COMPLETIONS OF NOETHERIAN HEREDITARY PRIME RINGS

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If R is a Noetherian hereditary prime ring with Jacobson radical $J \neq (0)$ then, it has been shown that the J-adic completion of R is a Noetherian hereditary semi-prime ring; it is prime if and only if the maximal ideals of R form a single cycle. Among other things, one also finds the result: If R is a right Noetherian semi-local ring with $\bigcap_{n=1}^{\infty} J^n = (0)$ then, R is J-adic complete if and only if it is right linearly compact and J has the right AR-property.

As is clear from the literature, it is not known whether $\hat{R}$, the J-adic completion of a Noetherian hereditary prime ring R with $J \neq (0)$, is (right) Noetherian. It has been shown here that $\hat{R}$ is Noetherian. The approach taken proves not only this but also that $\hat{R}$ is hereditary and semi-prime. In view of Michler's structure theorem for semi-local Noetherian hereditary prime rings [13; 6.5], the structure of $\hat{R}$ then is completely determined since, in the present case, $\hat{R}$ decomposes into prime rings each complete in its radical topology. Thus the theorem generalizes the corresponding result for Dedekind prime rings obtained in [6]. (It is immediate that if R is a Noetherian semi-local ring which is either hereditary or serial then $\hat{R}$ is Noetherian, and respectively hereditary or serial.) Section 3 is concerned with the above result on HNP-rings (Hereditary Noetherian prime rings), and § 4 gives a necessary and sufficient condition so that $\hat{R}$ be an HNP-ring. Section 1 gives preliminaries. Section 2 contains some general results needed for the main theorem; in particular, it contains some properties of the endomorphism ring of a quasi-injective module over a semi-local ring satisfying some additional conditions.

1. Preliminaries. All rings considered have unity, need not be commutative and the modules are unitary. J denotes the Jacobson radical of a ring R. R is said to be *semi-local* if R/J is Artinian. An ideal I of a ring has the right AR-property if for each right ideal B there is an n such that $B \cap I^n \subset BI$. The socle of a module M is denoted by soc$M$. The socle series of a module M is the ascending sequence $\{\text{soc}_n M : n \geq 0\}$ of submodules of M defined as follows: $\text{soc}_0 M = (0)$; $\text{soc}_{n+1} M = \pi_n^{-1}(\text{soc}(M/\text{soc}_n M))$ where $\pi_n : M \to M/\text{soc}_n M$ is the canonical map. If M is a right R-module and if T and N are subsets of R and M respectively, then $\text{ann}_R N$ and

$\text{ann}_M T$ are defined as:

$$\text{ann}_R N = \{r \in R \,|\, nr = 0 \quad \forall\, n \in N\}\,;$$
$$\text{ann}_M T = \{m \in M \,|\, mt = 0 \quad \forall\, t \in T\}\,.$$

If R is a semi-local ring with $\bigcap_{n=1}^{\infty} J^n = (0)$ then the J-adic completion $\hat{R}$ of R is defined as in the commutative case. One takes for a neighborhood base of zero the set of all the powers of J. This defines a topology on R with respect to which it is a topological ring. This topology gives rise to a metric on R. $\hat{R}$ is the completion of this metric space R and it turns out that $\hat{R}$ is a ring containing R as a subring, and is complete in its radical topology; further, $\hat{R}$ is semi-perfect. For more details one is referred to [1] and [12]. Note that $\hat{R}$ can also be realized as the inverse limit, $\varprojlim (R/J^n)$, of the system $\cdots \to R/J^3 \to R/J^2 \to R/J$ where the maps are canonical.

Other definitions will be given whenever needed. Unexplained terms and notations will be standard and well known.

2. Some results, which are needed in the sequel, are proved in this section.

The main ideas in the proof of the following theorem may be found in [17; 5.15]. For convenience the details are given.

THEOREM 1. *Let E_R be an injective module over a ring R and $S = \text{End} E_R$, the ring of R-endomorphisms of E_R. If E_R is Artinian then S is left Noetherian.*

Proof. Since E is finite-dimensional, S is semi-perfect and the radical of S, say H, consists of all $f \in S$ which annihilate essential submodules of E. In particular, if f is in H, $\ker f \neq (0)$. Now let T be a left ideal of S contained in H. Suppose $A_R = \text{ann}_E T = \bigcap_{f \in T} \ker f$. Since E is Artinian, the family of all submodules of E of the form

$$(*) \qquad\qquad \ker f_1 \cap \ker f_2 \cap \cdots \cap \ker f_n\,,$$

where $f_i \in T$, has a minimal member. Suppose $(*)$ is a minimal member of this family. Then for any $f \in T$, one has

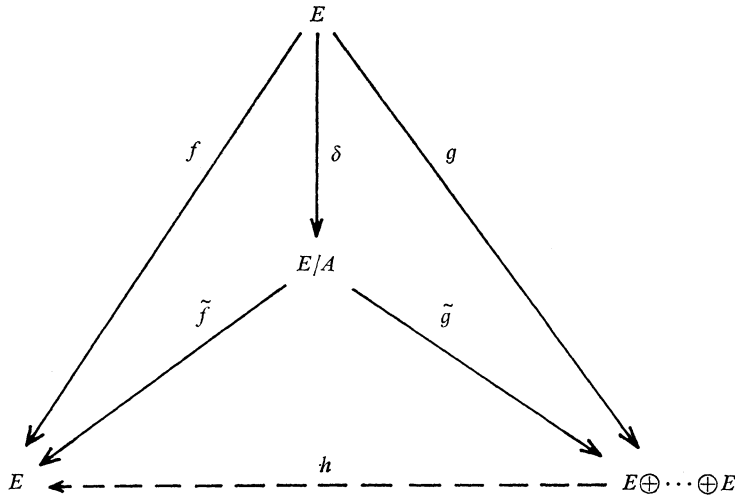$$\ker f \cap \ker f_1 \cap \cdots \cap \ker f_n = \ker f_1 \cap \cdots \cap \ker f_n\,.$$

This implies that $\ker f \supset \ker f_1 \cap \cdots \cap \ker f_n$ and hence $A_R = \ker f_1 \cap \cdots \cap \ker f_n$.

Define $g(e) = (f_1(e), \cdots, f_n(e))$, where $e \in E$. Then g is an R-homomorphism of E into $E \oplus E \oplus \cdots \oplus E$, the direct sum of n copies of E. Clearly, $\ker g = A_R$. Now, let $f \in T$. Then $f(A) = (0)$

and f and g both give rise to induced mappings:

$$\tilde{f}\colon E/A \longrightarrow E\,, \quad \tilde{g}\colon E/A \longrightarrow E \oplus \cdots \oplus E\,.$$

Further, $\tilde{g}$ is a monomorphism. Consider [the following commutative diagram, where $\delta\colon E \to E/A$ is the natural map.

Since E is injective, there is an R-homomorphism $h\colon E \oplus \cdots \oplus E \to E$, such that $h\tilde{g} = \tilde{f}$.

For $1 \leqq i \leqq n$, denote by α_i the ith injection: $E \to E \oplus \cdots \oplus E$ and by π_i, the projection. Then for each $e \in E$,

$$f(e) = \tilde{f}\delta(e) = h\tilde{g}\delta(e) = hg(e)$$
$$= h\left(\sum_{i=1}^{n} \alpha_i \pi_i g(e)\right) = h\left(\sum_{i=1}^{n} \alpha_i f_i(e)\right) = \sum_{i=1}^{n} h\alpha_i f_i(e)\,.$$

This implies that $f = \sum_{i=1}^{n}(h\alpha_i)f_i$ and hence $T = Sf_1 + \cdots + Sf_n$. Since T is an arbitrary left ideal of S contained in H, $_SH$ is Noetherian. Since the ring S/H is Artinian, it follows that S is left Noetherian. This proves the theorem.

Let M_R be a module over a ring R, $\{x_i\}_{i \in I} \subset M$, and $\{N_i\}_{i \in I}$ a set of submodules of M. Then the set of congruences $\{x \equiv x_i \bmod N_i\}_I$ is said to be solvable (finitely solvable) if there is a $y \in M$ (a $y_F \in M$, for each finite subset F of I) such that $y - x_i \in N_i$ for each $i \in I (y_F - x_i \in N_i$ for each $i \in F)$. M_R is said to be *Linearly Compact* if every finitely solvable set of congruences in M is solvable. A ring R is *right linearly compact* if the module R_R is linearly compact.

The following theorem establishes a connection between completeness and the linear compactness for the semi-local rings with

$\bigcap_{n=1}^{\infty} J^n = (0)$.

THEOREM 2. *Let R be a semi-local right Noetherian ring with* $\bigcap_{n=1}^{\infty} J^n = (0)$, *where J denotes the Jacobson radical of R. Then R is J-adic complete if and only if it is right linearly compact and J has the right AR-property.*

Proof. Suppose R is J-adic complete. By [12; 2.8], J has the right AR-property. R is right linearly compact is proved by Hinohara [7; Cor. to Thm. 2].

Conversely, suppose R is right linearly compact and J has the right AR-property. By [9; 4.7], the right AR-property for J implies that $\widehat{R}$ is isomorphic with the biendomorphism ring of E_R, where E_R is the injective hull of the right R-module R/J. It is clear that E_R is an injective cogenerator in the category of right R-modules. In [16; p. 342], it has been shown that a ring is right linearly compact if and only if every injective cogenerator in the category of right modules over it satisfies the doule centralizer property and is quasi-injective over its R-endomorphism ring. (A module M_R has the double centralizer property means the canonical ring homomorphism of R into the biendomorphism ring of M_R is onto.) Hence, in the present case, it follows that $R \cong \widehat{R}$, since E_R is faithful. This completes the proof.

It should be noted that the above theorem is no longer true if one drops the Noetherian hypothesis. For example, a semi-primary ring is always J-adic complete; but a right linearly compact semi-primary ring has to be right Artinian. In a slightly different direction one can show the following: If R is a semi-local ring with $\bigcap_{n=1}^{\infty} J^n = (0)$ such that it is J-adic complete, then R is right Noetherian if and only if it is right linearly compact. For, if R is right linearly compact then R_R and each homomorphic image of it have finite uniform dimension [16]. Hence, if $A \neq 0$ is a right ideal of R, then $A \neq AJ$ and A/AJ is finite direct sum of simple modules. Completeness of R implies that A is finitely generated [12; 2.3 (i)].

Some lemmas and propositions, needed later on, are given below.

LEMMA 3. *Let R be a semi-local ring with Jacobon radical J and M, a right R-module with essential socle. If $S = \operatorname{End} M_R$ and if H denotes the ideal of S consisting of the elements which annihilate essential submodules of M, then for each n,*

$$H^n \subset \operatorname{ann}_S(\operatorname{ann}_M J^n) \, .$$

If further, $M = \bigcup_{n=1}^{\infty} \mathrm{soc}_n M,$ *then* $\bigcap_{n=1}^{\infty} H^n = (0).$

Proof. It is well known that $H = \mathrm{ann}_S(\mathrm{soc}\, M_R) = \mathrm{ann}_S(\mathrm{ann}_M J)$. Suppose the statement of the lemma is true for n, and consider $s \in H$ and $t \in H^n$. Put $T_k = \mathrm{ann}_M J^k$. Then $sT_{n+1}J^n \subset sT_1 = (0)$, since $T_{n+1}J^nJ = (0)$. This implies $sT_{n+1} \subset T_n$. By hypothesis, $H^n \subset \mathrm{ann}_S T_n$. Thus for $t \in H^n$, it follows that $tT_n = (0)$. Hence $tsT_{n+1} \subset tT_n = (0)$. Since $t \in H^n$ and $s \in H$ are arbitrary, $H^{n+1}T_{n+1} = (0)$, that is, $H^{n+1} \subset \mathrm{ann}_S T_{n+1}$. This proves the first statement. For the second statement note that R semi-local implies $\mathrm{soc}_n M = \mathrm{ann}_M J^n$. Let $B_n = \mathrm{ann}_M H^n$. By the first statement, it is clear that $B_n \supset \mathrm{ann}_M J^n = \mathrm{soc}_n M$. Since $M_R = \bigcup_{n=1}^{\infty} \mathrm{soc}_n M_R$, it follows that $_S M = \bigcup_{n=1}^{\infty} B_n$. Let $K = \bigcap_{n=1}^{\infty} H^n$. K annihilates each B_n and hence $_S M$. Since $_S M$ is faithful over S, $K = (0)$ and the lemma is proved.

LEMMA 4. *Let R be a semi-local ring, M_R a quasi-injective module with essential socle and $S = \mathrm{End}\, M_R$. If $M_R = \bigcup_{n=1}^{\infty} \mathrm{soc}_n M_R$ and if M_R is finite-dimensional, then $\mathrm{soc}_S M$ is essential in $_S M$ and $_S M = \bigcup_{n=1}^{\infty} \mathrm{soc}_{nS} M$.*

Proof. Let $H = \mathrm{rad}\, S$. Then H consists of those elements of S which annihilate essential submodules of M_R. Put $B_n = \mathrm{ann}_M H^n$. Since S is semi-perfect, $B_n = \mathrm{soc}_{nS} M$. As seen in the proof of the previous lemma, $M_R = \bigcup_{n=1}^{\infty} \mathrm{soc}_n M_R$ implies $_S M = \bigcup_{n=1}^{\infty} \mathrm{soc}_{nS} M$. Now, let $m \in M$. Then $m \in B_n$ for some n and hence $H^n m = (0)$. Since H is a two-sided ideal, $H^n Sm = (0)$ and the cyclic submodule Sm of $_S M$ is a module over the semi-primary ring S/H^n. Since every module over a semi-primary ring contains a minimal submodule, it follows that Sm intersects $\mathrm{soc}_S M$ nontrivially. (Since $\mathrm{soc}\, M_R \neq (0)$, $\mathrm{ann}_M H = \mathrm{soc}_S M \neq (0)$.) Thus $\mathrm{soc}_S M$ is essential in $_S M$.

LEMMA 5. *Let M_R be a quasi-injective module over a ring R and $S = \mathrm{End}\, M_R$. If M_R has finitely generated essential socle, then $_S M$ contains a copy of each simple left S-module.*

Proof. It is sufficient to note that S is a semi-perfect ring and $\mathrm{rad}\, S = \mathrm{ann}_S(\mathrm{soc}\, M_R)$, the $\mathrm{soc}\, M_R$ being a left module over S.

PROPOSITION 6 [9; 4.3]. *The following conditions on an arbitrary ring R are equivalent:*

(i) *If M is a cyclic right R-module with essential socle then $M = \mathrm{soc}_n M$ for some positive integer n.*

(ii) *If E is an injective right R-module with essential socle then $E = \bigcup_{n=1}^{\infty} \mathrm{soc}_n E$.*

If R is a semi-local ring then the above two conditions are equivalent to the following one:

(iii) *J has the right AR-property.*

It is well known that if T is a direct sum of the injective hulls of nonisomorphic copies of simple right R-modules then it is a (minimal) cogenerator in the category of right R-modules. This means every right R-module can be embedded in a direct product of the copies of T.

PROPOSITION 7. *If R is an arbitrary ring for which the equivalent conditions* (i) *and* (ii) *of Prop. 6 hold then $\bigcap_{n=1}^{\infty} J^n = (0)$. In particular, if R is a semi-local ring whose Iacobson radical J has the right AR-property, then $\bigcap_{n=1}^{\infty} J^n = (0)$ even if R is not right Noetherian.*

Proof. Suppose $K = \bigcap_{n=1}^{\infty} J^n \neq (0)$. Since for each injective right R-module E with essential socle, $E = \bigcup_{n=1}^{\infty} \operatorname{soc}_n E$, it is clear that K annihilates injective hull of each simple right R-module. Consequently, K annihilates the (minimal) cogenerator in the category or right R-modules. This implies K annihilates every right R-module. Thus $K \neq (0)$ leads to a contradiction.

Before stating one more theorem of this section a definition is needed. For each $n = 1, 2, \cdots, M^n$ denotes a product of n copies of a module M_R. If each module, which is isomorphic to a submodule of a factor module of M^n for some n, can be embedded in a product of copies of M, then M is said to be a *self-cogenerator*. Obviously any cogenerator in the category of R-modules is a self-cogenerator.

THEOREM 8. *Let R be a semi-local ring whose Jacobson radical J has the right AR-property, M_R a quasi-injective module with essential socle, $S = \operatorname{End} M_R$ and $H = \operatorname{rad} S$. Then*

(i) $\bigcap_{n=1}^{\infty} H^n = (0)$;

(ii) *if M_R is injective and Artinian, then S is a left Noetherian semi-perfect ring which is left linearly compact;*

(iii) *if M_R is injective, Artinian and a self-cogenerator, then $_S M$ is an injective cogenerator in the category of left S-modules and H has the left AR-property; consequently, S is complete in its radical topology.*

Proof. (i) Since J has the right AR-property, by Proposition 6 $M_R = \bigcup_{n=1}^{\infty} \operatorname{soc}_n M_R$. By Lemma 3 the conclusion follows.

(ii) Since M_R is Artinian, by Theorem 1 S is left Noetherian

semi-perfect. Further, every Artinian module is linearly compact. Then as an immediate consequence of [16; 3.7], it follows that S is left linearly compact.

(iii) Since M_R is quasi-injective, self-cogenerator and linearly compact, by [16; p. 342 Cor. 2] it follows that $_sM$ is injective. By Lemma 5, $_sM$ contains a copy of each simple left S-module and hence $_sM$ is a cogenerator in the category of left S-modules. Further, by Lemma 4, $_sM = \bigcup_{n=1}^{\infty} \operatorname{soc}_{nS}M$. If N is a cyclic left S-module with essential socle, since S is left Noetherian, N can be embedded in a product of finite number of copies of $_sM$. This leads to $_sN = \operatorname{soc}_{nS}N$ for some n. Thus by Proposition 6, H has the left AR-property. Now, since S is also left linearly compact, by Theorem 2 S is complete in the H-adic topology.

Some known examples of rings, over which the injective hull of any simple module is Artinian, are: (1) Any commutative Noetherian ring [17; 4.30]; (2) Noetherian PI-rings [10]; (3) Noetherian serial rings [8; Thm. 2], and (4) HNP-rings [18]. For all of these rings the Jacobson radical has the right, as well as left, AR-property. For this property of HNP-rings one may refer to [2; 3.3].

3. HNP-rings. It has been shown by Chatters that any hereditary Noetherian ring R satisfies the restricted minimum condition for both left and right ideals. As a consequence, the Jacobson radical J of such a ring has both left and right AR-property. If the ring R is prime with $J \neq (0)$, then J is essential right ideal in R and therefore R/J is Artinian. Thus any HNP-ring with $J \neq (0)$ is semi-local and its radical J has the AR-property.

In [18; Thm. 4], S. Singh shows that if R is an HNP-ring which is not right primitive then the injective hull E of a simple right R-module has a unique ascending chain of submodules

$$(0) = x_0 R \subset x_1 R \subset \cdots \subset x_n R \subset \cdots ,$$

whose union is E, and the members of this chain are the only submodules of E. This, in particular, clearly shows that E is Artinian. It has been also shown that the endomorphism ring of such a module E is a local Dedekind domain which is complete (and left and right PID). With these remarks, one can have the following.

THEOREM 9. *Let R be an HNP-ring with $J \neq (0)$. Then the injective hull of any simple R-module is Artinian over its endomorphism ring.*

Proof. Let E be the injective hull of a simple right R-module.

Then $D = \operatorname{End} E_R$ is a complete local domain which is both left and right PID [18; Thm. 9]. Since D is an HNP-ring, injective hull of any simple D-module is Artinian and so is each injective D-module with finitely generated essential socle. Thus it is sufficient to show that $_D E$ has finite-dimensional essential socle.

Since E_R is Artinian and Jacobson radical of R has the AR-property, $\operatorname{soc}_D E$ is essential in $_D E$ by Lemma 4. Now if H denotes the Jacobson radical of D, clearly $\operatorname{ann}_E H$ is the socle of $_D E$ and is a bisubmodule of $_D E_R$. Since E_R has a unique ascending chain of submodules

$$(0) = x_0 R \subset x_1 R \subset \cdots \subset x_n R \subset \cdots ,$$

whose union is E, it follows that for some $m \geqq 1$, $\operatorname{soc}_D E = x_m R$. (Note that $\operatorname{soc}_D E$ cannot be equal to $_D E$ since E is faithful over D.) As $x_m R = \operatorname{ann}_R J^m$, $x_m R$ is an injective module over the Artinian ring R/J^m. Therefore by [5; Thm. 2], $x_m R$ is finitely generated over its R/J^m − endomorphism ring, and so over its R-endomorphism ring. As $x_m R$ is fully-invariant submodule of E_R, its R-endomorphism ring is isomorphic to the ring $D/\operatorname{ann}_D(x_m R)$. Clearly as a module over $D/\operatorname{ann}_D(x_m R)$ and as a module over its R-endomorphism ring, $x_m R$ has the same submodule structure. Hence over $D/\operatorname{ann}_D(x_m R)$, and therefore over D, $x_m R$ is finitely generated left module; consequently the socle of $_D E$ is finitely generated. This completes the proof.

LEMMA 10. *Let M be a right module over a ring R and $S = \operatorname{End} M_R$. Suppose S contains a finite set of orthogonal idempotents $e_1, \cdots, e_n$ such that $1 = e_1 + \cdots + e_n$. If for each $i = 1, \cdots, n$, $e_i M$ is Artinian (Noetherian) over its endomorphism ring $e_i S e_i$, then $_S M$ is Artinian (Noetherian).*

Proof. $A = e_1 S e_1 \oplus \cdots \oplus e_n S e_n$ is a subring of S containing the unity. Also $M_R = e_1 M \oplus \cdots \oplus e_n M$. It is easy to see that $_A M$, and therefore $_S M$, is Artinian (Noetherian).

One is now in a position to have the following

THEOREM 11. *Let R be an HNP-ring with $J \neq (0)$. Then the J-adic completion $\hat{R}$ of R is a Noetherian hereditary semi-prime ring.*

Proof. Let E_R be an injective hull of the right R-module R/J, and $S = \operatorname{End} E_R$. Clearly E_R is an injective cogenerator with finitely generated essential socle. Since E_R is a finite direct sum of inde-

composable injectives, each Artinian over its endomorphism ring (Theorem 9), by Lemma 10 $_sE$ is Artinian. Further, E_R itself is Artinian and a self-cogenerator. Thus by Theorem 8 (iii), $_sE$ is an injective cogenerator in the category of left S-modules and S is left Noetherian and complete. Now, Theorem 8 (iii) applies also to $_sE$ and it follows that End_sE is right Noetherian and complete in its radical topology. Since the radical J of R has the AR-property, by [9; 4.7] $\hat{R} \cong \text{End}_sE$. This shows that $\hat{R}$ is right Noetherian. By symmetry, and the uniqueness of $\hat{R}$ (up to isomorphism), it is both left and right Noetherian.

Now, as R is an HNP-ring, each factor module of E_R is injective. Thus all requirements of [14; 2.1] are satisfied by the module E_R, namely, E_R is injective self-cogenerator and every finitely cogenerated factor of E_R is injective. Therefore S is left semi-hereditary. Since S is left Noetherian, it is left hereditary. Now the same arguments hold for S and $_sE$, and hence $\hat{R} \cong \text{End}_sE$ is right hereditary. Being Noetherian, $\hat{R}$ is both left and right hereditary.

It remains to see that $\hat{R}$ is a semi-prime ring. For this, first let $x \in R$ be a right regular element. Since $xR \cong R$ under the map $xr \to r$, it follows that $x\hat{R} \cong \hat{R}$ under the map $x\hat{r} \to \hat{r}$ (see the proof of [6; 3.4]). Thus x is also a right regular element in $\hat{R}$. In particular, every regular element of R remains regular in $\hat{R}$.

Now as $\hat{R}$ is a Noetherian hereditary ring, by a well known theorem of Chatters [3], $\hat{R} = A \oplus B$ where A is an Artinian and B a semi-prime ring. Since R is prime, J is essential in R and hence contains a regular element t. Therefore t is regular in $\hat{R}$ and is contained in its Jacobson radical $\hat{J}$. This implies that $\hat{R}$ cannot have a minimal left or right ideal. Thus $A = (0)$ and $\hat{R}$ is semi-prime.

COROLLARY 13. *Let R be an HNP-ring with $J \neq (0)$ and M_R an injective cogenerator with finitely generated essential socle. The endomorphism ring of M_R is left hereditary, left Noetherian and complete in its radical topology.*

COROLLARY 14. *If R is an hereditary Noetherian semi-local ring with Jacobson radical J, then the J-adic completion $\hat{R}$ is a product of an Artinian ring and a finite number of HNP-rings, each complete in its radical topology.*

Proof. Since R is Noetherian and hereditary, by the theorem of Chatters [3], $R = A \oplus B_1 \oplus \cdots \oplus B_n$, where A is an Artinian ring and each B_i is an HNP-ring. It is easy to verify that $\hat{R} = A \oplus \hat{B}_1 \oplus \cdots \oplus \hat{B}_n$, where $\hat{B}_i$ is the J-adic completion of B_i. Each

$\hat{B}_i$ is a Noetherian hereditary complete semi-prime ring and will decompose as a direct product of HNP-rings, each complete in its radical topology.

In [20; 5.11], Warfield has shown that if R is a Noetherian serial ring, with Jacobson radical J, then $\bigcap_{n=1}^{\infty} J^n = 0$, and R is the product of an Artinian serial ring and a hereditary serial ring with no simple left or right ideals. Clearly the latter type of serial ring is semi-prime; (see the proof of [20; 5.14]). Moreover, by [20; 4.7] any semi-perfect, semi-prime Noetherian ring is hereditary if and only if it is a serial ring. Recalling that any complete semi-local ring is semi-perfect, one obtains the following

COROLLARY 15. *Let R be a Noetherian serial ring with Jacobson radical J. Then the J-adic completion of R is also a Noetherian serial ring.*

4. Theorem 11 states that if R is an HNP-ring with $J \neq (0)$, then the J-adic completion $\hat{R}$ of R is a hereditary Noetherian semi-prime ring. One may obviously sak: "Why isn't $\hat{R}$ prime?"

Firstly, it can be easily seen that, in general, $\hat{R}$ need not be prime. For this, let S be a commutative Dedekind domain, say the ring of integers, and $P_1, P_2, \cdots, P_m$ a finite number of nonzero prime ideals of S; (obviously P_i are maximal). Let M be the complement of the union of $P_1, \cdots, P_m$. The quotient ring S_M, with respect to the multiplicative system M, is again a Dedeking domain with exactly m maximal ideals [21; Example 2, p. 227, Example 2, p. 270]. This ring S_M serves the purpose; (when S is the ring of integers and p_i prime numbers such that $P_i = (p_i)$, S_M is the ring consisting of rationals with denominators not divisible by any of the p_i). Set $R = S_M$, and let $M_1, \cdots, M_m$ be the maximal ideals of R; then its Jacobson radical $J = M_1 \cap \cdots \cap M_m = M_1 M_2 \cdots M_m$. In general, $J^n = M_1^n \cap M_2^n \cap \cdots \cap M_m^n$. This leads to $R/J^n \cong R/M_1^n \oplus \cdots \oplus R/M_m^n$, hence $\lim_{\leftarrow} (R/J^n) \cong \lim_{\leftarrow} (R/M_1^n) \oplus \cdots \oplus \lim_{\leftarrow} (R/M_m^n)$. Thus $\hat{R}$ is not indecomposable, and so cannot be a prime ring.

For describing conditions intrinsic to R, which render $\hat{R}$ of Theorem 11 prime or not prime, a theorem of B.J. Muller is helpful. Before stating this theorem, some definitions are needed.

Let R be a right Noetherian ring, $S = \bigcap_{i=1}^{n} P_i$ a semi-prime ideal, where the prime ideals $\{P_1, \cdots, P_n\}$ are supposed to be incomparable. S is called *right-localizable* if the multiplicative set $\mathscr{C}(S) = \{c \in R \mid cx \in S \text{ implies } x \in S\}$ is right Ore. Let ρ_S denote the left exact radical corresponding to the torsion theory determined by $\mathscr{C}(S)$. $\rho_S(R)$ is an ideal of R. The image $\overline{\mathscr{C}(S)}$ of $\mathscr{C}(S)$ under the canonical map $R \to \bar{R} = R/\rho_S(R)$ is a regular right Ore set in $\bar{R}$ when

$\mathscr{C}(S)$ is right Ore in R. The (classical) right quotient ring of $\bar{R}$ w.r.t. $\overline{\mathscr{C}(S)}$ is defined to be the right quotient ring of R at the semi-prime ideal S, and is denoted by R_S. R_S happens to be a right Noetherian semi-local ring (when S is right-localizable).

S is called *right-classical* if it is right-localizable and if, in addition, the Jacobson radical of R_S has the right AR-property.

A ring R is Noetherian, means it is left and right Noetherian; the same applies to the terms like localizable, classical etc.

A nonempty finite set $\{P_1, \cdots, P_n\}$ of mutually incomparable prime ideals is called a *Clan* if its associated semi-prime ideal $\bigcap_{i=1}^n P_i$ is classical but the associated semi-prime ideals of all proper subsets are not.

Now the Muller's theorem may be stated.

THEOREM 16 [15; Thm. 4]. *Let R be a Noetherian ring with the classical semi-prime ideal $S = \bigcap_{i=1}^n P_i$. Then there is a one-to-one correspondence between the central idempotents of $\hat{R}_S$ and the localizable subsets of $\{P_1, \cdots, P_n\}$ (where $\hat{R}_S$ denotes the $J(R_S)$-adic completion of R_S). Such subsets are automatically classical.*

Since $\hat{R}_S$ is semi-perfect, Theorem 16 implies that $\{P_1, \cdots, P_n\}$ is uniquely partitioned into clans, and clans correspond to centrally indecomposable central orthogonal idempotents of $\hat{R}_S$. For these results, and the above preliminaries, one may refer to [15, 9].

Now coming to the main objective of this section, first note that if R is an HNP-ring with $J \neq (0)$, then its maximal ideals fall into disjoint cycles [4]; a clan of maximal ideals is a cycle and vice versa [15; §3.3]. Further, for any semi-local ring R with Jacobson radical J, $\mathscr{C}(J)$ consists of the units of R; J is localizable, and $_J R = R_J = R$. Thus, in Theorem 16, if R is also a semi-local ring and S its radical then $\hat{R}$ and $\hat{R}_S$ coincide. The following theorem is now obvious.

THEOREM 17. *Let R be an HNP-ring with $J \neq (0)$, and $\hat{R}$ the J-adic completion of R. The hereditary Noetherian semi-prime ring $\hat{R}$ decomposes into n prime rings if and only if the maximal ideals of R fall into n distinct cycles.*

COROLLARY 18. *Let R be an HNP-ring with $J \neq (0)$, then the completion $\hat{R}$ is also an HNP-ring if and only if the maximal ideals of R form a single cycle.*

REMARKS. (i) The referee has brought to the author's notice a recent paper of P. Vamos [19] in which, among other things, it

has been shown that if R is a semi-local ring satisfying a poly-
nomial identity, then $\hat{R}$ is a Noetherian semi-local PI-ring satisfying
all the identities of R, and $\hat{R}$ has Morita duality.

If R is an HNP-ring with $J \neq (0)$, by Theorem 2, $\hat{R}$ is (both
left and right) linearly compact. Since $\hat{R}$ is a direct sum of HNP-
rings, it is easy to see that the $\hat{R}$-injective hull of $\hat{R}/\hat{J}$ is Artinian.
Thus, by [16; Cor 2, p. 344], $\hat{R}$ has Morita duality in this case also.
In fact, any complete Noetherian semi-local ring whose minimal in-
jective cogenerator is Artinian, has Morita duality.

(ii) The present paper lists commutative Noetherian rings,
Noetherian PI-rings, Noetherian serial rings and HNP-rings as the
known classes of rings over which the injective hull of any simple
module is Artinian. As is now clear, the completion of a Noetherian
semi-local ring belonging to any of the above classes is Noetherian.
Hence one would be tempted to ask the following question (which
will be left as open):

If R is a Noetherian semi-local ring over which the injective
hull of any simple module is Artinian, is $\hat{R}$ Noetherian?

Note that for such a ring R, the Jacobson radical J has the
AR-property.

(iii) In [11], it has been shown that if M is a maximal ideal
of a Dedekind prime ring R and P is any maximal right ideal con-
taining M, then the M-adic completion $\hat{R}$ of R can be realized as
the biendomorphism ring of the injective hull of the right R-module
R/P (and $\hat{R}$ is a local principal ideal domain). In view of [9; 4.7]
and [13], this result generalizes, with appropriate modifications, to
the case where R is an HNP-ring and M its Jacobson radical. An
obvious question is: Can same thing be said when R is an arbit-
rays HNP-ring and M any invertible ideal?

The author proposes to further exploit Theorem 16 in a separate
communication; as a consequence of the results ontained there, one
will find an affirmative answer to the above question.

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