PACIFIC JOURNAL OF MATHEMATICS
Vol. 91, No. 1, 1980

THE RADIUS OF STARLIKENESS FOR A CLASS OF REGULAR FUNCTIONS DEFINED BY AN INTEGRAL

V. KARUNAKARAN AND M. R. ZIEGLER

Let $F(z)$, $f(z)$, and $g(z)$ be regular in the unit disc $E = \{z: z < 1\}$, be normalized by $F(0) = f(0) = g(0) = 0$ and $F'(0) = f'(0) = g'(0) = 1$, and satisfy the equation $z^{c-1}(c + 1)f(z) = [F(z)g(z)^c]'$, $c \geqq 0$. This paper is concerned with studying relationships between the mapping properties of these functions. The principle result is the determination of the radius of β-starlikeness of $f(z)$ when $F(z)$ and $g(z)$ are restricted to certain classes of univalent starlike functions. Conversely, a lower bound for the radius of β-starlikeness of $F(z)$ is obtained when $f(z)$ and $g(z)$ satisfy similar conditions.

Problems of this nature were first studied by Libera [9], where he showed that if $f(z)$ is a convex, starlike, or close-to-convex univalent function and $F(z)$ is defined by

$$F(z) = \frac{2}{z}\int_0^z f(t)dt\ , \tag{1}$$

then $F(z)$ is also convex, starlike, or close-to-convex, respectively. Livingston then considered the converse of this problem and determined that if $F(z)$ satisfies one of these geometric conditions in E and $f(z) = (F(z) + zF'(z))/2$, then $f(z)$ satisfies the same condition in $\{z: |z| < 1/2\}$ [11]. Refinements of Livingston's results can be found in [1], [2], [10], [12], and [13], while results dealing with generalizations of (1) appear in [3], [4], [5], [6], [7], and [8]. Most recently, Lewandowski et al have shown that if $f(z)$ is starlike in E and $F(z)$ is the solution of

$$cF(z) + zF'(z) = (1 + c)f(z)\ , \tag{2}$$

then $F(z)$ is starlike whenever $\operatorname{Re} c \geqq 0$ [8].

Before proceeding any further, it will be convenient to introduce the following notation. Let $S^*(\alpha)$ denote the collection of functions $f(z)$ which are regular in E, are normalized by $f(0) = 0$ and $f'(0) = 1$, and satisfy $\operatorname{Re}[zf'(z)/f(z)] \geqq \alpha$ for z in E. Such functions are said to be starlike of order α. Normally one only considers α in the interval $[0, 1)$, however, in order to relate the results presented here to earlier works, it is advantageous to allow $\alpha = 1$, with the understanding that $S^*(1)$ consists only of the function $f(z) = z$.

In this paper we continue the investigation of a generalization of (1) which was introduced by the first author in [7]. Let $\mathscr{F}_1(\alpha, \gamma, c)$ denote the family of functions $F(z)$ which satisfy

$$(3) \qquad F(z) = \frac{c+1}{[g(z)]^c} \int_0^z t^{c-1} f(t) dt ,$$

where $f(z)$ is in $S^*(\alpha)$, $g(z)$ is in $S^*(\gamma)$ and $c \geqq 0$. Let $\mathscr{F}_2(\alpha, \gamma, c)$ denote the family of functions $f(z)$ which satisfy

$$(4) \qquad (c+1)f(z) = c[g(z)/z]^{c-1} g'(z) F(z) + [g(z)/z]^c z F'(z)$$

for $F(z)$ in $S^*(\alpha)$, $g(z)$ in $S^*(\gamma)$ and $c \geqq 0$. Theorem 1 provides a lower bound for the radius of β-starlikeness of $\mathscr{F}_1(\alpha, \gamma, c)$ and Theorem 3 gives the radius of β-starlikeness of $\mathscr{F}_2(\alpha, \gamma, c)$.

We begin by stating a slight generalization of the result obtained by Lewandowski et al mentioned above. Since our result follows directly from the techniques used in [8], the proof will be omitted.

LEMMA 1. *If $F(z)$ and $f(z)$ satisfy (2), $f(z)$ is in $S^*(\alpha)$ and $c \geqq 0$, then $F(z)$ is in $S^*(\alpha)$.*

This lemma now enables us to determine a lower bound for the radius of β-starlikeness of $\mathscr{F}_1(\alpha, \gamma, c)$.

THEOREM 1. *If $F(z)$ is in $\mathscr{F}_1(\alpha, \gamma, c)$, then $F(z)$ is β-starlike for $|z| < \sigma = \sigma(\alpha, \beta, \gamma, c)$, where σ is the least positive root of the equation*

$$(5) \qquad \begin{aligned} 1 - \beta - r[2(1-\alpha) + 2c(1-\gamma)] \\ - r^2[2\alpha - 1 - \beta + 2c(1-\gamma)] = 0 . \end{aligned}$$

Proof. If $h(z) = [(c+1)/z^c] \int_0^z t^{c-1} f(t) dt$ then $F(z) = [z/g(z)]^c h(z)$ and Lemma 1 implies $h(z)$ is in $S^*(\alpha)$. Differentiating logarithmically and applying the usual inequalities we obtain

$$\operatorname{Re}\left\{\frac{zF'(z)}{F(z)}\right\} \geqq \frac{1 + (2\alpha - 1)r}{1 + r} + \frac{2c(1-\gamma)r}{1 - r} .$$

Thus $\operatorname{Re}\{zF'(z)/F(z)\} \geqq \beta$ whenever $|z| < \sigma$ where σ is the least positive root of (5).

Before turning our attention to the principal result of this paper, we state without proof two lemmas which appear in [7] and are fundamental to what follows.

LEMMA 2. *If $\omega(z)$ is analytic and satisfies $|\omega(z)| \leqq |z|$ in E and* *if* $p(z) = (1 + D\omega(z))/(1 + B\omega(z))$, $-1 \leqq D < B \leqq 1$, *then for* $|z| = r < 1$ *we have*

$$\operatorname{Re}\left\{\frac{z\omega'(z)}{(1 + D\omega(z))(1 + B\omega(z))}\right\}$$
$$\leqq \frac{-1}{(B - D)^2}\left[\operatorname{Re}\left\{\frac{D}{p(z)} + Bp(z)\right\} - \frac{r^2|Bp(z) - D|^2 - |p(z) - 1|^2}{(1 - r^2)|p(z)|}\right] + \frac{B + D}{(B - D)^2}\,.$$

LEMMA 3. *If* $p(z)$ *and* $\omega(z)$ *satisfy the conditions of Lemma* 2, *then for any* $K \geqq B$ *we have on* $|z| = r$

$$\operatorname{Re}\left\{Kp(z) + \frac{D}{p(z)}\right\} - \left[\frac{r^2|Bp(z) - D|^2 - |1 - p(z)|^2}{(1 - r^2)|p(z)|}\right]$$
$$\geqq \begin{cases} P_1(r) & \textit{for} \quad R_0 \leqq R_1 \\ P_2(r) & \textit{for} \quad R_0 \geqq R_1 \end{cases}$$

where

$$P_1(r) = P_1(r, K, B, D) = K\frac{1 + Dr}{1 + Br} + D\frac{1 + Br}{1 + Dr}\,,$$
$$P_2(r) = P_2(r, K, B, D)$$
$$= \frac{2}{(1 - r^2)}[(1 + D)(1 + K - (B^2 + K + D(1 + K))r^2 + D(B^2 + K)r^4)]^{1/2} - \frac{2(1 - BDr^2)}{1 - r^2}\,,$$
$$R_0^2 = [(1 + D)(1 - Dr^2)]/[(1 + K) - (K + B^2)r^2]\,,$$

and

$$R_1 = (1 + Dr)/(1 + Br).$$

The above estimates are sharp.

THEOREM 2.

$$\min_{f \in \mathscr{F}_2(\alpha,\gamma,c)} \min_{|z|=r} \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} = \begin{cases} Q_1(r)\,, & R_0 \leqq R_1 \\ Q_2(r)\,, & R_0 \geqq R_1 \end{cases}$$

where

$$(6) \qquad Q_1(r) = \frac{1 + r(1 + 2D - K) + r^2D(2 - K)}{(1 + r)(1 + Dr)}\,,$$

$$(7) \qquad Q_2(r) = \frac{2}{1 - D}\left[\left\{\frac{(1 + D)(1 + K)(1 - Dr^2)}{1 - r^2}\right\}^{1/2} - \frac{1 - Dr^2}{1 - r^2}\right] - \frac{K - 1 + 2D}{1 - D}\,,$$

(8) $$R_0^2 = [(1 + D)(1 - Dr^2)]/[(1 + K)(1 - r^2)] ,$$

(9) $$R_1 = (1 + Dr)/(1 + r) ,$$

$\delta = (\alpha + c\gamma)/(1 + c)$, $D = 2\delta - 1$, *and* $K = 1 + (c + 1)(1 - D)$.

Proof. Let $s(z) = z[F(z)/z]^{1/(c+1)}[g(z)/z]^{c/(c+1)}$ where in each multi-valued expression we choose the branch which has value 1 at $z = 0$. Combining this with (4) yields

(10) $$f(z) = [s(z)/z]^c zs'(s) .$$

Since

$$\frac{zs'(z)}{s(z)} = \frac{1}{(1 + c)}\left[\frac{zF'(z)}{F(z)} + c\frac{zg'(z)}{g(z)}\right],$$

$s(z)$ is in $S^*(\delta)$ for $\delta = (\alpha + c\gamma)/(1 + c)$, $p(z) = zs'(z)/s(z)$ is analytic in E, $p(0) = 1$ and $\operatorname{Re}[p(z)] \geqq \delta$, z in E. Consequently, there exists a function $\omega(z)$ analytic in E and satisfying $|\omega(z)| \leqq |z|$, $z \in E$, such that

(11) $$p(z) = \frac{1 + D\omega(z)}{1 + \omega(z)} , \quad D = 2\delta - 1 .$$

Now differentiating (10) and making use of (11), we have

$$\begin{aligned}\frac{zf'(z)}{f(z)} &= (c + 1)p(z) + \frac{zp'(z)}{p(z)} - c \\ &= (c + 1)p(z) + \frac{z\omega'(z)(D - 1)}{(1 + D\omega(z))(1 + \omega(z))} - c\end{aligned}$$

and Lemma 2 now yields

$$\begin{aligned}\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqq \frac{1}{(1 - D)}\Bigg[&\operatorname{Re}\left\{Kp(z) + \frac{D}{p(z)}\right\} \\ &- \frac{r^2|p(z) - D|^2 - |p(z) - 1|^2}{(1 - r^2)|p(z)|}\Bigg] - c - \frac{1 + D}{1 - D} ,\end{aligned}$$

where $B = 1$ and $K = 1 + (c + 1)(1 - D)$. An application of Lemma 3 now completes the proof. Sharpness follows directly from the sharpness of Lemma 3.

In [7] the radius of β-starlikeness of $\mathscr{F}_2(\alpha, \gamma, c)$ is determined in the case $c = 1$ and $\alpha + \gamma \leqq 1$. The following result extends this to include all permissible values of α, γ and c.

THEOREM 3. *Let* $r_* = r_*(\alpha, \gamma, c, \beta)$ *be the radius of* β*-starlikeness of* $\mathscr{F}_2(\alpha, \gamma, c)$. *Let* $D = 2\delta - 1$, $\delta = (\alpha + c\gamma)/(1 + c)$, $c \geqq 0$, $0 \leqq \alpha < 1$, *and* $0 \leqq \gamma \leqq 1$. *For each fixed* c *in* $[0, \infty)$, *let* $r(D)$ *be the* *unique solution in* (0, 1] *of the equation*

$$(12) \quad (2 + c) - (4 - 2D - 2Dc + c)r - D(5 - D + 2c - Dc)r^2 + D(1 - D - Dc)r^3 = 0 .$$

If $Q_1(r)$ *and* $Q_2(r)$ *are defined by* (6) *and* (7) *and* $\mu(D) = Q_1(r(D))$, *then the equation* $\mu(D) = 0$ *has a unique solution* D_0 *in* $(-1, 1)$. *Furthermore, if* D *satisfies* $D_0 < D < 1$ *and* $0 \leqq \beta \leqq \mu(D)$, *then* r_* *is the unique root in* (0, 1) *of the equation* $Q_2(r) = \beta$. *For all other values of* D, r_* *is the unique root in* (0, 1) *of the equation* $Q_1(r) = \beta$.

Proof. Let $I(r) = \min_{f \in \mathscr{F}_2(\alpha,\gamma,c)} \min_{|z|=r} \operatorname{Re}\{zf'(z)/f(z)\}$ and let R_0 and R_1 be defined by (8) and (9). A differentiation shows R_0 is a decreasing function of r and R_1 is an increasing function of r, hence the equation $R_0 = R_1$ has a unique solution $r(D, c)$ which is the unique root in (0, 1] of (12). Thus

$$I(r) = \begin{cases} Q_1(r) & 0 \leqq r < r(D, c) \\ Q_2(r) & r(D, c) \leqq r < 1 \end{cases} ,$$

with the understanding that the second inequality holds vacuously when $r(D, c) = 1$. An examination of (12) shows this happens only when $D = -1$, in which case r_* is the solution of $Q_1(r) = \beta$. Since $\alpha < 1$ implies $D < 1$, we can now restrict our attention to $D \in (-1, 1)$.

It follows from the minimum principle and the compactness of $\mathscr{F}_2(\alpha, \gamma, c)$ that $I(r)$ is a continuous, decreasing function of r. [In fact one can show $Q_1'(r(D, c)) = Q_2'(r(D, c))$ so that $I(r)$ is differentiable and $I'(r) < 0$ on (0, 1).] Since $r(D, c) < 1$ for $D > -1$, $\lim I(r)$ $(r \to 1^-) = \lim Q_2(r)$ $(r \to 1^-) = -\infty$, and, since $I(0) = 1$, the equation $I(r) = \beta$ will always have a unique solution r_* in (0, 1). Clearly r_* is always the solution of either $Q_1(r) = \beta$ or $Q_2(r) = \beta$, depending on the relationship between the roots of these equations and $r(D, c)$, or equivalently, on the relationship between $I(r(D, c))$ and β. The remainder of this argument is concerned with determining this relationship.

Let $c \in [0, \infty)$ be fixed, let $r(D) = r(D, c)$ and let $\mu(D) = Q_1(r(D)) = Q_2(r(D))$. We will show $\mu(D)$ is a strictly increasing function of D mapping $(-1, 1)$ onto $(-\infty, 1)$. Now

$$\mu'(D) = \frac{d}{dD} Q_1(r(D)) > 0$$

if and only if

$$(13) \quad r'(D) < \frac{r(D)(1 + r(D))}{1 - D} \cdot \frac{(1 + c)(1 + Dr(D))^2 + 1 + r(D)}{(1 + c)(1 + Dr(D))^2 + 1 - Dr(D)^2} .$$

Since the second factor in the right hand side of (13) is clearly greater than 1, it is sufficient to show

$$r'(D) < r(D)(1 + r(D))/(1 - D) . \tag{14}$$

Differentiating (12) implicitly yields

$$\begin{aligned} r'(D) = [&2(1 + c)r(D) + (2D - 5 - 2c + 2Dc)r(D)^2 \\ &+ (1 - 2D - 2Dc)r(D)^3]/[(4 - 2D - 2Dc - c) \\ &+ 2D(5 - D + 2c - Dc)r(D) - 3D(1 - D - Dc)r(D)^2] , \end{aligned} \tag{15}$$

and, before substituting (15) in (14), we must determine the sign of the denominator in (15). Let

$$\begin{aligned} p(r) &= (1 + K)(1 - r)(1 + r)^2(R_0^2 - R_1^2) \\ &= (2 + c) - (4 - 2D - 2Dc + c)r \\ &\quad - D(5 - D - 2c - Dc)r^2 + D(1 - D - Dc)r^3 \end{aligned}$$

so that $p(r(D)) = 0$ and the denominator in (15) is $-p'(r(D))$. Since R_0 is decreasing and R_1 is increasing, $p(r)$ changes sign at $r(D)$ and must have a zero of order 1 or 3 at $r(D)$. If $r(D)$ is a root of order 3 then $p''(r(D)) = 0$ which implies

$$r(D) = (5 + 2c - D - Dc)/(3(1 - D - Dc)) .$$

However this last expression is not in (0, 1) for $D \in (-1, 1)$ and $c \in [0, \infty)$, hence $r(D)$ is a root of order 1 and, since $P(r)$ is decreasing at $r(D)$, $p'(r(D)) < 0$. Thus the denominator in (15) is positive and substituting (15) in (14) then shows that (14) is equivalent to

$$\begin{aligned} &(2 - c) + (9 + D + 3c - 2Dc)r(D) \\ &\quad + (6Dc - D^2c + 10D - 1 - D^2)r(D)^2 \\ &\quad - 3D(1 - D - Dc)r(D)^3 > 0 . \end{aligned} \tag{16}$$

Using the fact that $r(D)$ satisfies (12) to elminate $r(D)^3$ in (16), we find that (16) is equivalent to

$$\begin{aligned} r(D)(7 - 5r(D))(D + 1) + (8 - 10r(D) + 4r(D)^2) \\ + 2D^2r(D)^2 + 2c(1 + Dr(D))^2 > 0 , \end{aligned}$$

which is obviously valid for $r(D)$ in (0, 1), D in $(-1, 1)$ and $c \geqq 0$. Thus $\mu(D)$ is increasing on $(-1, 1)$.

An examination of (12) shows $r(D) \to 1$ when $D \to 1$ or $D \to -1$, hence $\mu(D) \to -\infty$ as $D \to -1$, $\mu(D) \to 1$ as $D \to 1$, and the equation $\mu(D) = 0$ has a unique solution D_0 in $(-1, 1)$. If $-1 < D \leqq D_0$, then $\mu(D) = Q_1(r(D)) \leqq 0$ and r_* is the root of $Q_1(r) = \beta$. If $D_0 < D < 1$, then r_* is the root of $Q_1(r) = \beta$ when $\mu(D) \leqq \beta$ and r_* is the root of $Q_2(r) = \beta$ when $\beta \leqq \mu(D)$. This completes the proof.

If we take $c = \gamma = 1$ and $\alpha = \beta = 0$, then we obtain as a special case Livingston's result [11]. If we let $\gamma = 1$ and $\alpha = \beta = 0$, then we obtain Theorem 1 in [4]. Letting $c = \gamma = 1$ yields results found in [1], [2], [10], [13] and, as we have already noted, the case $c = 1$ and $\alpha + \gamma \leqq 1$ appears in [7].

REFERENCES

1. H. S. Al-amiri, *On the radius of starlikeness of certain analytic functions*, Proc. Amer. Math. Soc., **42** (1974), 466-474.
2. P. L. Bajpai and P. Singh, *The radius of starlikeness of certain analytic functions* Proc. Amer. Math. Soc., **44** (1974), 395-402.
3. S. D. Bernardi, *Convex and starlike univalent functions*, Trans. Amer. Math. Soc., **135** (1969), 429-446.
4. ———, *The radius of univalence of certain anaytic functions*, Proc. Amer. Math. Soc., **24** (1970), 312-318.
5. R. M. Goel, *On radii of starlikeness, convexity, close-to-convexity for p-valent functions*. Arch. Rat. Mech. and Anal., **44** (1972), 320-328.
6. R. M. Goel and V. Singh, *On radii of univalence of certain analytic functions*, Indian J. Pure and App. Math.. **4** (1973), 402-421.
7. V. Karunakaran, *Certain classes of regular univalent functions*, Pacific J. Math., **61** (1975), 173-182.
8. Z. Lewandowski, S. Miller and E. Zlotkiewicz, *Generating functions for some classes of univalent functions*, Proc. Amer. Math. Soc., **56** (1976), 111-117.
9. R. J. Libera, *Some classes of regular univalent functions*, Proc. Amer. Math. Soc., **16** (1965), 755-758.
10. R. J. Libera and A. E. Livingston, *On the univalence of some classes of regular functions*. Proc. Amer. Math. Soc., **30** (1971), 327-336.
11. A. E. Livingston, *On the radius of univalence of certain analytic functions*, Proc. Amer. Math. Soc., **17** (1966), 352-357.
12. K. S. Padmanabhan, *On the radius of univalence of certain classes of analytic functions*, J. London Math. Soc., (2) **1** (1969), 225-231.
13. V. Singh and R. M. Goel, *On radii of convexity and starlikeness of some classes of functions*, J. Math. Soc. Japan, **23** (1971), 323-339.

Received June 27, 1977 and in revised form February 7, 1978. The second author was supported in part by a Marquette University Summer Faculty Fellowship.

NATIONAL COLLEGE
TIRCHIRAPALLI 620001, INDIA
AND
MARQUETTE UNIVERSITY
MILWAUKEE, WI 53233