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LINEAR OPERATORS FOR WHICH T^*T AND TT^* COMMUTE III

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Let T be a bounded linear operator on a Hilbert space H. Let $[T] = T^*T - TT^*$. The structure of operators such that T^*T and TT^* commute and $\operatorname{rank}[T] < \infty$ is studied.

1. Introduction. Let T be a bounded linear operator acting on a separable Hilbert space H. Let $[T] = T^*T - TT^*$ and $(BN) = \{T \mid T^*T$ and TT^* commute$\}$. As in [1] let $(BN)^+ = \{T \mid T \in (BN)$ and T is hyponormal$\}$.

In [2] it is shown that if $T \in (BN)$ and rank $[T] = 1$, (hence either T or T^* is in (BN^+), then $T = T_1 \oplus T_2$ where T_1 is normal and T_2 is a special type of weighted bilateral shift.

The purpose of this note is to examine the extension of this result to those $T \in (BN)$ for which rank $[T] < \infty$. The simple example [1]

$$T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}, \quad TT^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad T^*T = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

of a $T \in (BN)$, $T^2 \notin (BN)$, shows that if rank $[T] = 2$, then different behavior is possible.

2. Notation and preliminary results. The notation of this section will be kept throughout the paper. Suppose that $T \in (BN)$ and rank $[T] = r$. Then, for the correct choice of orthonormal basis we have

$$(1) \qquad T^*T = \begin{bmatrix} D_1 & 0 \\ 0 & P \end{bmatrix}, \quad TT^* = \begin{bmatrix} D_2 & 0 \\ 0 & P \end{bmatrix}$$

where $D_1 = \operatorname{Diag}\{\alpha_1, \cdots, \alpha_r\}$, $D_2 = \operatorname{Diag}\{\beta_1, \cdots, \beta_r\}$ with $\alpha_i \neq \beta_i$ for all i. Let $T = U(T^*T)^{1/2}$ be the polar factorization of T. Thus U is a partial isometry with $R(U) = R(T)$, $N(U) = N(T)$. Note that $U(T^*T)^{1/2} = (TT^*)^{1/2}U = T$ and T^*T and TT^* have identical spectrum except for zero eigenvalues. Also UU^* is the orthogonal projection onto $R(T) = N(T^*)^\perp$ while U^*U is the orthogonal projection onto $R(T^*) = N(T)^\perp$. Now $(T^*T)^{1/2} = U^*(TT^*)^{1/2}U$. Thus for any polynomial $p(\lambda)$,

$$(2) \qquad p((T^*T)^{1/2}) = U^*p((TT^*)^{1/2})U + p(0)(I - U^*U),$$

$$p((TT^*)^{1/2}) = Up((T^*T)^{1/2})U^* + p(0)(I - UU^*)\ . \tag{3}$$

Taking uniform limits shows that (2), (3) hold for any $p \in C[0, \|T\|^2]$.

Let p be such that $p(0) = p(\alpha_i) = p(\beta_i) = 0$, $p(x) > 0$ otherwise. By construction and (1)

$$p(T^*T) = \begin{bmatrix} 0 & 0 \\ 0 & p(P) \end{bmatrix}. \tag{4}$$

But then from (1), $Up(T^*T) = p(TT^*)U$. Thus we have the following.

PROPOSITION 1. *If $T \in (BN)$ and T is completely nonnormal, rank $[T] = r$ and T^*T and TT^* are written as in* (1) *then $\sigma(T^*T) \subseteq \sigma(D_1) \cup \sigma(D_2)$.*

Proof. Proposition 1 follows immediately from the observation that for the p of (4), $\mathscr{M} = R(p(T^*T))$ is a reducing subspace for T on which T is normal and $\sigma((T^*T)|\mathscr{M}^\perp) \subseteq \{\alpha_1, \cdots, \beta_r, 0\}$, $\sigma((TT^*)|\mathscr{M}^\perp) \subseteq \{\beta_1, \cdots, \beta_r, 0\}$. If $0 \neq \alpha_i$ for all i, and $0 \neq \beta_i$ for all i, then $N(T^*T) = N(TT^*)$ and $0 \notin \sigma(T^*T)$, $0 \notin \sigma(TT^*)$ by the complete nonnormality of T. □

For a self-adjoint operator C, let $E_C(\cdot)$ denote its spectral measure.

We shall say $\alpha \sim \beta$ if α_i, $\beta = \beta_i$ for some i. A *web* is a lattice of relations, for example

$$\left.\begin{matrix} k \sim \\ \alpha \sim \end{matrix}\right\} \beta \sim \gamma\ .$$

The relation $\sim$ is not an equivalence relation. A web is maximal if there is no larger web (larger in the sense of cardinality of elements or relations) that contains it as a subweb.

PROPOSITION 2. *Suppose that $T \in (BN)$, T is completely nonnormal, and rank $[T] = r$. Suppose that W is a maximal web. Let Δ be the set of all α_i and β_i that are elements of the web. Then*

(i) *$E_{T^*T}(\Delta) = E_{TT^*}(\Delta)$, and*

(ii) *$R(E_{T^*T}(\Delta))$ is a reducing subspace for T.*

Proof. If $\Delta = \sigma(T^*T) \cup \sigma(TT^*)$, then the result is trivial. So suppose not. Rearranging the basis in (1) we get

$$T^*T = \begin{bmatrix} D_1' & 0 & 0 & 0 \\ 0 & P_1 & 0 & 0 \\ 0 & 0 & D_1'' & 0 \\ 0 & 0 & 0 & P_2 \end{bmatrix} \qquad TT^* = \begin{bmatrix} D_2' & 0 & 0 & 0 \\ 0 & P_1 & 0 & 0 \\ 0 & 0 & D_2'' & 0 \\ 0 & 0 & 0 & P_2 \end{bmatrix}$$

where

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & P_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & P_2 \end{bmatrix}$$

is the P of (1), D_1', D_1'' make up the D_1 of (1) and D_2', D_2'' make up the D_2 of (1). Furthermore, $\sigma(P_1) \subseteqq \Delta$, $\sigma(P_2) \subseteqq \Delta^c$, $\sigma(D_1') \cup \sigma(D_2') \subseteqq \Delta$, and $\sigma(D_1'') \cup \sigma(D_2'') \subseteqq \Delta^c$. Now only one of Δ or Δ^c can contain zero. Let Σ be the one that does not cotain zero. We shall show that the range of $E_{T^*T}(\Sigma) = E_{TT^*}(\Sigma)$ is a reducing subspace for T. Note that either

$$E_{T^*T}(\Sigma) = E_{TT^*}(\Sigma) = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ or they equal } \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}.$$

In either case, the fact that $E_{T^*T}(\Sigma) = E_{TT^*}(\Sigma)$ and $0 \notin \Sigma$, implies from (2) that $R(E_{T^*T}(\Sigma))$ reduces U and hence T. □

From Proposition 2, we now get the following.

THEOREM 1. *If $T \in (BN)$, rank $[T] < \infty$, and T^*T, TT^* have distinct eigenvalues on the reducing subspace $R([T])$, then T is a sum of rank $[T]$ copies of the weighted shift in* [2]. *In particular $T = S_1 \oplus S_2 \oplus N$ where S_1 is hyponormal, S_2 is cohyponormal, and N is normal.*

3. An example. In some sense Theorem 1 is the strongest general result possible. Recall that an operator T is *centered* if $\{T^nT^{*n}, T^{*m}T^m\}$ is commutative. Centered operators have been studied in [3]. All weighted shifts are centered. A consequence of Del Valle's result is that if $T \in (BN)$ and rank $[T] = 1$, then T is centered. (Actually, this is essentially equivalent to it.) We shall now given an example of an operator T, such that $T \in (BN)^+$ (hence T^2 is hyponormal), T isinvertible, rank $[T] = 2$, but $T^2 \notin (BN)$. Hence T is not centered. That $T \in (BN)^+$ does not imply $T^2 \in (BN)$ is known [1]. However, the example of [1] has a self-commutator of infinite rank.

First we need a proposition.

PROPOSITION 3. *Let $P = (T^*T)^{1/2}$, $Q = (TT^*)^{1/2}$, $T = UP$ be the polar decomposition of T. Assume U is unitary. If $T \in (BN)$, then*

$T^2 \in (BN)$ if and only if P^2Q^2 commutes with $U^2P^2Q^2U^{*2}$.

Proof. For operators X, Y. let $[X, Y] = XY - YX$. Assume that T satisfies the assumptions of Proposition 3. Note that $UP = QU$. Now $T^2 = UPUP = UPQU$, so that $T^{*2} = U^*PQU^*$. Hence $T^2 \in (BN)$

$$\begin{aligned} &\Longleftrightarrow [T^2T^{*2}, T^{*2}T^2] = 0 \\ &\Longleftrightarrow [UP^2Q^2U^*, U^*P^2Q^2U] = 0 \\ &\Longleftrightarrow [U^2P^2Q^2U^{*2}, P^2Q^2] = 0 . \end{aligned}$$

□

EXAMPLE. Let $\mathscr{H} = \boldsymbol{C} \oplus H \oplus \boldsymbol{C} \oplus H \oplus H$ where H is a separable Hilbert space. Let P, Q be the operators $P = \mathrm{Diag}\{3, 3I, 2, 2I, I\}$, $Q = \mathrm{Diag}\{2, 3I, 1, 2I, I\}$. Define the unitary operator U as follows, $U(\boldsymbol{C} \oplus H \oplus 0 \oplus 0 \oplus 0) = 0 \oplus H \oplus 0 \oplus 0 \oplus 0$, $U(0 \oplus 0 \oplus 0 \oplus 0 \oplus H) = 0 \oplus 0 \oplus \boldsymbol{C} \oplus 0 \oplus H$. Let ϕ be a vector in H. Let $U[0, 0, 1/\sqrt{2}, \phi/\sqrt{2}, 0] = [1, 0, 0, 0, 0]$. If M is the orthogonal complement of $[0, 0, 1/\sqrt{2}, 1/\sqrt{2}\phi, 0]$ in $0 \oplus 0 \oplus \boldsymbol{C} \oplus H \oplus 0$, let $UM = 0 \oplus 0 \oplus 0 \oplus H \oplus 0$. Now let $T = UP^{1/2}$. It is easy to verify that $P = T^*T$, $Q = TT^*$, and rank $[T] = 2$.

To show that $T^2 \notin (BN)$ we shall use Proposition 3 and show that $[PQ, U^2PQU^*] \neq 0$. Suppose that $[U^2PQU^{2*}, PQ] = 0$. Then every spectral projection of PQ must be a reducing subspace of U^2PQU^*. Now $PQ = \mathrm{Diag}\{6, 9I, 2, 4I, I\}$. Thus $\mathscr{M} = \boldsymbol{C} \oplus 0 \oplus 0 \oplus 0 \oplus 0$ must be a reducing subspace. But

$$\begin{aligned} U^2PQU^2[1, 0, 0, 0, 0] &= U^2PQU^*[0, 0, 1/\sqrt{2}, 1/\sqrt{2}\phi, 0] \\ &= U^2PQ[0, 0, 0, \psi_1, \psi_2](\|\psi_1\| = \|\psi_2\| = 1/\sqrt{2}) = U^2[0, 0, 0, 4\psi_1, \psi_2] \\ &= U[0, 0, 4/\sqrt{2}, 1/\sqrt{2}\phi, 0] = U\{5/2[0, 0, 1\sqrt{2}, 1/\sqrt{2}\phi, 0] \\ &\quad +3/2[0, 0, 1/\sqrt{2}, -1/\sqrt{2}\phi, 0]\} \\ &= 5/2[1, 0, 0, 0, 0] + 3/2[0, 0, 0, \psi_3, 0] \end{aligned}$$

where $\|\psi_3\| = 1/\sqrt{2}$. Which contradicts the fact that $\mathscr{M}$ is reducing.

In a certain sense, Example 1 is a canonical example. We shall conclude by showing how to construct all $T \in (BN)^+$ such that rank $[T] = 2$. The generalization to rank $[T] < \infty$ is straightforward. A similar argument works for $T \in (BN)$, rank $[T] < \infty$, but the large number of cases would make the statement of the theorem unreasonably messy. Suffice it to say that the same type of analysis will handle $T \in (BN)$, rank $[T] < \infty$.

Suppose then that $T \in (BN)^+$, rank $[T] = 2$, T is completely nonnormal, and there is a single maximal web. The possibilities are then ($\hat{\sigma}(T^*T) = \sigma(T^*T \mid R([T]))$, $\hat{\sigma}(TT^*) = \hat{\sigma}(TT^* \mid (R[T]))$)

(I) $\hat{\sigma}(T^*T) = \{\alpha\}$, $\hat{\sigma}(TT^*) = \{\beta\}$, $\alpha > \beta \geqq 0$
(II) $\hat{\sigma}(T^*T) = \{\alpha\}$, $\hat{\sigma}(TT^*) = \{\beta_1, \beta_2\}$, $\alpha > \beta_1 > \beta_2 \geqq 0$
(III) $\hat{\sigma}(T^*T) = \{\alpha_1, \beta_2\}$, $\hat{\sigma}(TT^*) = \{\beta\}$, $\alpha_i > \beta \geqq 0$, $\alpha_1 > \alpha_2 > 0$.
(IV) $\hat{\sigma}(T^*T) = \{\alpha_1, \alpha_2\}$, $\hat{\sigma}(TT^*) = \{\alpha_2, \beta\}$, $\alpha_1 > \alpha_2 > \beta \geqq 0$.

By assumption U is an isometry.

Case I. By the correct choice of orthonormal basis we have if $\beta \neq 0$, $H = \boldsymbol{C}^2 \oplus H_2 \oplus H_3$,

$$(5) \qquad T^*T = \begin{bmatrix} \alpha I_1 & 0 & 0 \\ 0 & \alpha I_2 & 0 \\ 0 & 0 & \beta I_3 \end{bmatrix}, \quad TT^* = \begin{bmatrix} \beta I_1 & 0 & 0 \\ 0 & \alpha I_2 & 0 \\ 0 & 0 & \beta I_3 \end{bmatrix}$$

where I_1 is a 2×2 identity. Both I_2, I_3 must operate on an infinite dimensional space since there exists an isometry of $\boldsymbol{C}^2 \oplus H_2$ onto H_2 and on isometry of H_3 onto $\boldsymbol{C}^2 \oplus H_3$.

Pick an orthonormal basis for $\boldsymbol{C}^2$. Forward iteration under V gives an orthonormal basis for H_2. Iteration under $V^* = V^{-1}$ gives an orthonormal basis for H_3. Thus V is just two copies of the bilateral shift and $T = T_1 \oplus T_2$ where $T_i \in (BN)^+$, Rank $[T_i] = 1$.

If $\beta = 0$, then

$$T^*T = \begin{bmatrix} \alpha I_1 & 0 \\ 0 & \alpha I_2 \end{bmatrix}, \quad TT^* = \begin{bmatrix} 0 & 0 \\ 0 & \alpha I_2 \end{bmatrix},$$

and $T = \sqrt{\alpha}(S \oplus S)$, S the unilateral shift.

Case II. In this case, if $\beta_2 \neq 0$ on $\boldsymbol{C} \oplus \boldsymbol{C} \oplus H_1 \oplus H_2 \oplus H_3$.

$$(6) \quad T^*T = \begin{bmatrix} \alpha & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & \beta_1 I_1 & 0 & 0 \\ 0 & 0 & 0 & \beta_2 I_2 & 0 \\ 0 & 0 & 0 & 0 & \alpha I_3 \end{bmatrix}, \quad TT^* = \begin{bmatrix} \beta_1 & 0 & 0 & 0 & 0 \\ 0 & \beta_2 & 0 & 0 & 0 \\ 0 & 0 & \beta_1 I_1 & 0 & 0 \\ 0 & 0 & 0 & \beta_2 I_2 & 0 \\ 0 & 0 & 0 & 0 & \alpha I_3 \end{bmatrix}.$$

$$(7) \qquad U = \begin{bmatrix} 0 & 0 & U_{13} & 0 & 0 \\ 0 & 0 & 0 & U_{24} & 0 \\ 0 & 0 & U_{33} & 0 & 0 \\ 0 & 0 & 0 & U_{44} & 0 \\ U_{51} & U_{52} & 0 & 0 & U_{55} \end{bmatrix}$$

where

$$\begin{bmatrix} U_{13} \\ U_{33} \end{bmatrix} \text{ is an isometry of } H_1 \text{ onto } \boldsymbol{C} \oplus H_1$$

$$\begin{bmatrix} U_{24} \\ U_{44} \end{bmatrix} \text{ is an isometry of } H_2 \text{ onto } \boldsymbol{C} \oplus H_2$$

$$[U_{51} \quad U_{52} \quad U_{55}] \text{ is an isometry of } \boldsymbol{C} \oplus \boldsymbol{C} \oplus H_3 \text{ onto } H_3 .$$

If $\beta_2 = 0$, the fourth row and column are deleted from both matrices in (6) and $H = \boldsymbol{C} \oplus \boldsymbol{C} \oplus H_1 \oplus H_3$. The essential difference is that whereas (7) is unitary, U is only an isometry if $\beta_2 = 0$.

Case III. In this case, we have for $\beta \neq 0$, $H = \boldsymbol{C} \oplus \boldsymbol{C} \oplus H_1 \oplus H_2 \oplus H_3$

$$(8) \quad T^*T = \begin{bmatrix} \alpha_1 I_1 & 0 & 0 & 0 & 0 \\ 0 & \alpha_2 I_2 & 0 & 0 & 0 \\ 0 & 0 & \alpha_1 I_3 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 I_4 & 0 \\ 0 & 0 & 0 & 0 & \beta I_5 \end{bmatrix}, \quad TT^* = \begin{bmatrix} \beta I_1 & 0 & 0 & 0 & 0 \\ 0 & \beta I_2 & 0 & 0 & 0 \\ 0 & 0 & \alpha_1 I_3 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 I_4 & 0 \\ 0 & 0 & 0 & 0 & \beta I_5 \end{bmatrix}.$$

Then

$$(9) \qquad U = \begin{bmatrix} 0 & 0 & 0 & 0 & U_{15} \\ 0 & 0 & 0 & 0 & U_{25} \\ U_{31} & 0 & U_{33} & 0 & 0 \\ 0 & U_{42} & 0 & U_{44} & 0 \\ 0 & 0 & 0 & 0 & U_{55} \end{bmatrix}$$

where

$$\begin{bmatrix} U_{15} \\ U_{25} \\ U_{55} \end{bmatrix} \text{ is an isometry of } H_3 \text{ onto } \boldsymbol{C} \oplus \boldsymbol{C} \oplus H_3$$

$$[U_{31} \quad U_{33}] \text{ is an isometry of } \boldsymbol{C} \oplus H_1 \text{ onto } H_1$$

$$[U_{42} \quad U_{45}] \text{ is an isometry of } \boldsymbol{C} \oplus H_2 \text{ onto } H_2 .$$

If $\beta = 0$, then the fifth column and row of the matrices in (9) are deleted and $T = T_1 \oplus T_2$ where $T_i \in (BN)^+$, rank $[T_i] = 1$.

Case IV. In this case if $\beta \neq 0$.

$$(10)\quad T^*T = \begin{bmatrix} \alpha_1 & 0 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & 0 \\ 0 & 0 & \alpha_1 I_1 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 I_2 & 0 \\ 0 & 0 & 0 & 0 & \beta I_3 \end{bmatrix}, \quad TT^* = \begin{bmatrix} \alpha_2 & 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 & 0 \\ 0 & 0 & \alpha_1 I_1 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 I_2 & 0 \\ 0 & 0 & 0 & 0 & \beta I_3 \end{bmatrix}$$

on $C \oplus C \oplus H_1 \oplus H_2 \oplus H_3$. If $\beta = 0$, of course, $H_3 = 0$. However, there is no restriction at all on H_2. It may be zero, finite, or infinite dimensional. Suppose $\beta \neq 0$. Then

$$(11)\qquad U = \begin{bmatrix} 0 & U_{12} & 0 & U_{14} & 0 \\ 0 & 0 & 0 & 0 & U_{25} \\ U_{31} & 0 & U_{33} & 0 & 0 \\ 0 & U_{41} & 0 & U_{44} & 0 \\ 0 & 0 & 0 & 0 & U_{55} \end{bmatrix}$$

where

$\begin{bmatrix} U_{12} & U_{14} \\ U_{41} & U_{44} \end{bmatrix}$ is an isometric map of $C \oplus H_2$ onto $C \oplus H_2$.

$[U_{31} \quad U_{33}]$ is an isometric map of $C \oplus H_1$ onto H_1

$\begin{bmatrix} U_{25} \\ U_{55} \end{bmatrix}$ is an isometric map of H_3 onto $C \oplus H_3$.

If $\beta = 0$, then the fifth row and column is deleted from the matrices in (10), (11). ($H_3 = 0$).

If one is interested in constructing a particular example then in (7), (9), (11), the indicated isometries are completely arbitrary as long as they have the correct initial and final spaces. The example was constructed in this way from (11).

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