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ASYMPTOTIC PRIME DIVISORS AND GOING DOWN

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Let I be an ideal in a commutative Noetherian domain R, and let $\hat{I}$ be the integral closure of I. It is known that the sequences of sets of primes $\operatorname{Ass}(R/I^n)$ and $\operatorname{Ass}(R/\hat{I}^n)$ both eventually become constants, those constants denoted $A^*(I)$ and $\hat{A}^*(I)$ respectively. The main result of this paper is that if T is an integral extension of a local domain (R, M), and if I is an ideal of R such that T/IT contains a height 0 maximal ideal, then $M \in \hat{A}^*(I)$. This fact is then used to study when $\hat{A}^*(P) = \{P\}$ for P a prime of R. (This is a variation of the question when does $P^n = P^{(n)}$ for all large n?) It is shown that if $\hat{A}^*(P) = \{P\}$, then "going down to P" holds. Finally, the main argument is used to produce an example of an n-dimensional local domain, (R, M) such that for any $P \in \operatorname{Spec} R - \{0\}$, and any $m \geq 2$, $M \in \operatorname{Ass}(R/P^m)$. Also the analytic spread of any such P is n.

Background. The question concerning the asymptotic behavior of $\operatorname{Ass}(R/I^n)$ and $\operatorname{Ass}(R/\hat{I}^n)$ was posed in [12] which essentially showed the existence of $\hat{A}^*(I)$ (also see [9, Proposition 7]). The existence of $A^*(I)$ was proved in [1].

Notation. Throughout, R will be a Noetherian domain with integral closure R'. If I is an ideal of R, $\hat{I}$ will be the integral closure of I. If R is local, $v(I)$ will be the minimal number of generators of I and $l(I)$ will be the analytic spread of I. Finally, "$\subset$" will denote proper containment.

THE MAIN ARGUMENT.

THEOREM 1.1. *Let I be an ideal in a local domain (R, M). Let T be an integral extension domain of R and assume that T/IT contains a height 0 maximal ideal. Then there is an integer $n \geqq 1$ with the following property: If J is any ideal of R satisfying $I^m \subseteqq J \subseteqq \hat{M}^n$ (any $m \geqq 1$), then $M \in \operatorname{Ass}(R/J)$.*

Proof. We first reduce to the case that T is a finite R-module contained in R'. By a simple going up argument, T'/IT' contains a height 0 maximal ideal. Since $R' \subseteqq T'$ satisfies going down, R'/IR' has a height 0 maximal, say N'/IR'. Choose $u \in N'$ but u in no other maximal ideal of R'. Let $N = N' \cap R[u]$. Then N' is the only prime of R' lying over N. We now easily see that $N/IR[u]$ is a height 0 maximal of $R[u]/IR[u]$. Thus we may assume that $T = R[u]$ is a finite R-module contained in R'.

We have a maximal ideal N of T with N minimal over IT. Let (V, P) be a D.V.R. overring of T with $P \cap T = N$. Choose $b \in R$ with $bT \subseteqq R$, and select n sufficiently large that $b \notin P^n$. Since N is minimal over $I^m T$, for some $s \in T - N$ and integer $k \geqq 1$, we have $sN^k \subseteqq I^m T$. Thus $bsM^k \subseteqq bsN^k \subseteqq I^m(bT) \subseteqq I^m R = I^m$. We also claim that $bs \notin \hat{M}^n$, since if $bs \in \hat{M}^n \subseteqq \hat{N}^n \subseteqq \hat{P}^n = P^n$, since s is a unit of V we have $b \in P^n$, a contradiction. Now if $I^m \subseteqq J \subseteqq \hat{M}^n$ then clearly $bsM^k \subseteqq J$ and $bs \in R - J$, showing that M^k consists of zero divisors modudo J. Thus $M \in \operatorname{Ass}(R/J)$.

REMARK. Our choice of n actually only depended on b and P (that is, on b and N). In particular this n will work for any ideal I such that N is minimal over IT.

COROLLARY 1.2. *Let I be an ideal in a local domain (R, M). If there exists an integral extension domain T of R such that T/IT contains a height 0 maximal ideal, then $M \in \hat{A}^*(J)$ for any ideal J of R satisfying $I \subseteqq \operatorname{Rad}(J)$.*

Proof. With n as in Theorem 1.1, clearly for all $m \geqq n$ we have $I^m \subseteqq \hat{I}^m \subseteqq \hat{M}^n$. Thus $M \in \operatorname{Ass}(R/\hat{I}^m)$, so that $M \in \hat{A}^*(I)$. Since $I \subseteqq \operatorname{Rad}(J)$, clearly T/JT contains a height 0 maximal and so similarly we get $M \in \hat{A}^*(J)$.

The converse of Corollary 1.2 if false. That is, there is a local domain (R, M) containing an ideal I, such that $M \in \hat{A}^*(J)$ for any ideal J whose radical contains I, but such that R is integrally closed and $\dim R/I > 0$ (so that no T is in Corollary 1.2 exists, using going down). For this, suppose that (R, M) is a 2-dimensional integrally closed local domain, and that $I = P$ is a height 1 prime ideal with the property that P is not the radical of any principal ideal. Let $P \subseteqq \operatorname{Rad}(J)$. If $P = \operatorname{Rad}(J)$, then J is not principal and so by [9, Proposition 21] and [7, Theorem 6] we have $M \in A^*(J) = \hat{A}^*(J)$. If instead $P \neq \operatorname{Rad}(J)$, then clearly $M = \operatorname{Rad}(J)$ and $M \in \hat{A}^*(J)$. As $\dim R/P = 1$ and R is integrally closed, the converse of Corollary 1.2 fails. There are domains (R, M) with a prime P as just described. A. Sathaye has shown that $K[X, Y, Z]/(ZY^2 - X^3 - Z^3)$ is such a domain, and [4] constructs a 2-dimensional integrally closed local domain in which no height 1 prime is the radical of a principal ideal.

Ratliff has proved the following lovely pair of theorems. Here R^* denotes the completion of the local domain R.

THEOREM A ([13, Theorem 9]). *The following are equivalent for a local domain* (R, M).

(1) R^* *contains a depth* 1 *yrime divisor of* 0.

(2) *There is an integer* $n \geqq 1$ *such that for any ideal* $I \subseteqq M^n$, $M \in \mathrm{Ass}\,(R/I)$.

THEOREM B ([14, Theorem 1]). *The following are equivalent for a local domain* (R, M).

(1) R^* *contains a depth* 1 *minimal prime.*

(2) R' *contains a height* 1 *maximal prime.*

(3) *There is an integer* $n \geqq 1$ *such that for any ideal* $I \subseteqq M^n$, $M \in \mathrm{Ass}\,(R/\hat{I})$.

(Note: In Theorem B, Ratliff actually only assumes Rad $R = 0$.)

The equivalence of (1) and (2) in Theorem B follows from an earlier theorem of Ratliff [11, Proposition 3.5]. Ratliff's proof of Theorem B argues that (1) is equivalent to (3). We will now give a more elementary proof that (2) is equivalent to (3). (Note: Condition (iii) below is new.)

PROPOSITION 1.3 ([14, *Theorem* 1]). *Let* (R, M) *be a local domain with integral closure* R' *and completion* R^*. *The following are equivalent.*

(i) R^* *contains a depth* 1 *minimal prime.*

(ii) R' *contains a height* 1 *maximal prime.*

(iii) *There is an* $n \geqq 1$ *such that for any ideal* $I \subseteqq \hat{M}^n$, $M \in \mathrm{Ass}\,(R/I)$.

(iv) *There is an* $n \geqq 1$ *such that for any ideal* $I \subseteqq M^n$, $M \in \mathrm{Ass}\,(R/\hat{I})$.

(v) *For any ideal* I *of* R, *there is an* $n \geqq 1$ *with* $M \in \mathrm{Ass}\,(R/\hat{I}^n)$.

(vi) $M \in \mathrm{Ass}\,(R/\hat{a})$ *for some* $a \in R$.

Proof. (i) $\Leftrightarrow$ (ii): By [11, Proposition 3.5]:

(ii) $\Rightarrow$ (iii): Let N' be a height 1 maximal of R'. If $u \in N'$ but u is not in any other maximal prime of R', then let $T = R[u]$ and $N = N' \cap T$, so that height $N = 1$. For any ideal I of R, N is minimal over IT. By the remark following the proof of Theorem 1.1, we see that (iii) is satisfied by the n constructed in Theorem 1.1.

(iii) $\Rightarrow$ (iv): If $I \subseteqq M^n$ then $\hat{I} \subseteqq \hat{M}^n$.

(iv) $\Rightarrow$ (v): Straightforward.

(v) $\Rightarrow$ (vi): Straightforward.

(vi) $\Rightarrow$ (ii): Since $\widehat{aR} = aR' \cap R$, M can be lifted to a prime divisor of aR' in R'. This prime will have height 1.

REMARK. Theorem A together with [13, Remark 12.1] show that in conditions (iv), (v), and (vi) above we must use the integral closures of the ideals.

Going down and $\hat{A}^*(P)$. A question which has received some attention is when does $P^n = P^{(n)}$ for P a prime ideal. For this to hold for all large n is obviously equivalent to $A^*(P) = \{P\}$. We pose the question, when does $\hat{A}^*(P) = \{P\}$? (Since $\hat{A}^*$ appears to be better behaved than A^*, as evidenced by [7, Theorem 3] for example, our version of the question might be more tractable than the other version.) In this section, we will show that $\hat{A}^*(P) = \{P\}$ implies a pleasant going down property. We will then show that that going down property often fails.

DEFINITION. Let $P \subset Q$ be primes in a domain R. We will say that $P \subset Q$ *satisfies going down* if for any integral extension domain T of R and any prime q of T with $q \cap R = Q$, there is a prime p of T with $p \subset q$ and $p \cap R = P$.

PROPOSITION 2.1. *Let* $P \subset Q$ *be primes in a Noetherian domain* R. *If* $P \subset Q$ *does not satisfy going down, then there is a prime* Q_1 *of* R *with* $P \subset Q_1 \subseteqq Q$, $P \subset Q_1$ *does not satisfy going down, and* $Q_1 \in \hat{A}^*(P)$.

Proof. Let T be an integral extension domain of R and let q be prime in T with $q \cap R = Q$ such that there is no prime p of with $p \subset q$ and $p \cap R = P$. Choose q_1 prime in T with $PT \subseteqq q_1 \subseteqq q$ and q_1 minimal over PT. Let $Q_1 = q_1 \cap R$. Clearly $P \subset Q_1$ fails going down. Letting $S = R - Q_1$ and considering $R_S \subset T_S$, since $(q_1)_S$ is minimal over $P_S T_S$, by Corollary 1.2 we have $(Q_1)_S \in \hat{A}^*(P_S)$. Thus $Q_1 \in \hat{A}^*(P)$.

COROLLARY 2.2. *Let* P *be prime in a Noetherian domain* R. *If* $\hat{A}^*(P) = \{P\}$, *then* $P \subset Q$ *satisfies going down for any prime* Q *containing* P.

Proof. Immediate.

REMARK. The converse of Corollary 2.2 fails. Let (R, M) be a 2-dimensional integrally closed local domain, and let P be a non-principal height 1 prime of R. Since R is integrally closed, $P \subset M$ satisfies going down. However, by [9, Proposition 21] and [7, Theorem 6], $M \in A^*(P) = \hat{A}^*(P)$. (Notice that such an (R, M) is Macaulay, and hence satisfies the altitude formula.)

COROLLARY 2.3. *Let* $P \subset Q$ *be primes in a Noetherian domain* R *with height* $(Q/P) = 2$. *If* $P \subset Q$ *fails to satisfy going down, then for all but finitely many primes* p *with* $P \subset p \subset Q$, *we have* $Q \in \hat{A}^*(p)$. *For those* p *with* $Q \notin \hat{A}^*(p)$ *we have* $p \in \hat{A}^*(P)$.

Proof. Suppose $P \subset p \subset Q$, p prime. If $p \notin \hat{A}^*(P)$ then Proposition 2.1 easily shows that $P \subset p$ satisfies going down. Since $P \subset Q$ fails going down, obviously $p \subset Q$ must fail going down. Now by Proposition 2.1, we see that $Q \in \hat{A}^*(p)$ for all $p \notin \hat{A}^*(P)$. As $\hat{A}^*(P)$ is finite, we are done.

It is not difficult to produce situations in which $P \subset Q$ fails to satisfy going down, as is illustrated by [5, Theorem 2] or [6]. As an example of how the arguments in [5] or [6] can be combined with the present arguments, we present the following.

PROPOSITION 2.4. *Let* R *be a Noetherian domain with integral closure* R'. *Let* Q *be a prime of* R *with height* $Q \geqq 2$. *Suppose that in* R', *more than one prime lies over* Q. *Then there are infinitely many primes* P *of* R *satisfying* $P \subset Q$, *height* $(Q/P) = 1$ *and* $Q \in \hat{A}^*(P)$.

Proof. A simple variation of the proof of [5, Theorem 2] shows that there are infinitely many primes $P \subset Q$ with $P \subset Q$ failing to satisfy going down. If infinitely many such P also satisfy height $(Q/P) = 1$ then for each of these we have $Q \in \hat{A}^*(P)$ by Proposition 2.1, and we are done. Otherwise for some $P \subset Q$ which fails going down, wehave height $(Q/P) > 1$. We claim that such P can be found with height $(Q/P) = 2$. Since infinitely many primes p satisfy $P \subset p \subset Q$ and height $(Q/p) =$ height $(Q/P) - 1$, we may pick such a p not in $\hat{A}^*(P)$. By Proposition 2.1, (since height $(p/P) = 1$) we have $P \subset p$ satisfies going down. Thus $p \subset Q$ must fail going down. Our claim now follows by induction. We now have $P \subset Q$, heigh $(Q/P) = 2$ and $P \subset Q$ fails going down. The result follows from Corollary 2.3.

We close this section with an observation concerning local domains which satisfy the altitude formula. Recall that $l(I)$ is the analytic spread of the ideal I.

PROPOSITION 2.5. *Let* (R, M) *be a local domain which satisfies the altitude formula. Let* P *be a prime ideal of* R. *Consider the statements* (a) $l(P) =$ *height* P; (b) $\hat{A}^*(P) = \{P\}$. *Then* (a) *implies* (b) *but not conversely.*

Proof. Suppose that $P \neq Q \in \hat{A}^*(P)$. By [7, Theorem 3], $l(P_Q) =$

height $Q >$ height P. However it is not difficult to see that $l(P) \geqq l(P_Q)$. Thus (a) fails. Therefore (a) implies (b).

To see that (b) does not imply (a), we consider a 3-dimensional integrally closed local domain (R, M), satisfying the altitude formula and having R/M infinite, with a height 1 prime P whose minimal number of generators $v(P) = 2$, but such that $v(P_Q) = 1$ for all height 2 primes Q. We first see that (b) holds. Since $v(P_Q) = 1$, clearly $l(P_Q) = 1 <$ height $Q = 2$. By [7, Theorem 3], $Q \notin \hat{A}^*(P)$ for any height 2 prime Q. Also $l(P) \leqq v(P) = 2 <$ height $M = 3$. Thus $M \notin \hat{A}^*(P)$. Therefore $\hat{A}^*(P) = \{P\}$ and (b) holds. To show (a) fails, we need $l(P) \neq$ height $P = 1$. If $l(P) = 1$ then (since R/M is infinite) there is a principal reduction, cR, of P. Thus $cR \subseteqq P \subseteqq \widehat{cR}$. As R is integrally closed, $cR = \widehat{cR}$, contradicting $v(P) = 2$.

It remains to be seen that such an (R, M) and P exists. P. Eakin provided the following example. With K an infinite field and X, Y, Z, W indeterminates, let $R = K[X, Y, Z, W]_{(X,Y,Z,W)}/(XY - ZW)_{(X,Y,Z,W)}$. This is integrally closed using [15, Theorem 1]. Let P be the image of (X, Z). Since $\bar{X} = \bar{Z}(\bar{W}/\bar{Y})$ and $\bar{Z} = \bar{X}(\bar{Y}/\bar{W})$ and since any height 2 prime Q containing P fails to contain one of $\bar{W}$ or $\bar{Y}$, $v(P_Q) = 1$. Since $(XY - ZW) \subset (X, Y, Z, W)^2$, the maximal ideal of R requires 4 generators. Thus P must not be principal, so $v(P) = 2$.

REMARK. We now have that if P is prime in a local domain satisfying the altitude formula, then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) and none of the reverse, with

(i) $l(P) =$ height P

(ii) $\hat{A}^*(P) = \{P\}$

(iii) $P \subset Q$ satisfies going down for any prime Q containing P.

We mention that the proof in (i) $\Rightarrow$ (iii) is not difficult. Thus the significance of the above is that (ii) lies properly between (i) and (iii).

AN EXAMPLE. Let (R, M) be a local domain with completion R^*. In [13, 10.2] Ratliff asks if the following two conditions are equivalent.

(I) R^* contains a depth 1 prime divisor of 0.

(II) For $P \in \operatorname{Spec} R - \{0, M\}$, $P^n \neq P^{(n)}$ for all large n. That is, $A^*(P) \neq \{P\}$.

((I) $\Rightarrow$ (II) is immediate from Theorem A.) The following example shows that (II) does not imply (I), as well as having other interesting characteristics.

EXAMPLE. Let $n \geqq 2$ be an integer. By [3, Theorem A] it is possible to construct a Noetherian domain T with exactly two maximal ideals N_1 and N_2 with height $N_1 =$ height $N_2 = n$, such that there is a field F and indeterminates $X_{i1}, \cdots, X_{in}$ $i = 1, 2$ with $T_{N_i} = F[X_{i1}, \cdots, X_{in}]_{(X_{i1}, \cdots, X_{in})}$. Thus $T/N_i \approx F$, $i = 1, 2$. Also, for each $0 \neq Q \in \operatorname{Spec} T$, Q is in exactly one of N_1 or N_2. Now let φ be a field isomorphism from T/N_1 onto T/N_2 and let $R = \{t \in T \mid \varphi(t + N_1) = t + N_2\}$. By [2, Theorem A], R is a local domain with maximal ideal $M = N_1 \cap N_2$. Since $MT \subseteqq R$, T is a finite R-module in the quotient field of R, and for each $P \in \operatorname{Spec} R - \{M\}$, there is a unique prime of T lying over P.

(a): R is analytically unramified. By [10, §36], T_{N_i} is analytically unramified $i = 1, 2$. Thus T is analytically unramified and since T is a finite R-module, R is analytically unramified.

(b): R satisfies the altitude formula. For this, Ratliff has shown that it is equivalent to see that $R[X]_{(M,X)}$ is catenary [8, Corollary 2.5]. However this follows easily from the fact that $T[X]_{(N_i,X)}$ is $n + 1$-dimensional and catenary, $i = 1, 2$.

(c): In the completion , R^*, each prime divisor of 0 is minimal and has depth n. Since R is analytically unramified, $\operatorname{Rad} R^* = 0$ and so each prime divisor of 0 is minimal. Since R satisfies the altitude formula, R is quasi-unmixed so that each minimal prime in R^* has depth n [11, Theorem 3.1].

(d): For $P \in \operatorname{Spec} R - \{0, M\}$, $M \in \hat{A}^*(P) \subseteqq A^*(P)$. Let Q be the unique prime of T lying over P. Without loss, we may assume $Q \subset N_1$ so that $Q \nsubseteqq N_2$. We claim that N_2 is minimal over PT. If $PT \subseteqq q \subset N_2$ with q prime in T, let $p = q \cap R$. Since $P \subseteqq p$, by going up, Q can be enlarged to a prime lying over p. However q is the unique prime of T lying over p. Thus $Q \subseteqq q \subset N_2$. This contradiction proves our claim that N_2 is minimal over PT. By Corollary 1.2, $M \in \hat{A}^*(P)$.

(e): For $0 \neq P \in \operatorname{Spec} R$, $l(P) = n$. This follows from (b), (d) and [7, Theorem 3].

(f): If $P \in \operatorname{Spec} R - \{0, M\}$ we in fact have $M \in \operatorname{Ass}(R/P^m)$ for all $m \geqq 2$. Pick $b_1 \in N_1 - (N_1^2 \cup N_2)$ and $b_2 \in N_2 - (N_2^2 \cup N_1)$. Then $b = b_1 b_2 \in (N_1 \cap N_2) - (N_1^2 \cup N_2^2)$. As was argued in (d), we have (say) N_2 minimal over PT. Thus for some $s \in T - N_2$ and integer $k \geqq 1$, $sN_2^k \subseteqq P^m T$. Therefore $sbM^k \subseteqq sbN_2^k \subseteqq P^m(bT) \subseteqq P^m R = P^m$ (since $b \in N_1 \cap N_2 = M$ and $MT \subseteqq R$). Now $sb \in R - P^m$ since if $sb \in P^m \subseteqq M^m \subseteqq M^2 \subseteqq N_2^2$, then since $s \in T - N_2$ we would have $b \in N_2^2$, a contradiction. As $sbM^k \subseteqq P^m$ and $sb \in R - P^m$, M^k consists of zero divisors modulo P^m so that $M \in \operatorname{Ass}(R/P^m)$ for any $m \geqq 2$.

REMARK. In fact (II) $\Rightarrow$ (I) in Ratliff's question, even if R is integrally closed. [4] constructs a 2-dimensional integrally closed local domain (R, M) in which each height 1 prime is nonprincipal. By [9, Proposition 21] R satisfies (II). Also 2-dimensional integrally closed implies R is Macaulay so that (I) is known to fail.

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