectral sequence of $j\colon W - V \subset W$ collapses at E_{2N+1}.

However, it is easy to give topological examples where the spectral sequence does not degenerate at the approprimate codimension. E.g., let S_1 and S_2 be two 2-dimensional spheres in R^4 which intersect transversely in two points P and Q. Let $T(S)$ be a regular tubular neighborhood of $S = S_1 \cup S_2$ in R^4 and form $DT(S)$, the double of $T(S)$, by glueing $T(S)$ to itself along its boundary. Then $DT(S)$ is a compact 4-dimemsional manifold and S is two submanifolds in general position of real codimension two.

Let γ_1 be the one cycle formed by tracing a line from P to Q in each of the S_i. Then $0 \neq [\gamma_1] \in H_1(S)$. If $i\colon S \subset DT(S)$, then it is easy to see that $i_*[\gamma_1] \neq 0$ in $DT(S)$. Hence by (2.3.3), the Leray sequence of $j\colon DT(S) - S \subset DT(S)$ does not degenerate at $E_3^{0,2} \neq E_4^{0,2}$, but at $E_4^{p,q} \cong E_\infty^{p,q}$.

The obvious examples one would consider for W a compact non-Kähler surface do not yield examples where the spectral sequence collapses at E_3. For example, if W is an elliptic surface with singular fibre of type ${}_mI_b$ with $b > 1$, cf. Kodaira [10], then the spectral sequence associated to this singular fibre always degenerates at E_3, i.e., the cycle formed by joining all the double points always bounds in W. If such an example exists for surfaces, it would probably have to be of type VII_0 with $b_1 \equiv 1(2)$ and $b_2 > 0$.

3. Arbitrary singularities in the compact Kähler case.

3.1. If $V \subset W$ is a variety of complex codimension N and W is a complex manifold, then one can resolve the singularities of V by a finite number of monoidal transforms with nonsingular centers, cf., Hironaka [8]. That is, one has another complex manifold W' and a holomorphic map $\pi\colon W' \to W$ such that if $\pi^{-1}(V) = V'$, then V' is a union of complex submanifolds in general position with $\operatorname{codim}_C V' = N$. Furthemore, by a theorem of Blanchard [1], if W is compact Kähler, then so is W'.

Then one has two spectral sequences with regards to the two maps $j\colon W - V \subset W$, $j'\colon W' - V' \subset W'$ with $E_2^{p,q} \Rightarrow H^{p+q}(W - V)$ and ${}'E_2^{p,q} \Rightarrow H^{p+q}(W' - V')$. Now, by the general theory of monoidal transforms one has that π is a bianalytic homeomorphism of $W - V$ onto $W' - V'$. Furthermore, by [5, p. 56] and by [4, Proposition 3.1], we have the following:

3.1.1. PROPOSITION. $\operatorname{Im} E_\infty^{p,q} \xrightarrow{\sim} \ker\{H_{p+q}(W - V) \to H_{p+q}(W)\}$ and $\operatorname{Im} E_\infty^{p,q} \cong \operatorname{Im} {}'E_\infty^{p,q}$ *under the identification* $\pi\colon W - V \xrightarrow{\sim} W' - V'$.

3.2. Of course, if V is a variety of complex codimension N whose singular locus is of sufficiently high codimension in V, then then spectral sequence will not collapse at E_{2N+1}, even if W is compact Kähler.

For example, if in CP^3, we let $C(T)$ be the cone over the torus in CP^2, i.e., $C(T) = \{[x, y, z, w] | x^3 + y^3 + z^3 = 0\}$, then $E_2^{4,1} \cong H^4(C(T))$ and $E_2^{0,4} \cong H^3(L)$, where L is the intersection of $C(T)$ and a sufficiently small 5-sphere about the singular point of $C(T)$. This follows because $E_2^{p,q} \cong H^p(W; R^q)$, where R^q is the sheaf associated to the presheaf which sends the open set $U \rightarrow H^q(U - V \cap U)$, cf., Swan [11, p. 129]. Hence, $d_4^{0,4}: E_4^{0,4} = E_2^{0,4} \rightarrow E_4^{4,1} = E_2^{4,1}$ is dual to the intersection mapping of $H_4(C(T))$ onto $H_3(L)$, which is nonzero, i.e., $E_5^{0,4} = 0$.

But, when we do not have normal crossings, even in the compact Kähler case, we can have counterexamples to (2.3.2). For example, let T be the torus in $C(T)$ as above, and let W be the projective, algebraic manifold gotten by a monoidal transform π in CP^3 with center T. Let $V \subset W$ be the strict transform of $C(T)$ by π. Then $V \cong C(T)$, since T is of codimension one in $C(T)$. Hence V has one singular point, which we shall call P_∞; and a Whitney stratification of V is given by $M_1 = V - P_\infty$, $M_2 = P_\infty$.

Consider $[\gamma] \in H_1(T)$, $T \subset V$. Then $\gamma \sim 0$ in V; in fact $\gamma = \tau_2(P_\infty)$ where $\tau_2: H_*(M_2) \rightarrow H_*(M_1)$ is the tube over cycle map, [4, p. 158]. Hence, $0 \neq [\gamma] \in E_2^{0,2}$. But $\tau_1\tau_2 P_\infty = 0$ in $W - V$, since there is a $[\gamma_3] \in H_3(W)$ such that $\gamma_3 \cap V = \gamma$. Topologically, $\gamma_3 = \pi^{-1}(\gamma)$. Here $E_4^{0,2} \cong E_\infty^{0,2}$.

Similarly, taking $V_m = \underbrace{V \times \cdots \times V}_{m \text{ times}} \subset \underbrace{W \times \cdots \times W}_{m \text{ times}} = W_m$, we can form a Whitney stratification of V_m by letting M_1 be the smooth points of V_m, M_2 be the smooth points of $V_m - M_1$, M_3 be the smooth points of $V_m - M_1 \cup M_2$, etc., so that $M_{m+1} = P_\infty \times \cdots \times P_\infty = P$. Then $\tau_2 \cdots \tau_{m+1}(P) \not\sim 0$ in M_1 (topologically, it is $\gamma \times \cdots \times \gamma$), where τ_i are the iterated tubes maps defined in Gordon [4, p. 158]. But $\tau_1\tau_2 \cdots \tau_{m+1}(P) \sim 0$ in $W_m - V_m$, i.e., $d_{3m}^{0,3m-1}$ is not the zero map.

What happens that we have a product of circle each of which bounds a disk in V, but when one resolves the singularities, this product of circle defines a nonzero homology class in V'. What (2.3.2) states is that when the ambient space is compact Kähler this always happens:

3.2.1. PROPOSITION. *Let W be a compact Kähler manifold and V an arbitrary variety. If an iterated tube map is not injective, then when one resolves the singularties via nonsingular centers, those cycles (in the kernel of the iterated tube map) represent nonzero homology classes in the proper transform.*

References

1. A. Blanchard, *Sur les variëtes analytiques complexes*, Ann. Ec. Norm (3), **73** (1956), 157-202.
2. P. Deligne, *Theorie de Hodge II*, IHES Publications Mathématiques, **40** (1972), 5-57.
3. D. Fotiadi, M. Froissard, J. Lascoux and F. Pham, *Applications of an isotropy theorem*, Topology, **4** (1964), 159-191.
4. G. L. Gordon, *The residue calculus in several complex variables*, Trans Amer. Math. Soc., **213** (1975), 127-176.
5. ———, *A Poincare' duality type theorem for polyhedra*, Ann. Inst. Fourier (Grenoble), **22** (1972), 47-58.
6. ———, *Differentials of the second kind* (preprint), 1976.
7. P. A. Griffiths and W. Schmid, *Recent Developments in Hodge Theory*, mimeographed notes from Tata Institute.
8. H. Hironaka, *Bimeromorphic mappings*, mimeographed from Warwick University, 1971.
9. J. Leray, *Cauchy Probleme III*, Bull. Soc. Math. France, **87** (1959), 81-180.
10. K. Kodaira, *On compact analytic surfaces II*, Annals of Math., **77**, no. 3 (1963), 563-626.
11. R. Swan, *The Theory of Sheaves*, Un. of Chicago Press, Chicago, 1964.

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